Stochastic Financial Models[1] MRT/PKF/LCGR

Example sheet 4.3 solution - Lent 2010

Problem 3. A European lookback call option entitles the holder to buy one unit of stock at the expiry time T at the lowest price reached by the stock during the life of the option. Thus, if it is purchased at time 0, at time T it pays off the amount $S_T - \inf_{0 \le u \le T} S_u$. In the Black-Scholes model show that the price at time 0 of such an option is

$$S_0 \left[\left(\frac{2r + \sigma^2}{2r} \right) \Phi \left(\frac{2r + \sigma^2}{2\sigma/\sqrt{T}} \right) - e^{-rT} \left(\frac{2r - \sigma^2}{2r} \right) \Phi \left(\frac{2r - \sigma^2}{2\sigma/\sqrt{T}} \right) - \frac{\sigma^2}{2r} \right],$$

if $r > 0$.

Solution. Let $X_t = W_t + at$ and $Y_T = \inf_{0 \le t \le T} X_t$, where W is a Brownian motion. From the reflection principle and the Cameron–Martin theorem, we know the joint density f of (X_T, Y_T) is

$$f(x,y) = \frac{2(x-2y)}{\sqrt{2\pi T^3}} e^{ax - \frac{1}{2}a^2 T - \frac{1}{2T}(x-2y)^2} \quad x > y, y < 0$$

For this example, we have $S_t = S_0 e^{(r - \sigma^2/2)t + \sigma W_t}$. Hence, the price is given by

$$e^{-rT} \mathbb{E}(S_T - \inf_{0 \le t \le T} S_t) \;=\; S_0 - e^{-rT} S_0 \iint e^{\sigma y} f(x,y) dx\ dy$$

where f is given above with $a = r/\sigma - \sigma/2$.

Here are the details:

$$\iint e^{\sigma y} f(x,y) dx dy \;=\; \frac{2e^{-a^2 T/2}}{\sqrt{2\pi T^3}} \iint_{x>y, y<0} (x-2y) e^{\sigma y + ax - \frac{1}{2T}(x-2y)^2} dx\ dy$$

Make the change of variables

$$\begin{array}{rcl} x & = & \sqrt{T} u \\ y & = & \frac{1}{2}\sqrt{T}(u - v) \end{array} \Leftrightarrow \begin{array}{rcl} u & = & \frac{1}{\sqrt{T}} x \\ v & = & \frac{1}{\sqrt{T}}(x - 2y) \end{array}$$

in the integral, remembering to put in the Jacobian factor $T/2$:

$$\iint_{x>y, y<0} (x-2y) e^{\sigma y + ax - \frac{1}{2T}(x-2y)^2} dx\ dy$$

$$= \frac{T^{3/2}}{2} \iint_{-v<u<v} v e^{(\sigma/2 + a)\sqrt{T}u - \sigma\sqrt{T}v/2 - \frac{1}{2}v^2} du\ dv$$

Now, recalling that $\sigma/2 + a = r/\sigma$, we do the integration with respect to u.

$$\iint_{-v<u<v} v e^{(\sigma/2 + a)\sqrt{T}u - \sigma\sqrt{T}v/2 - \frac{1}{2}v^2} du\ dv$$

$$= \frac{\sigma}{r\sqrt{T}} \int_{v>0} v(e^{(r/\sigma)\sqrt{T}v} - e^{-(r/\sigma)\sqrt{T}v}) e^{-\sigma\sqrt{T}v/2 - v^2/2} dv$$

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Now, do the integrals separately:

$$\begin{aligned}\int_{v>0} ve^{(r/\sigma-\sigma/2)\sqrt{T}v-v^2/2}dv &= \int_{v>0} ve^{a^2T/2-(v-a\sqrt{T})^2/2}dv \\ &= e^{a^2T/2}\int_{s>-a\sqrt{T}}(s+a\sqrt{T})e^{-s^2/2}ds \\ &= e^{a^2T/2}[e^{-a^2T/2}+a\sqrt{2\pi T}(1-\Phi(-a\sqrt{T}))] \\ &= 1+a\sqrt{2\pi T}e^{a^2/T}\Phi(a\sqrt{T})\end{aligned}$$

Similarly, letting $b = \sigma/r + \sigma/2$

$$\begin{aligned}\int_{v>0} ve^{-(r/\sigma+\sigma/2)\sqrt{T}v-v^2/2}dv &= \int_{v>0} ve^{b^2T/2-(v+b\sqrt{T})^2/2}dv \\ &= 1-b\sqrt{2\pi T}e^{b^2T/2}\Phi(-b\sqrt{T})\end{aligned}$$

Putting it all together yields:

$$e^{-rT}\mathbb{E}(S_T-\inf_{0\le t\le T}S_t) \;=\; S_0-e^{-rT}S_0\iint e^{\sigma y}f(x,y)dx\ dy =$$

$$S_0\left\{1-e^{-rT}\left(\frac{2e^{-a^2T/2}}{\sqrt{2\pi T^3}}\right)\left(\frac{T^{3/2}}{2}\right)\left(\frac{\sigma}{r\sqrt{T}}\right)\left[1+a\sqrt{2\pi T}e^{a^2/T}\Phi(a\sqrt{T})-1+b\sqrt{T}\sqrt{2\pi}e^{b^2T/2}\Phi(-b\sqrt{T})\right]\right\}$$

$$= S_0\left(1-e^{-rT}(\sigma/r)a\Phi(a\sqrt{T})-e^{-rT-a^2T/2+b^2T/2}(\sigma/r)b\Phi(-b\sqrt{T})\right)$$

But $-a^2+b^2 = -(\sigma/r-\sigma/2)^2+(\sigma/r+\sigma/2)^2 = 2r$, so that everything equals

$$S_0[1-e^{-rT}(\sigma/r)a\Phi(a\sqrt{T})-(\sigma/r)b+(\sigma/r)b\Phi(b\sqrt{T})].$$

We're done now since $-(\sigma/r)b+1 = -(\sigma/r)(r/\sigma+\sigma/2)+1 = -\sigma^2/(2r)$.