Origins of scaling and power law fluctuations in a competitive equilibrium

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Plan of the talk

- The challenge of Critical Phenomena in Economics (CPE)
- A reappraisal of Rational Expectations Economics (REE).
- What's wrong with dynamic stochastic general equilibrium (DSGE) models.
- Rational decisions in turbulent times: what we learn from volatility modeling.
- Scale Invariant Dynamic Stochastic General Equilibria (SI-DSGE).
- Scale Invariance vs Time consistency.
- Conclusions and a Research Agenda

The challenge of Critical Phenomena in Economics

- Quantitative Finance community is setting new benchmarks and new challenges to mainstream economic sciences!!
- It is not unfair to say that interdisciplinary contributions that are nowaday universally recognized as relevant, would have not been published without the effort of QF founders and former editors.
- Today talk: An attempt to introduce a notion of SI-DSGE. The minimum goal is to establish a common language to popularize critical phenomena in macro but.... a scale invariant, time inconsistent version of the dynamic programming principle makes its appearance.

Litterature

- Physics: Renormalization group and Critical Phenomena, Complexity and Spin Glasses: K. G. Wilson, G. Jona Lasinio, P.W. Anderson, G. Parisi, P. Bak.
- Econophysics and Mathematical Finance: our editors Michael, Doyne Famer, Jean Philippe, and Jim, editorial and advisory board as representatives of the full community!
- Economics and Finance: L. Hansen, J. Scheinkman, M. Woodforde, P. Kyle, X. Gabaix.

Normative vs Descriptive Models in Economics

'*Economists should aspire to be like dentists*' (Keynes)

- Physicist analysis of economic phenomena is mostly relevant only from a descriptive point of view.
- Lucas Critique (1976) '*Parameters of the traditional unrestricted macroeconometric models are unlikely to remain invariant in a changing economic environment*'.
- L. P. Hansen (2014 Nobel Lecture) '*Uncertainty is inside and outside an economic model*' To generate reliable counterfactuals, a model has to endogenize the impact of policy rules on individual and collective expectations.
  - Individual level: a macro model requires a proper microfoundation in terms of the description of individual decision making.
  - Collective level: a game solution concept is necessary to explore the counterfactuals in an equilibrium of agents that act strategically

Basics of Rational Expectations Economics.

- A proper model discussion of the selection of the optimal policy requires a precise model of:
  1. The law of motions of the underlying drivers of dynamic uncertainty.
  2. The formation of agent beliefs.
  3. Agent decision functions.
- Rational expectations (very strong assumption):
  1. Agents use publicly available information in an efficient manner. Thus, they do not make systematic mistakes when formulating expectations.
  2. They understand the structure of the model economy and base their expectations on this knowledge.

Example: Asset pricing in discrete time

- Asset pricing models commonly considered in finance are based on the assumption that agents take their decisions based on their risk-adjusted expectations about security cash flows and discount rates.
- In discrete time, models in structural form that describe also the process of expectation formation read as follows:

$$
\begin{aligned}
y_t &= a\left(x_{t+1} + Ey_{t+1}\right) \leftarrow \text{ Backward Expectation Dynamics} \\
x_t &= (1-c)\, x_{t-1} + v\varepsilon_t \leftarrow \text{ Forward Driver of Economic Uncertainty}
\end{aligned}
$$

Example: Asset pricing in discrete time II

- A variation of the so called dividend discount model: set $P_t = y_t$, $d_{t+n} = d_0 (1+g)^n x_{t+n}$, $a = (1+r)^{-1}$
- Assume v is sufficiently small that $Prob(x_t < 0)$ is negligible, take expectations under the historical measure and solving for $g, r, c > 0$ and $r - (1-c) g + c > 0$. Then:

$$P_t \stackrel{BED}{=} \sum_{n=1}^{+\infty} \frac{E_t [d_{t+n}]}{(1+r)^n} \stackrel{FDEU}{=} \frac{d_0 x_t}{r - (1-c) g + c}$$

- Reduced form (VAR) representation of the price dynamics:

$$P_t = (1-c) P_{t-1} + \frac{d_0 v}{r - (1-c) g + c} \varepsilon_t$$

- Campbell-Shiller (1988) log-linearized model is a (more sophisticate) example derived following the same logic.

Stability of Stationary Equilibria and Learning

- *In standard macroeconomic model, rational expectations can emerge in the long run, provided the agents' environment remains stationary for a sufficiently long period.* (Evans and Honkapohja 2013, 68)
- Within our example $E^*_{t-1}P_t$ denotes the subjective expectation:

$$
\begin{aligned}
P_t &= \alpha E^*_{t-1}P_t + \delta x_{t-1} + \eta_t \\
E^*_{t-1}P_t &= \beta_{t-1}x_{t-1}
\end{aligned}
$$

where β_{t-1} is the estimated parameter value given the past history. Then the actual update will be:

$$P_t = (\alpha\beta_{t-1} + \delta)\, x_{t-1} + \eta_t$$

while the parameter estimate may be updated according for example accoring to a generalized stochastic gradient.

Stability of Stationary Equilibria and Learning

$$\begin{aligned}
\beta_t &= \beta_{t-1} + \frac{\left(P_t - \beta_{t-1} x_{t-1}\right)}{t R_t} \\
R_t &= R_{t-1} + \frac{\left(x_{t-1}^2 - R_{t-1}\right)}{t}
\end{aligned}$$

Set $T(\beta_\tau) := \delta + \alpha\beta_\tau$, $M := E\left[x_{t-1}^2\right]$, $S(t-1) := R_{t-1}$ then:

$$\begin{aligned}
\frac{d\beta_\tau}{d\tau} &= \frac{M}{S(\tau)}\left(T(\beta_\tau) - \beta_\tau\right) \\
\frac{dS(\tau)}{d\tau} &= M - S(\tau)
\end{aligned}$$

Expectational stability

Noting that $\lim_{\tau\to+\infty} S(\tau) = M$, in order to achieve convergence of β_t to the true value, it is sufficient that the equilibrium point for the dynamical system:

$$\frac{d\beta_\tau}{d\tau} = \left(T(\beta_\tau) - \beta_\tau\right)$$

$$\beta^* = \frac{\delta}{1-\alpha}$$

is stable, which is equivalent to require the condition $\alpha < 1$.

Proposition (Evans and Honkapohja): The economy converges to the REE under RLS learning iff the REE is E-stable. The latter occurs iff $\alpha < 1$.

Lucas Critique

- Investors adjust their expectations based on the current observations, e.g. they may expect that a raise in the interest rate r may lower expected dividend rate of growth g and this may determine a change in the demand for stocks.
- To specify this change, you need to specify decision functions to be optimized by the monetary authority, the producer and the investor.
- In a stationary environment we can write the Bellman's equations as a fixed point equation, e.g.:

$$V(x,z) = \max_{c \in C(x,z)} \left\{ u(x,c) + \beta \int_Z V\left(f(x,z,c), z'\right) dQ\left(z', z\right) \right\} \tag{1}$$

The solution to this problem will be a stationary (time-invariant) policy function $c^* = g^*(x,z)$.

Preferences

- Exponential discounting, β^n of preference is a necessary condition to achieve **time consistency** of the preference functional which implies the possibility to rely on the **dynamic programming principle** to identify an (autonomous) value function in a stationary environment.
- Hansen-Scheinkman (PNAS2012) discuss existence and uniqueness of this fixed point equation in the context of recursive utilities in a Markov enviroment with stochastic growth that can accomodate exotic preferences.
- Controlled state variable dynamics includes forward and backward components to take into account the impact of expectations on current values.

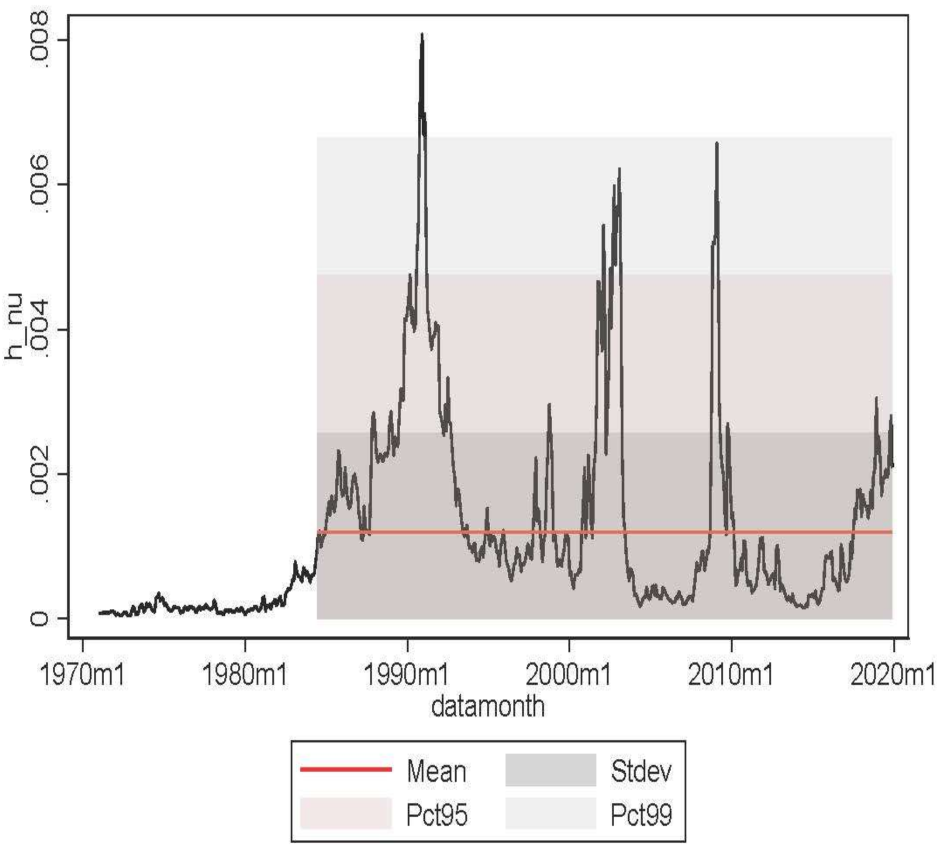
What's wrong with DSGE

In the words of Bouchaud and Farmer (2023):
In the real world, the random events that influence our lives are **neither stationary nor ergodic**

- Buraschi and Tebaldi (MS2023) DSGE with network distress propagation.... the empirical evidence is consistent with an equilibrium located above but in the vicinity of a tipping point so that critical fluctuations of aggregate distress h_t^{ν} are scale invariant....:

$$\Pr\left[h_t^{\nu} > H\right] \simeq H^{-\alpha}$$

- Bouchaud and Farmer (JPE2023) Quasi non-ergodicity .... ergodicity breaks down on a timescale at which realizations from the process might realistically be observed by a human agent..... the wealth distribution multiplicative, thereby generating a Pareto-tailed wealth distribution.



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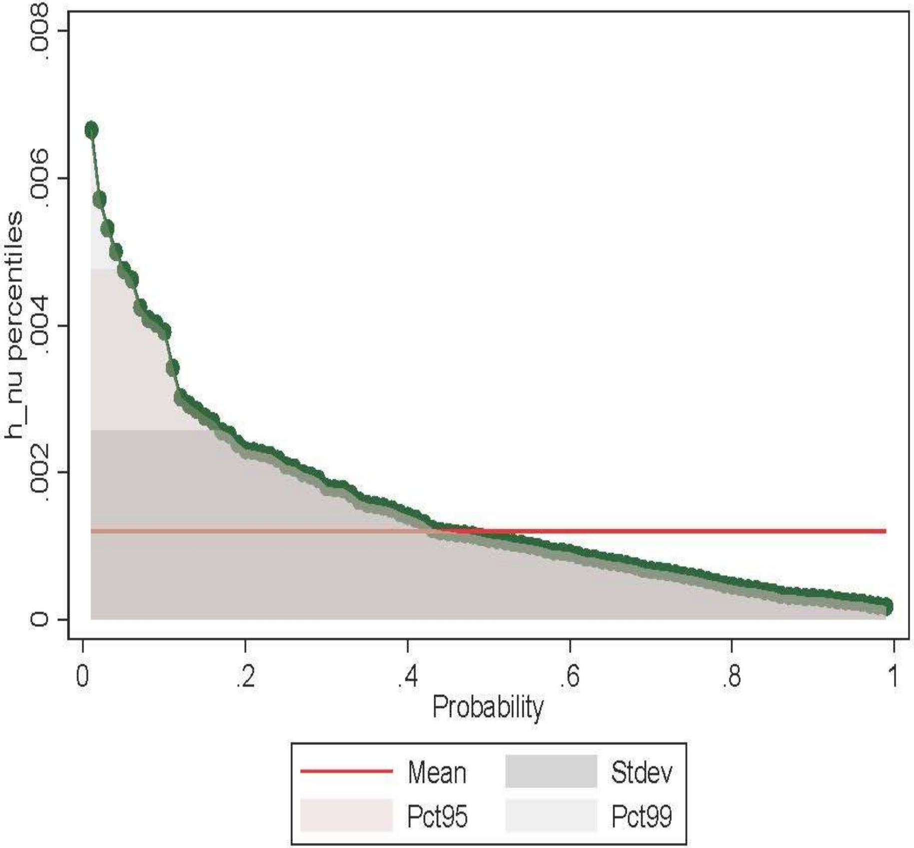

h_nu percentiles
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Probability
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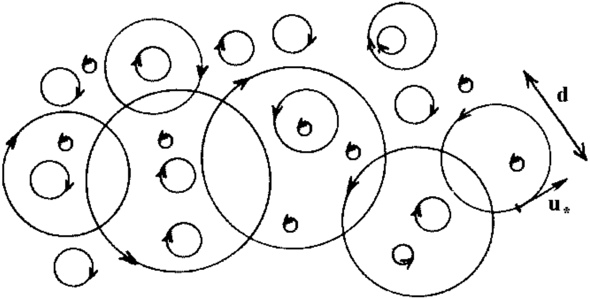
Stdev
Pct95
Pct99

Rationality in turbulent times

Proposal: first steps toward a definition of a Scale-Invariant DSGE.



As Kolmogorov hypothesized that equipartion of energies in turbulence takes place over log scales, Lamperti shows that a self-similar process w.r.t. t, if properly rescaled is stationary in log(t).

Lamperti isomorphism.

- The shift operator $\mathcal{S}_\tau$, $\tau \in \mathbb{R}$:

$$\left(\mathcal{L}_\tau X^T\right)(t) := X^T(t+\tau).$$

- Let $H > 0$ and $\lambda > 0$, the renormalized dilation operator $\mathcal{R}_{H,\lambda}$:

$$\left(\mathcal{R}_{H,\lambda} X^D\right)(t) := \lambda^{-H} X^D(\lambda t)$$

- The Lamperti transform $\mathcal{U}_H$:

$$\left(\mathcal{U}_H X^T\right)(t) := t^H X^T(\log t),\, t > 0$$

- The inverse Lamperti transform $\mathcal{U}_H^{-1}$:

$$\left(\mathcal{U}_H^{-1} X^D\right)(t) := e^{-Ht} X^D\left(e^t\right),\, t \in \mathbb{R}.$$

For any $\lambda > 0$:

$$\mathcal{U}_H^{-1} \mathcal{R}_{H,\lambda} \mathcal{U}_H = \mathcal{L}_{\log \lambda}$$

Renormalization operator

- Shift operator $\mathcal{L} \to L$ lag operator

$$L\varepsilon_{t-k} = \varepsilon_{t-k-1}$$

- Dilation operation $\mathcal{R}_{H,\lambda} \to R_H$ Haar Scale operator

$$\begin{aligned} R_H \varepsilon_t^{(j)} &= \frac{\varepsilon_t^{(j)} + \varepsilon_{t-2^j}^{(j)}}{2^H} \\ \varepsilon_t^{(j+1)} &:= \frac{\varepsilon_t^{(j)} - \varepsilon_{t-2^j}^{(j)}}{2^H} \end{aligned}$$

- Lamperti isometry in discrete time: the rescaling operator R_H acts as a shift operator L acting with respect to the scale index $j \to j+1$!
- R_H and L do not commute: you cannot be simultaeously translation and scale invariant.

Renormalization Group in Structural Econometrics: Extended Wold Decomposition

- Structural Vector AutoRegression:

$$L y_t = y_{t-1} \qquad A\left(L\right) y_t = \epsilon_t$$

Let $A^{-1}\left(L\right) = A_0 - \sum_{k=1}^{+\infty} A_k L^k$.

- Cerreia-Vioglio Ortu Severino Tamoni T. (QE19, DEF23) introduce a Wold decomposition w.r.t. rescaling operator R

$$A\left(L, R\right) y_t = \epsilon_t$$

where $A^{-1}\left(L, R\right) = A_0 - \sum_{j,k=1}^{+\infty} A_{k,j} R^j L^k$.

- After Lamperti transformation the relevant information filtration is across scales not across time.

Brownian motion

Consider a unit variance Brownian Motion $t_1 \to B_{t+t_1}(\omega)$, ,$t_1 = e^{\tau}$, $B_{t+e^{\tau}}(\omega)$ and using the scaling invariance of the brownian motion we can observe that setting:

$$O_{\tau}^{t}(\omega) := e^{-\frac{\tau}{2}} B_{t+e^{\tau}}(\omega), \ \tau > 0$$

and taking the differential one gets:

$$dO_{\tau}^{t}(\omega) = -\frac{1}{2} O_{\tau}^{t}(\omega)\, d\tau + dW_{\tau}(\omega)$$

A self-similar, non stationary process to an Ornstein-Uhlenbeck process with an ergodic steady state. MA representation:

$$O_{\tau}^{t}(\omega) = \int_{-\infty}^{\tau} e^{-(\tau-\sigma)\frac{1}{2}} dW_{\sigma}(\omega) \tag{2}$$

Multiscale autoregressive dynamics

OU differential formula is a scale version of an AR(1) operator

$$G^{(j+1)}_{t+2^{j+1}} = 2^{-\frac{1}{2}} G^{(j)}_{t+2^j} + \varepsilon^{(j+1)}_{t+2^{j+1}}$$

Then, by recursion, I can prove that the forward process is determined by the moving average:

$$G^{(J)}_{t+2^J} = \sum_{j=J_{\min}}^{J} 2^{-\frac{J-j}{2}} \varepsilon^{(j)}_{t+2^j}$$

Time scale flow inversion $\{t > 0, \log(t)\} \rightarrow \{t < 0, -\log\left(-\frac{1}{t}\right)\}$

$$G^{(j-1)}_{t-2^{j-1}} = 2^{-\frac{1}{2}} G^{(j)}_{t-2^j} + \varepsilon^{(j-1)}_{t-2^{j-1}}$$

Then, by recursion, I can prove that the backward process is determined by the moving average:

$$G^{(J)}_{t-2^J} = \sum_{j=J}^{J^{\max}} 2^{-\frac{j-J}{2}} \varepsilon^{(j)}_{t-2^j}$$

Forward Multiscale autoregressive dynamics

- The best estimate of the time t at level J is obtained by considering a weighted average of the shocks over different (lower) frequencies scaled by an appropriate weight.
- Notice that the above formulation applied to volatility fully motivates a structure in line with the Corsi (2009) HAR volatility predictor:

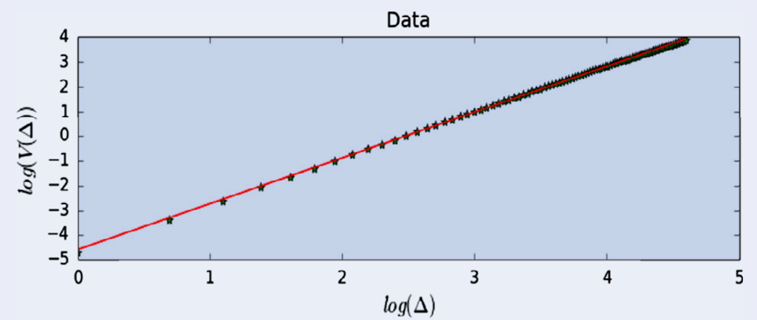
$$RV_t = c + \beta^{(d)} RV_{t-1} + \beta^{(w)} RV_t^{(w)} + \beta^{(m)} RV_t^{(m)} + \varepsilon_t$$

- A multiscale version the Rough Vol model: the joint process of log-prices and log-variances obeys scale similarity. A scale-stationary VAR would read:

$$\begin{bmatrix} \log\left(R\right)^{(j)}_{t-2^j} \\ \log\left(\sigma^2\right)^{(j)}_{t-2^j} \end{bmatrix} = \underbrace{A}_{2\times 2} \begin{bmatrix} \log\left(R\right)^{(j+1)}_{t-2^{j+1}} \\ \log\left(\sigma^2\right)^{(j+1)}_{t-2^{j+1}} \end{bmatrix} + \begin{bmatrix} \varepsilon^{(j)}_{R,t-2^j} \\ \varepsilon^{(j)}_{\sigma,t-2^j} \end{bmatrix}$$

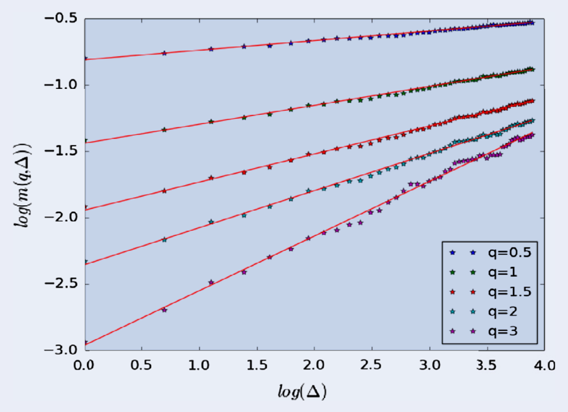
where $\varepsilon^{(j)}_{t-2^j}$ is a multivariate white noise on a time grid 2^j.

Scaling



Integrated Return Variance is scale invariant
(Data on S&P500 from Oxford Man Institute)

Scaling



Log-Variance moments are scale invariant and compatible with a Fractional BM evolution with $H = 0.1$
(Data on S&P500 from Oxford Man Institute)

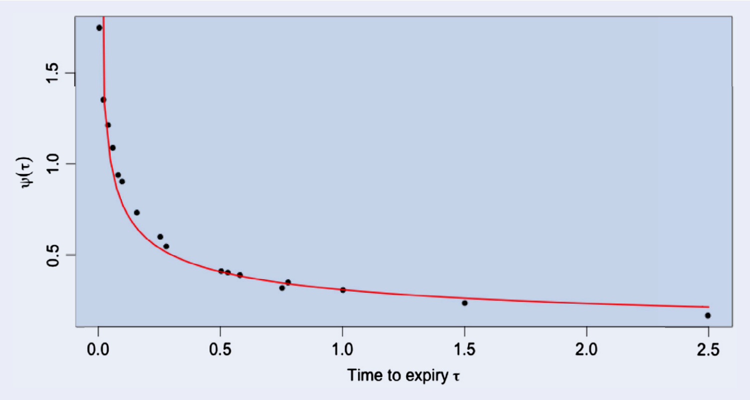
Backward Multiscale autoregressive dynamics

- To compute expectations of investors, you have to compute the backward component.
- Intuition: computation of the backward component is equivalent to carry out a pricing exercise. Observation from the Rough Vol paper: Skew, the price of an option strategy, is also scale-stationary. Backward variables must also be scale stationary.
- A scale consistent version of the dividend-discount formula:

$$P_t = \sum_{j=1}^{+\infty} q^j E_t^F \left[d_{t+2^j}^{(j)} \right], \quad q < 1$$

  where $E_t^F \left[d_{t+2^j}^{(j)} \right]$ is the persistence-j component of the (forward measure) expected dividend.
- 'Excess volatility puzzle' is circumvented: volatility of low frequency components is reduced by the averaging and rescaling procedure and then properly rescaled according to q.

Scaling



Option Markets: Term Structure of Implied Skewness vs $\tau^{H-\frac{1}{2}}$
(Gatheral Jaisson and Rosenbaum 2018)

Backward Multiscale Autoregressive dynamics

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- Madan and Wang (2022) state that risk neutral variance term structures are characterized by their time elasticities:

$$\gamma\left(T-t\right) = -\frac{\left(T-t\right)v_t\left(T-t\right)}{v\left(T-t\right)} = \frac{dln(v(T-t))}{dln(T-t)}$$

An additional month at one month is not comparable to an additional month at five years or sixty months.

Important conclusion for 'ergodic economists': there is a second ergodic problem which is relevant if you consider a dynamics w.r.t. the time-changed scale $\tau = \log\left(t\right)$.

Learning dynamics across scales

- The stationary point in the Lamperti transformed dynamics corresponds to a fixed point for the Renormalization Operator.
- Beyond scaling, a second important property implied by the presence of a fixed point of a renormalziation group operator is Universality, the fact that upon averaging different models converge to the same fixed point.
- Universality property is equivalent to analyzing dynamics in model space and focusing on fixed points in the parameter space. This is in full analogy REE where rational expectations coincide with fixed points for the learning dynamics.
- Universality in financial markets: investors with large trading interests recognize that their trades can move the market-clearing price, which reduces their profits and split their orders into child ones.

Universality

Kyle and Obizhaeva (2016) hypothesize that the quantity:

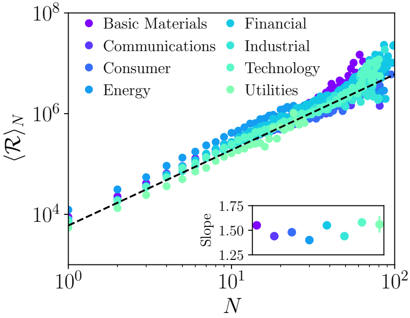
$$I = Q_B \frac{\sigma P}{N_B^{1/2}} = \frac{\sigma P V}{N_B^{3/2}} = \frac{W}{N_B^{3/2}}.$$

does not depend on time and on assets.

- A ‘bet’ or metaorder is a sequence of correlated orders driven by the same information. Very similar to the notion of cascades/avalanches.
- Q_B quantity of dollars per ‘bet’.
- σP volatility of price per unit period.
- N_B number of ‘bets’ per unit time is business time.
- $V = Q_B N_B$ dollar volume.
- $W = \sigma P V$ trading activity.

Universality

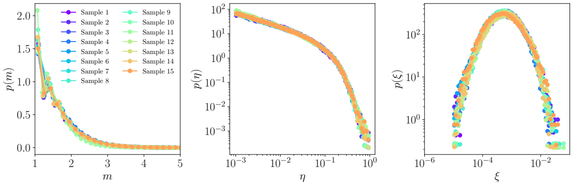
- Market Microstructure Invariance, Kyle and Obizhaeva (2016):



$log(W)$ vs $log(N_B)$
Benzaquen Bouchaud Bucci Lillo (2019)

Universality

- Market Microstructure Invariance, weak universality hypothesis:



Invariance requires normalization of the Kyle Obizhaeva invariant by an index reflecting total trading costs
Benzaquen Bouchaud Bucci Lillo (2019)

Renormalization Group

- 'Scaling and efficiency determine the irreversible evolution of a market' Stella Baldovin PNAS (2007) and Challet and Peirano (2008)

$$p_{t,T}\left(r\right) = \frac{1}{\sqrt{\left(T+t\right)^{2D} - t^{2D}}} g\left(\frac{r}{\sqrt{\left(T+t\right)^{2D} - t^{2D}}}\right)$$

- Omori Law Lillo and Mantegna (2001), Baldovin Stella et al. (2013) Omori Law, $N(t)$ cumulative number of aftershocks:

$$N\left(T\right) = \frac{K}{1-p}\left[\left(T+t\right)^{1-p} - \left(t\right)^{1-p}\right]$$

- All these equations, in order to exploit scale invariance, become non-time translation invariant.

The formation of agent beliefs and preferences

- To avoid Lucas critique, it is necessary to discuss how the shift from translation invariance to scale invariance impacts preferences
- We notice that this time change is fundamentally inconsistent with time translation-invariance of the preference, which is quite reasonable in relation to the fact that a process that is stationary in the log time will be self-similar in time.
- There is a large literature analyzing individual preferences w.r.t. time
  Hyperbolic preference: Lowenstein and Prelec (QJE1992)
  The discount function is a generalized hyperbola:

$$d(t) = (1 + \alpha t)^{-\frac{\beta}{\alpha}} \quad \alpha, \beta > 0$$

  The α-coefficient determines how much the function departs from constant discounting.

The formation of agent beliefs and preferences

A striking evidence of time inconsistency: procrastination of conference registration by Alfi Parisi Pietronero (Nature2007):

The probability $p(t)$ to register at time t is then $p(t) = C/\,(T^{*} - t)$, where T^{*} is the deadline and the constant C will be fixed by the total number of participants N_{tot} . The number of registrations at time t is given by $N(t) = C \ln\,(T^{\star}/\,(T^{\star} - t))$.

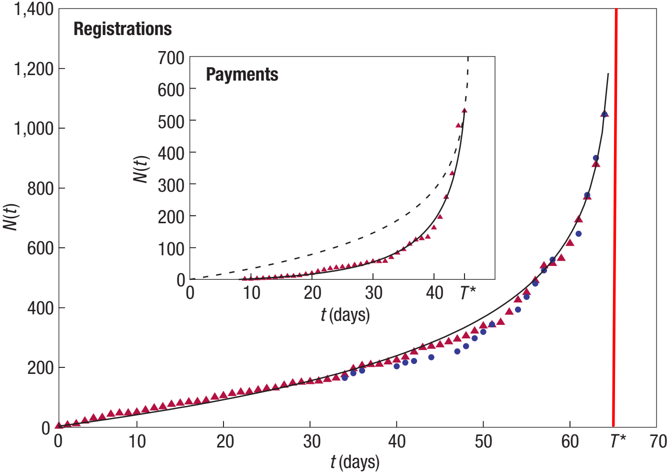


Figure 1 The distribution of registrations is shown for Statphys 23 (red triangles), up to the main deadline for abstract submission (T^*), and for the EP2DS 17 conference (blue circles), rescaled with respect to the total number of participants; the solid line corresponds to a simple model in which the pressure to register is inversely proportional to the time left before the deadline. The level of agreement between the data for the two conferences and the model suggests that there is a simple universal behaviour in response to a deadline. The inset shows the distribution in time of payment of the conference fee (credit-card payments only): the distribution is more peaked towards the deadline because, although registration is reversible, payment is irreversible. The simple model (dashed line) is not accurate in this case, and it is necessary to include an exponential utility function (solid line).

The formation of agent beliefs and preferences

- This results in a time inconsistent optimal control problem (a change in the initial point will change the definition of the optima) that deserves more attention:

$$
\begin{aligned}
& J_{t_0}(x; u(\cdot)) \\
\triangleq \quad & E_{(0,x)}\left[\int_{t_0}^{\infty} (\alpha t)^{-\frac{\beta}{\alpha}} \left\{ \begin{array}{c} P(t)\left[X(t)-c_1(t)\right]^2 \\ +Q(t)\left[Y(t)-c_2(t)\right]^2 \\ +R(t)u^2(t)+S(t)Z(t)^2 \end{array} \right\} dt \right].
\end{aligned}
$$

subject to the FBSDE constraint:

$$
\begin{aligned}
dX(t) &= b(t, X(t), Y(t), u(t))dt + \sigma(t, X(t), Y(t), u(t))dW(t), \\
dY(t) &= [-\widetilde{g}(t, X(t), Y(t), u(t))]dt + Z(t)dW(t), \quad t \geq 0, \\
X(0) &= x \in \mathbb{R}^n.
\end{aligned}
$$

The formation of agent beliefs and preferences

- If you shift from t to $\log(t)$ you restore a scale-invariant version of the time consistency property.
- You loose time-translational invariance of preference, but you can save the dynamic programming principle. It simply states that you do not want your current decision to be self-contraddictory over different horizons.
- A new dynamic programming principle equivalent to dynamic renormalization: you need to shift the time variable from t to $\log(t)$:

$$\frac{\partial}{\partial t} - HJB[.] \rightarrow t\frac{\partial}{\partial t} - HJB[.]$$

- The role of first order optimality conditions under this new information filtration to be understood.

Conclusions

If you take seriously the empirical evidence and are trying to set up a SI-DSGE........

shift from time-translation invariant to scale-consistent preferences.

a large body of work yet to be done!