Stochastic Calculus Michael Tehranchi
Example sheet 1 - Lent 2015

Problem 1. Let $X = (X_t)_{t \geq 0}$ be a continuous Gaussian process (i.e. the random variables $X_{t_1}, \dots X_{t_n}$ are jointly normal for every $0 \leq t_1 < \cdots < t_n$) such that

$$\mathbb{E}(X_t) = 0 \text{ and } \mathbb{E}(X_s X_t) = s \wedge t$$

for all $s, t \geq 0$. Show that X is a Brownian motion. (Recall the notation $a \wedge b = \min\{a, b\}$ for real a, b.)

Problem 2. Let W be a scalar Brownian motion. Show that $\hat{W}$ is also a Brownian motion when $\hat{W}$ is defined by

(1) $\hat{W}_t = \frac{1}{c} W_{c^2 t}$ for a constant $c \neq 0$.
(2) $\hat{W}_t = W_{t+a} - W_a$ for a constant $a \geq 0$.
(3) $\hat{W}_t = W_a - W_{a-t}$ for $t \in [0, a]$, for a constant $a > 0$.

Problem 3. Let W be a Brownian motion. Show that

$$\int_0^t \mathbb{E}\left(\frac{|W_s|}{s} \right) ds < \infty$$

and hence for any $t > 0$ the function $s \mapsto W_s(\omega)/s$ is Lebesgue integrable on $(0, t]$ for almost every $\omega \in \Omega$. Let $\hat{W}_0 = 0$ and

$$\hat{W}_t = W_t - \int_0^t \frac{W_s}{s} ds \text{ for } t > 0.$$

Show that $\hat{W}$ is Brownian motion. Furthermore, show that $(\hat{W}_s)_{s \in [0,t]}$ and W_t are independent for all $t > 0$.

Problem 4. Let W be a Brownian motion. Prove the Brownian strong law of large numbers $\frac{W_t}{t} \to 0$ a.s. as $t \uparrow \infty$. Hence show that $\hat{W}$ is a Brownian motion where $\hat{W}_0 = 0$ and $\hat{W}_t = t W_{1/t}$ for $t > 0$.

[Hint: One possibility is to use the ordinary strong law of large numbers and the square integrability of $\max_{0 \leq s \leq 1} |W_s|$.]

Problem 5. (a) Let A be the linear operator on $L^2[0, 1]$ defined by

$$(Af)(t) = \int_0^1 s \wedge t \, f(s) ds.$$

Find the eigenvalues and eigenfunctions of A, that is, numbers λ and functions g such that

$$Ag = \lambda g.$$

(b) Let $\xi_1, \xi_2, \dots$ be independent $N(0, 1)$ random variables. Show that the series

$$X_t = \sum_{n=1}^{\infty} \frac{\sqrt{2} \sin[(n - 1/2)\pi t]}{(n - 1/2)\pi} \xi_n$$

converges almost surely for every $t \in [0, 1]$. Show that the process $(X_t)_{t \in [0,1]}$ is Gaussian with mean $\mathbb{E}(X_t) = 0$ and covariance $\mathbb{E}(X_s X_t) = s \wedge t$.

[Hint: the Hilbert–Schmidt theorem says that a symmetric compact operator A has an orthonormal basis of eigenfunctions. It is known that operators of the form $Af(t) = \int_0^1 K(s,t)f(s)ds$ are compact when K is bounded.]

Problem 6. A Brownian sheet is a continuous Gaussian process $(B_{s,t})_{s\geq 0, t\geq 0}$ with the two-dimensional index set $\mathbb{R}_+^2$ such that

$$\mathbb{E}(B_{s,t}) = 0 \text{ and } \mathbb{E}(B_{s,t}B_{s',t'}) = (s \wedge s')(t \wedge t').$$

(1) Show for fixed $s > 0$, the process $(s^{-1/2}B_{s,t})_{t\geq 0}$ is a Brownian motion.
(2) Show that the process $(B_{\sqrt{t},\sqrt{t}})_{t\geq 0}$ is a Brownian motion.
(3) Show that a Brownian sheet exists.

Problem 7. Given a parameter $0 < \alpha \leq 1$, a function $f : [0,\infty) \to \mathbb{R}$ is said to be α-Hölder continuous at a point $s \geq 0$ iff there exist constants $C > 0$ and $\varepsilon > 0$ such that

$$|f(t) - f(s)| \leq C|t - s|^\alpha$$

for all $t \geq 0$ in the interval $(s - \varepsilon, s + \varepsilon)$. Let W be a scalar Brownian motion. Show that

$$\mathbb{P}\{\omega : t \mapsto W_t(\omega) \text{ is } \alpha\text{-Hölder continuous somewhere } \} = 0$$

for all $\alpha > 1/2$.

Problem 8. Let W be a scalar Brownian motion, and let

$$\mathcal{Z}_\omega = \{t \geq 0 : W_t(\omega) = 0\}$$

be the set of zeroes. Show that for almost all ω:

(1) $\mathcal{Z}_\omega$ is closed.
(2) $\mathcal{Z}_\omega \cap (0,\epsilon)$ is non-empty for all $\epsilon > 0$. [Hint: Use Brownian scaling.]
(3) Show $\mathrm{Leb}(\mathcal{Z}_\omega) = 0$ for almost every ω. [Hint: Use Fubini's theorem.]

Challenge: Draw a picture that illustrates simultaneously phenomena (1), (2) and (3) above.

Problem 9. The nth Hermite polynomial H_n is defined uniquely be the identity

$$\sum_{n=0}^\infty \frac{1}{n!} H_n(x)\theta^n = e^{\theta x - \theta^2/2}$$

for real x, θ. Find H_0, H_1, H_2, H_3 explicitly. Show that $t^{n/2}H_n(W_t/\sqrt{t})$ defines a martingale for each $n \geq 0$, where W is a scalar Brownian motion.

Problem 10. The Picard–Lindelöf theorem says that the ODE

$$\dot{y} = g(t, y)$$

has a unique solution if $t \mapsto g(t, y)$ is continuous and there is a constant $C > 0$ such that

$$\|g(t, y_1) - g(t, y_2)\| \leq C\|y_1 - y_2\| \text{ for all } y_1, y_2 \in \mathbb{R}^n \text{ and } t \geq 0.$$

Suppose there exists a $C > 0$ such that

$$\|f(x_1) - f(x_2)\| \leq C\|x_1 - x_2\| \text{ for all } x_1, x_2 \in \mathbb{R}^n.$$

Show that the integral equation

$$X_t = X_0 + \int_0^t f(X_s)ds + \sigma W_t$$

has a unique solution, where $\sigma > 0$ is a constant and W is an n-dimensional Brownian motion.

Problem 11. Let $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ be a filtration. Let T be a stopping time and let

$$\mathcal{F}_T = \{A \in \mathcal{F} : A \cap \{T \leq t\} \in \mathcal{F}_t \text{ for all non-random } t \geq 0\}.$$

(1) Prove that $\mathcal{F}_T$ is a sigma-algebra.
(2) Show that T is $\mathcal{F}_T$-measurable.

Problem 12. Given a filtration $\mathbb{F}$, a process $X = (X_t)_{t\geq 0}$ is called progressively measurable iff for all non-random $t \geq 0$ the mapping $(s, \omega) \mapsto X_s(\omega)$ from $[0, t] \times \Omega$ to $\mathbb{R}$ is $\mathcal{B}[0, t] \otimes \mathcal{F}_t$-measurable.

(1) Now suppose T is a stopping time such that $T(\omega) < \infty$ for all $\omega \in \Omega$. If X is progressively measurable, prove that the function $X_T : \Omega \to \mathbb{R}$ defined by $X_T(\omega) = X_{T(\omega)}(\omega)$ is $\mathcal{F}_T$-measurable.
(2) Suppose X is right-continuous and adapted. Show that X is progressively measurable.

Problem 13. Let $T_1, T_2, \ldots$ be stopping times for a filtration $\mathbb{F}$. Show that $\sup_n T_n$ is a stopping time. Show that $\inf_n T_n$ is a stopping time for the filtration $(\mathcal{F}_{t+})_{t\geq 0}$ where $\mathcal{F}_{t+} = \cap_{\varepsilon>0} \mathcal{F}_{t+\varepsilon}$.