Stochastic Calculus Michael Tehranchi
Example sheet 3 - Lent 2015

Problem 1. Let X be a continuous local martingale with $X_0 = 0$. For each $s \geq 0$, let

$$T(s) = \inf\{t \geq 0 : \langle X \rangle_t > s\}.$$

Show that the stopped process $X^{T(s)}$ is a square-integrable martingale. Hence, conclude that for almost every $\omega \in \{\langle X \rangle_\infty < \infty\}$ the limit $\lim_{t\uparrow\infty} X_t(\omega)$ exists.

Problem 2. Let $\mathbb{P} \sim \mathbb{Q}$. Show that $X_n \to X$ in $\mathbb{P}$-probability if and only if $X_n \to X$ in $\mathbb{Q}$-probability.

Problem 3. (Yet another martingale representation) Let X be a real continuous local martingale whose quadratic variation is

$$\langle X \rangle_t = \int_0^t a_s ds$$

for a non-negative, predictable process $(a_t)_{t\geq 0}$. Show that there exists a real Brownian motion W (possibly defined on an extended probability space) such that

$$X_t = \int_0^t \sqrt{a_s} dW_s$$

How could this be generalised to higher dimensions?

Problem 4. Let W be a standard two-dimensional Brownian motion, and let $a \in \mathbb{R}^2$, $\|a\| = 1$ be a constant. Let $Z_t = \log \|W_t - a\|$ and set $T_0 = 0$ and

$$T_k = \inf\{t \geq T_{k-1} : |Z_t - Z_{T_{k-1}}| > 1\}$$

(a) Show that for every k, the process $(Z_{t\wedge T_k})$ is a martingale.
(b) Let $X_k = Z_{T_k}$. Use the optional stopping theorem to show that $(X_k)_{k\geq 0}$ is a simple symmetric random walk on the integers.
(c) Conclude that for a two-dimensional Brownian motion W, the set $\{t \geq 0 : \|W_t - a\| < r\}$ is unbounded a.s. for any $a \in \mathbb{R}^2$ and $r > 0$.

Problem 5. Let W be a three-dimensional Brownian motion, and let $\xi \sim N_3(0, I)$ be independent of W. Using the fact that $\mathbb{P}(W_t = \xi \text{ for any } t \geq 0) = 0$, we define a positive process X by

$$X_t = \frac{1}{\|W_t - \xi\|}$$

(a) Use Itô's formula to show that X is a local martingale. Indeed, show that there exists a real Brownian motion B such that

$$dX_t = X_t^2 dB_t.$$

(b) Compute $\mathbb{E}(X_t)$. Why does this show that X is a *strictly* local martingale?
(c) Compute $\mathbb{E}(X_t^2)$. Conclude that X is a uniformly integrable local martingale. Compare this to example sheet 2 problem 1.
(d) Verify that $\mathbb{E}(\langle X \rangle_t) = \infty$ for all $t > 0$.
(e) Show that $X_t \to 0$ a.s. and conclude that for a three-dimensional Brownian motion W, we have $\|W_t\| \to \infty$ a.s.

Problem 6. If M is a scalar continuous local martingale with $\langle M\rangle_t \to \infty$ a.s., show that $M_t/\langle M\rangle_t \to 0$. Conclude that $\mathcal{E}(M)_t \to 0$ a.s.

Problem 7. Let $W = (W_t)_{0\le t\le T}$ be a real Brownian motion generating the filtration $\mathbb{F}$, where the non-random time-horizon T is finite. Also let θ be a bounded predictable process. Show that for any $\mathcal{F}_T$-measurable bounded random variable ξ there exists a constant x and a predictable process $(\alpha_t)_{0\le t\le T}$ such that

$$\xi = x + \int_0^T \alpha_t(dW_t + \theta_t\, dt).$$

Problem 8. Let W be a real Brownian motion and T a stopping time.
(a) Show that if $\mathbb{E}(T) < \infty$ then $\mathbb{E}(W_T^2 - T) = 0$.
(b) Show that if $\mathbb{E}(e^{\theta^2 T/2}) < \infty$ for some $\theta \in \mathbb{R}$, then $\mathbb{E}(e^{\theta W_T - \theta^2 T/2}) = 1$.

Problem 9. (Ornstein–Uhlenbeck process) Let W be a real Brownian motion, and let

$$X_t = e^{-t}\xi + \int_0^t e^{-(t-s)} dW_s$$

for some $\xi \in \mathbb{R}$.
(a) Verify that X satisfies the following stochastic differential equation:

$$dX_t = -X_t\, dt + \; dW_t, \quad X_0 = \xi.$$

(b) Show that

$$X_t \sim N\left(e^{-t}\xi, \frac{1}{2}(1 - e^{-2t})\right).$$

(c) Show that $\exp(X_t^2 - t)$ is a martingale.
(d) Now suppose $\xi \sim N(0, \sigma^2)$, independent of W. Find the unique σ^2 such that $X_t \sim N(0, \sigma^2)$ for all $t \ge 0$. Compute $\mathrm{Cov}(X_s, X_t)$ in this case.
(e) Let σ^2 and ξ be as in part (d), and let $Y_t = \sigma e^{-t} W_{e^{2t}}$. Show that Y has the same law as X.

Problem 10. (Stratonovich integral) Let X and Y be continuous semimartingales. The Stratonovich integral of X with respect to Y is defined by

$$\int_0^t X_s \circ dY_s = \int_0^t X_s dY_s + \frac{1}{2}\langle X, Y\rangle_t$$

where the integral on the right side of the equation is an Itô integral. Show that if f is three times continuously differentiable, then Itô's formula can be written

$$df(X_t) = f'(X_t) \circ dX_t.$$