

Problem 1. In the model of Question 8, Example sheet 2, find the time-zero price and optimal exercise policy for an American put option with strike 15 and expiry 2. Check that the value is strictly bigger than the value of the corresponding European put. Can you explain why, in contrast, the prices of American and Europeans calls are equal?

Problem 2. An investor with wealth X_0 at time 0 wishes to invest it in such a way as to maximise $\mathbb{E}[U(X_N)]$, where X_N is the wealth at the start of day N , and $U(x) = \frac{1}{1-R}x^{1-R}$ for a relative risk aversion coefficient $R > 0$, $R \neq 1$. He chooses the proportion θ_n of his wealth to invest in the single risky asset during the period $(n-1, n]$, so that his wealth at the start of day n will be

$$X_n = X_{n-1}[\theta_n \xi_n + (1 - \theta_n)(1 + r)]$$

where ξ_1, ξ_2, \dots are IID positive random variables, and r is the per-period riskless rate of interest. Find the form of the optimal policy in each of the two situations:

- (i) θ_n unrestricted,
- (ii) $0 \leq \theta_n \leq 1$.

Show that the solutions are the same if and only if

$$\frac{\mathbb{E}(\xi^{1-R})}{\mathbb{E}(\xi^{-R})} \leq 1 + r \leq \mathbb{E}(\xi)$$

where ξ is a random variable with the same distribution as the ξ_i 's

[Hint: look for a value function of form $V_n(x) = a_n U(x)$.]

Problem 3. At the start of year $n-1$, an investor receives income $Y_{n-1} \geq 0$. He chooses to invest a proportion $\theta_n \in [0, 1]$ of this in the stock market, and consumes the remainder. At the start of the next year, his income is

$$Y_n = Y_{n-1} + Y_{n-1} \theta_n \xi_n$$

where ξ_{n+1} is the return on the stock market through the period $(n, n+1]$. Finally, at the start of year N , the investor consumes all his income. Assuming ξ_1, ξ_2, \dots are IID positive random variables with $\mathbb{E}(\xi_1) = \mu$, The investor seeks to maximise

$$\mathbb{E} \left[Y_N + \sum_{n=1}^N (1 - \theta_n) Y_{n-1} \right]$$

How should he invest to achieve this? [Hint: look for a value function of form $V_n(y) = a_n y$.

Problem 4. A gambler has the chance to bet on a sequence of N coin tosses. Let $\xi_n = 1$ if the n -th toss is a head, otherwise $\xi_n = -1$ if a tail. The ξ_1, ξ_2, \dots are independent but *not* identically distributed; $\mathbb{P}(\xi_n = 1) = p_n \geq 1/2$. If the gambler's wealth just before the n -th toss is X_{n-1} , he may stake any amount $Y_n \in [0, X_{n-1}]$ on the toss of the coin; his wealth at time n is therefore

$$X_n = X_{n-1} + Y_n \xi_n.$$

Determine how the gambler should play so as to maximise his final expected utility $\mathbb{E} \log X_N$. Hint: look for a value function of form $V_n(x) = a_n + \log x$.

Problem 5. An agent starts with initial capital $C > 0$. At the beginning of day n he observes a new non-negative random variable Y_n , and then chooses c_n , the amount of his remaining capital to consume that day. The Y_n are independent with $\mathbb{E}(Y_n) = \mu_n$. His objective is to maximise

$$\mathbb{E} \left[\sum_{n=1}^N Y_n \log c_n \right]$$

subject to the constraint the constraint $\sum_{n=1}^N c_n = C$. Find his optimal consumption policy, and an expression for the maximal value of his objective. Hint: Given $Y_n = y$, and $x = C - \sum_{i=1}^{n-1} c_i$, look for a value function of form

$$V_n(x, y) = \left(y + \sum_{i=n+1}^N \mu_i \right) \log x + b_n(y).$$

Problem 6. If $(W_t)_{t \geq 0}$ is a Brownian motion, show that the following processes are martingales:

- (i) $W_t^2 - t$.
- (ii) $e^{\theta W_t - \theta^2 t/2}$ for any $\theta \in \mathbb{R}$.
- (iii) $W_t^3 - 3tW_t$.
- (iv) $\cosh(\theta W_t) e^{-\theta^2 t/2}$ for any $\theta \in \mathbb{R}$.

Problem 7. An agent holds a single share of a firm that will go bankrupt at time τ , an exponential random variable with mean $1/\lambda$. Suppose that the price of the share at time $0 \leq t < \tau$ is $S_t = S_0 + W_t$, where W is a Brownian motion independent of τ . The agent plans to sell the share at the first time H_a that the price exceeds a . The agent seeks to maximise

$$\mathbb{E}[a e^{-r H_a} \mathbb{1}_{\{H_a < \tau\}}]$$

Find the optimal choice of a .

Problem 8. Using a suitable martingale and the optional stopping theorem, show that if $T_a = \inf\{t \geq 0 : W_t \geq a\}$ is the first time that a Brownian motion exceeds level $a > 0$, then the Laplace transform of T_a is given by

$$\mathbb{E}[e^{-\lambda T_a}] = e^{-a\sqrt{2\lambda}}.$$

For the brave of heart: confirm this by integrating the density of T_a as derived from the reflection principle.

Problem 9. Suppose that $a < x < b$, and let $\tau = \inf\{t \geq 0 : W_t \geq b \text{ or } W_t \leq a\}$ Using an appropriate martingale and the optional stopping theorem, prove that

$$\mathbb{E}[e^{-\theta^2 \tau/2} | W_0 = x] = \frac{\cosh[\theta(x - \frac{1}{2}(a+b))]}{\cosh[\frac{1}{2}\theta(b-a)]}$$