

**Problem 1.** Show that the Black–Scholes price of a European call option is strictly convex in both the strike price  $K$  and the initial stock price  $S_0$ , decreasing in  $K$  and increasing in  $S_0$ . Show that the price increases with the interest rate  $r$ , and with the expiry  $T$ .

What are the corresponding statements for the Black–Scholes price of a European put option?

**Problem 2.** Consider a stock which pays a dividend at constant rate  $\delta \geq 0$ . The price of the stock is modelled by

$$S_t = S_0 e^{\sigma W_t + (r - \delta - \sigma^2/2)t}.$$

where  $r \geq 0$  is the risk-free interest rate. for a Brownian motion  $W$ . Show that

$$e^{-rt} S_t + \int_0^t e^{-ru} S_u \delta du$$

defines a martingale. Why then is  $S$  a sensible model for the stock price (at least under an equivalent measure)? Show that the time-0 value of a European put option with strike  $K$  and expiry  $T$  written on this asset is

$$K e^{-rT} \Phi\left(\frac{\log(K/S_0) - (r - \delta - \sigma^2/2)T}{\sigma\sqrt{T}}\right) - S_0 e^{-\delta T} \Phi\left(\frac{\log(K/S_0) - (r - \delta + \sigma^2/2)T}{\sigma\sqrt{T}}\right).$$

Can we deduce the price of the European call by put-call parity in the case when the stock pays dividends?

**Problem 3.** A European lookback call option entitles the holder to buy one unit of stock at the expiry time  $T$  at the lowest price reached by the stock during the life of the option. Thus, if it is purchased at time 0, at time  $T$  it pays off the amount  $S_T - \inf_{0 \leq u \leq T} S_u$ . In the Black-Scholes model show that the price at time 0 of such an option is

$$S_0 \left[ \left( \frac{2r + \sigma^2}{2r} \right) \Phi\left(\frac{2r + \sigma^2}{2\sigma/\sqrt{T}}\right) - e^{-rT} \left( \frac{2r - \sigma^2}{2r} \right) \Phi\left(\frac{2r - \sigma^2}{2\sigma/\sqrt{T}}\right) - \frac{\sigma^2}{2r} \right],$$

if  $r > 0$ .

**Problem 4.** Let  $EC(S_0, K, \sigma, r, T)$  denote the Black–Scholes price of a European call option with strike  $K$ , expiry  $T$  on an asset with initial price  $S_0$ , volatility  $\sigma$ , when the constant interest rate is  $r$ . Show that the price of a down-and-out call with strike  $K$  and a barrier at  $B$ , where  $B < \min\{S_0, K\}$ , can be expressed in terms of  $EC$  as

$$EC(S_0, K, \sigma, r, T) - (B/S_0)^{2r/\sigma^2 - 1} EC(B^2/S_0, K, \sigma, r, T).$$

**Problem 5.** In the Black–Scholes model, find the time-0 prices of European contingent claims which pay at time  $T$  the amounts:

- (a)  $\int_0^T S_u du$
- (b)  $(\log S_T)^2$ .

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**Problem 6.** The Black–Scholes model can be approximated by a binomial model, but even once we have fixed the step  $h > 0$  of the time discretisation there are several possible ways in which this might be done. Assuming that  $r \geq 0$ , consider the following possibilities:

(i) Choose  $d = 1/u$ , so that an up step followed by a down step returns us to the initial value. The probability  $p$  of an up step is chosen so as to match the first two moments of the share price move over the time interval of length  $h$ . Write down the two equations which determine  $u$  and  $p$ , and prove that  $u$  is the larger root of the quadratic

$$u^2 - u(e^{-rh} + e^{rh+\sigma^2h}) + 1 = 0.$$

Prove that  $u > 1$  always, and find the expression for  $p$ , proving that always  $0 < p < 1$ .

(ii) Choose  $d = 1/u = e^{-a}$ , and choose  $p$  and  $a$  so that the first two moments of the log price are matched. In this case, show that

$$\begin{aligned} a &= \sqrt{\sigma^2h + (r - \sigma^2/2)^2h^2} \\ p &= \frac{(r - \sigma^2/2)h + a}{2a} \end{aligned}$$

(iii) Take  $u = \exp(\sigma\sqrt{h} + (r - \sigma^2/2)h)$ , and  $d = \exp(-\sigma\sqrt{h} + (r - \sigma^2/2)h)$  with  $p = 1/2$  (this corresponds to approximating the underlying Brownian motion with a simple symmetric random walk).

**Problem 7.** The Black–Scholes PDE is given by

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2s^2\frac{\partial^2 V}{\partial s^2} + rs\frac{\partial V}{\partial s} - rV = 0.$$

(i) Set  $v(\tau, x) = V(T - 2\tau/\sigma^2, e^x)$ . Show that

$$\frac{\partial v}{\partial \tau} - \frac{\partial^2 v}{\partial x^2} + (1 - k)\frac{\partial v}{\partial x} + kv = 0$$

for constant  $k$  which you should determine.

(ii) Let  $\alpha, \beta$  be real constants and defined  $u(x, \tau)$  via

$$u(\tau, x) = e^{\alpha x + \beta \tau} v(\tau, x).$$

Find the PDE for  $u(x, \tau)$  and choose  $\alpha, \beta$  such that  $u$  satisfies the standard heat equation,

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}.$$

**Problem 8.** Consider the standard heat equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}.$$

Assume  $t \in [0, T]$ ,  $x \in [0, L]$  and initial/boundary data

$$u(0, x) = g(x), \quad u(t, 0) = a(t), \quad u(t, L) = b(t).$$

The grid  $\{(i\delta_t, j\delta_x) : i = 1, \dots, N_t, j = 1, \dots, N_x\}$  with  $\delta_x = L/N_x, \delta_t = T/N_t$  is given and we seek approximations  $U_j^i \approx u(i\delta_t, j\delta_x)$ . Set  $U^i = (U_j^i)_{j=1, \dots, N_x} \in \mathbb{R}^{N_x-1}$ .

(i) Formulate the FTCS-method as linear equations

$$U^{i+1} = FU^i + p^i$$

for some  $(N_x - 1) \times (N_x - 1)$ -matrix  $F$  and  $p^i \in \mathbb{R}^{N_x-1}$ .

(ii) Similarly, formulate the BTCS-method as

$$BU^{i+1} = U^i + q^i$$

for a matrix  $B$  and a vector  $q^i$  to be determined.

(iii) Show that adding these two linear equations yields exactly the Crank–Nicolson method.