Stochastic Calculus Michael Tehranchi

Example sheet 2 - Lent 2015

Problem 1. Let X be a local martingale. Prove that X is a uniformly integrable martingale if and only X is of class D.

Problem 2. Let X be a continuous local martingale. Show that if

$$\mathbb{E}[\sup_{0\le s\le t}|X_s|] < \infty$$

for each $t \ge 0$, then X is a true martingale.

Problem 3. Let X be a non-negative local martingale. Prove that X is a supermartingale.

Problem 4. Let X be a supermartingale. Show that if $\mathbb{E}(X_t) = \mathbb{E}(X_0)$ for all $t \ge 0$, then X is a martingale.

Problem 5. Suppose X is an adapted integrable process such that $\mathbb{E}(X_T) = \mathbb{E}(X_0)$ for all bounded stopping times. Show that X is a martingale.

[Hint: Fix non-random $0 \le s < t$ and an event $A \in \mathcal{F}_s$, and let $T = s\mathbb{1}_A + t\mathbb{1}_{A^c}$.]

Problem 6. Let X be a continuous square-integrable martingale with $X_0 = 0$, and A a continuous, adapted process of finite variation with $A_0 = 0$. Suppose for all bounded stopping times T that

$$\mathbb{E}(X_T^2) = \mathbb{E}(A_T).$$

Show that $A = \langle X\rangle$. Find a similar characterisation of $\langle X\rangle$ when X is only a local martingale.

Problem 7. For a continuous local martingale X and stopping time T, show that $\langle X\rangle^T = \langle X^T\rangle$.

Problem 8. Show that if X is a continuous local martingale and Y is predictable and such that $\int_0^t Y_s^2 d\langle X\rangle_s < \infty$ for all $t \ge 0$, then

$$\left\langle \int Y\,dX \right\rangle = \int Y^2 d\langle X\rangle.$$

Problem 9. Suppose that $f : [0,\infty) \to \mathbb{R}$ is absolutely continuous, in the sense that

$$f(t) = f(0) + \int_0^t f'(s)ds$$

for an integrable function f'. Show that

$$\|f\|_{t,\mathrm{var}} = \int_0^t |f'(s)|ds.$$

Problem 10. Let $A, B : [0,\infty) \to \mathbb{R}$ be bounded and measurable, and let $f : [0,\infty) \to \mathbb{R}$ be continuous and of finite variation. Show that

$$\int A\ d\left(\int B\ df\right) = \int AB\ df$$

in the sense of Lebesgue–Stieltjes integration.

Problem 11. Show that a sequence of processes $(X^n)_n$ converges uniformly on compacts in probability to X if and only if

$$\sum_{k=1}^{\infty} 2^{-k}\mathbb{E}[\sup_{0\le t\le k} |X_t^n - X_t| \wedge 1] \to 0.$$

Problem 12. Fix a $T \ge 0$ and let W be a scalar Brownian motion. Show that for any $p > 0$,

$$N^{p/2-1}\sum_{k=1}^{N} |W_{kT/N} - W_{(k-1)T/N}|^p \to c_p T^{p/2}$$

in probability, where $c_p = \pi^{-1/2}2^{p/2}\Gamma(\frac{p+1}{2})$. (Hint: Use the law of large numbers) Why is this conclusion not surprising when $p = 2$?

Problem 13. Let W be a Brownian motion and let

$$\hat{W}_t = W_t - \int_0^t \frac{W_s}{s} ds.$$

Show that $\hat{W}$ is not a martingale in the filtration generated by W, but that $\hat{W}$ is a martingale in its own filtration.

Problem 14. Let $(W_t)_{t\ge 0}$ be a scalar Brownian motion, and for $h \in H = L^2[0,\infty)$ let

$$W(h) = \int_0^\infty h(s)dW_s$$

Show that $\{W(h), h \in H\}$ is an isonormal process.

Problem 15. Let X be a continuous local martingale with $X_0 = 0$ such that $X_t^2 - t$ is also a local martingale. Show that X is a Brownian motion.

Problem 16. Let N be a Poisson process of rate $\lambda = 1$ and let $X_t = N_t - t$. Show that X is a finite variation. Show that both X and $X_t^2 - t$ are martingales. Why is does this not contradict the previous question?

Problem 17. (Burkholder inequality) Let $p \ge 2$ and M a continuous local martingale with $M_0 = 0$. Use Itô's formula, Doob's maximal inequality, and Hölder's inequality to show that there is a constant $C_p > 0$ such that

$$\mathbb{E}(\max_{0\le s\le t} |M|_s^p) \le C_p\mathbb{E}(\langle M\rangle_t^{p/2}).$$

Problem 18. Let $H_n(x)$ denote the nth Hermite polynomial as defined in example sheet 1. Let X be a continuous local martingale. Show that

$$\langle X\rangle_t^{n/2} H_n\left(\frac{X_t}{\langle X\rangle_t^{1/2}}\right)$$

defines a local martingale for each n. Hint: Use Itô's formula.