Quant(......) Finance

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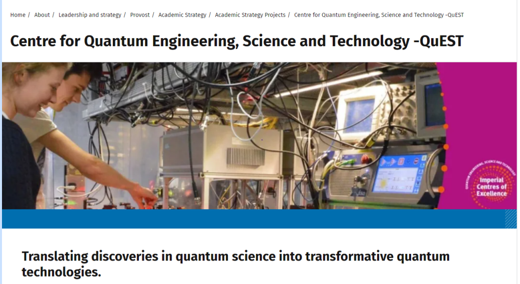
Celebrating *Quantitative Finance* and Michael Dempster
Cambridge, April 2023

Based on joint works / discussions with A. Assouel (ENS Paris), F. Fontanela (Lloyds), A. Kondratyev (ADIA), M. Oumgari (Lloyds), A. Bako (Bloomberg), A. Elkadi (Imperial), S. Laizet (Imperial), J. Dees (Imperial)

Support and Environment

- EPSRC grant on Distributed Quantum Computing and Applications
- EPSRC New Horizon Grant on Quantum algorithms for turbulent flows

- Imperial College QuEST Centre (Maths, Computings, Physics, Aeronautics, EEE), focusing on
  - Materials for Quantum Technologies
  - Quantum Internet
  - Applications of Quantum Computing



What this talk is NOT about

- Q advantage and Q supremacy – I am not a Sales guy

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Goal: Convince you that we, as (...applied...) Mathematicians have a role to play in the evolution of Q computing (in Finance).

Q mechanics or Q computing?

Q mechanics is a *framework* for the development of Physics theories, as originally proposed mid-1920s by N. Bohr$^{\aleph}$, L. de Broglie$^{\aleph}$, M. Born$^{\aleph}$, W. Heisenberg$^{\aleph}$, W. Pauli$^{\aleph}$, E. Schrödinger$^{\aleph}$, P. Dirac$^{\aleph}$.

The mathematics of Q mechanics allow for more general *computation*:

- more general definition of the *memory state* compared to classical computing;
- wider range of *transformations* / *evolution* of memory states.

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Why haven't we used this computation framework until now?
To perform Q computation efficiently we need actual Q mechanical systems, only proposed in the 1980s by P. Benioff, R. Feynman$^{\aleph}$, Y. Manin.

Q algorithms can be run on classical computers, but require enormous amount of memory, so that exponential gains in computing power are offset by exponential memory requirements.

Q formalism

Q formalism

- State space: complex Hilbert space $\mathfrak{H} = \mathbb{C}^N$. for $\mathrm{u}, \mathrm{v} \in \mathfrak{H}$, (*: complex conjugacy)

$$\text{(ket)} \quad |\mathrm{v}\rangle := \begin{bmatrix} v_1 \\ \vdots \\ v_N \end{bmatrix} \in \mathfrak{H}, \qquad \text{(bra)} \quad \langle \mathrm{u}| := (u_1^*, \dots, u_N^*) \in \mathfrak{H}^*,$$

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- 1-qubit quantum state: $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$, for $\alpha, \beta \in \mathbb{C}$ such that $|\alpha|^2 + |\beta|^2 = 1$.
- Given $\mathfrak{H}_N$ and $\mathfrak{H}_M$, tensor product $\mathfrak{H} := \mathfrak{H}_N \otimes \mathfrak{H}_M$ is the NM-dimensional Hilbert space spanned by $\{|i\rangle \otimes |j\rangle : i = 0, \ldots, N-1, j = 0, \ldots, M-1\}$.

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- For a 2-qubit system,

$$\{|0\rangle\otimes|0\rangle, |0\rangle\otimes|1\rangle, |1\rangle\otimes|0\rangle, |1\rangle\otimes|1\rangle\} = \{|00\rangle, |01\rangle, |10\rangle, |11\rangle\} =: \{|0\rangle, |1\rangle, |2\rangle, |3\rangle\}.$$

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- n-qubit quantum state: vector in $\mathbb{C}^{2^n}$ (with basis $(|0\rangle, \ldots, |2^n - 1\rangle)$), such that

$$|\psi\rangle = \sum_{i=0}^{2^n-1} \alpha_i\,|i\rangle, \quad \text{for } (\alpha_0, \ldots, \alpha_{2^n-1}) \in \mathbb{C}^{2^n}, \text{ such that } \sum_{i=0}^{2^n-1} |\alpha_i|^2 = 1.$$

Q formalism

The family $(|i\rangle)_{i=0,\dots,2^n-1}$ is an orthonormal basis of $\mathbb{R}^{2^n}$.

- $n = 1$; $|0\rangle = \begin{bmatrix}1\\0\end{bmatrix}, \quad |1\rangle = \begin{bmatrix}0\\1\end{bmatrix}$;
- $n = 2$;

$$|00\rangle = |0\rangle\otimes|0\rangle = \begin{bmatrix}1\\0\end{bmatrix}\otimes\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\cdot\begin{bmatrix}1\\0\end{bmatrix}\\0\cdot\begin{bmatrix}1\\0\end{bmatrix}\end{bmatrix} = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \quad |01\rangle = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \quad |10\rangle = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \quad \cdots$$

- $n \in \mathbb{N}$;

$$|0\cdots0\rangle = \begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}, \qquad \cdots \qquad, |1\cdots1\rangle = \begin{bmatrix}0\\\vdots\\0\\1\end{bmatrix}.$$

Q operations

- Q Gate: reversible quantum circuit (unitary matrix: $\mathrm{U}\mathrm{U}^* = \mathrm{U}^*\mathrm{U} = \mathtt{I}$).
- Standard gates:

$$\mathtt{X} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathtt{Y} = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}, \quad \mathtt{H} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \mathtt{R}_y(\theta) = \begin{bmatrix} \cos\left(\frac{\theta}{2}\right) & -\sin\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right) \end{bmatrix}$$

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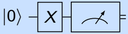
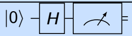
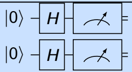
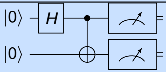
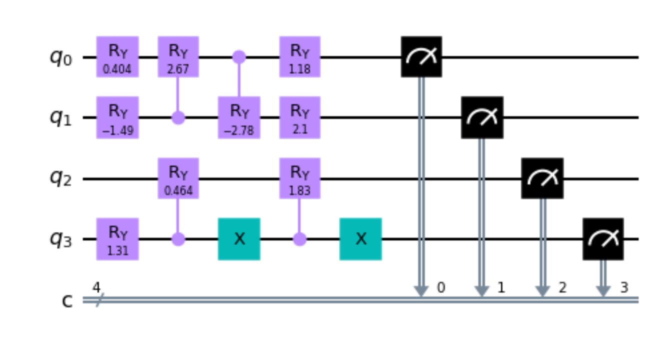
- Examples:

$\mathrm{X}\|0\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \|1\rangle.$	
$\mathrm{H}\|0\rangle = \frac{\|0\rangle + \|1\rangle}{\sqrt{2}} = \|+\rangle$ and $\mathrm{H}\|1\rangle = \frac{\|0\rangle - \|1\rangle}{\sqrt{2}} = \|-\rangle$	
$\mathrm{H}^{\otimes 2}\|00\rangle = (\mathrm{H}\|0\rangle) \otimes (\mathrm{H}\|0\rangle) = \frac{1}{2}\left\{ \|00\rangle + \|01\rangle + \|11\rangle + \|11\rangle \right\}$	
$_c\mathrm{X}\Big((\mathrm{H} \otimes \mathrm{I})\|00\rangle\Big) = {_c\mathrm{X}}\left(\frac{\|0\rangle + \|1\rangle}{\sqrt{2}} \otimes \|0\rangle\right) = \frac{\|0\rangle\|0\rangle + \|1\rangle\mathrm{X}\|0\rangle}{\sqrt{2}} = \frac{\|00\rangle + \|11\rangle}{\sqrt{2}}$ (EPR)	

Example of a Q circuit



Exciting example: Generating a uniform distribution

- 1 qubit, i.e. 2 values (discrete distribution over 2 points):

$$\mathtt{H}\,|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

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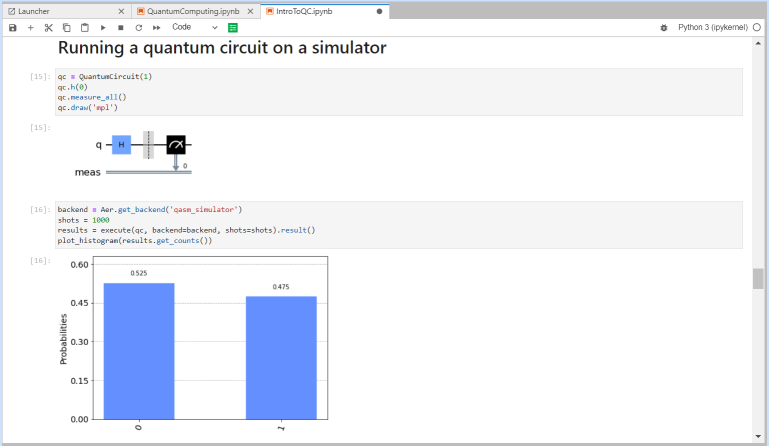
- n qubits, i.e. 2^n values (discrete distribution over 2^n points):

$$\begin{aligned}
\mathtt{H}^{\otimes n}|0\rangle^{\otimes n} &= \Big(\mathtt{H}|0\rangle\Big) \otimes \cdots \otimes \Big(\mathtt{H}|0\rangle\Big) \\
&= \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) \otimes \cdots \otimes \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) \\
&= \frac{1}{2^{n/2}}\Big(|0\rangle + |1\rangle\Big) \otimes \cdots \otimes \Big(|0\rangle + |1\rangle\Big) \\
&= \frac{1}{2^{n/2}}\Big(|0\cdots 0\rangle + |0\cdots 01\rangle + \cdots + |1\cdots 10\rangle + |1\cdots 1\rangle\Big) \\
&= \frac{1}{2^{n/2}}\sum_{i=0}^{2^n-1}|i\rangle.
\end{aligned}$$

Possible to code things up:

- Simulated quantum computer
- Actual (small-size) quantum computer

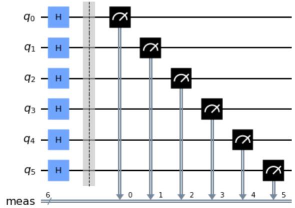
```
from qiskit import QuantumCircuit, Aer, execute
from qiskit.visualization import plot_histogram
```



6 qubits

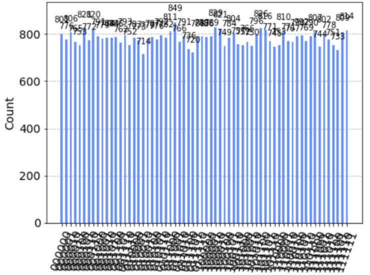
[10]:
```
n = 6
qc = QuantumCircuit(n)
for i in range(n):
    qc.h(i)
qc.measure_all()
qc.draw('mpl')
```

[10]:



[11]:
```
backend = Aer.get_backend('qasm_simulator')
shots = 50000
results = execute(qc, backend=backend, shots=shots).result()
plot_histogram(results.get_counts())
```

[11]:



Application: Quantum Random Number Generation

Sale!

Quantis QRNG: PCIe 240 Mbps

Limited-time offer: -15%

- Based on latest Quantis QRNG technology (IDQ20MC1 chip)
- Embedded NIST 800-90 A/B/C compliant DRBG
- 232 Mbps entropy source , 58 Mbps RNG data (after post-processing)
- Pass NIST SP800-22, SP800-90B and DieHarder statistical test suites
- Certification by Swiss National Laboratory (METAS): pending
- PCIe Express Base 1.0a
- OS support: Windows 10, Ubuntu 18.04, CentOS 7

Choose your support option:

Product only

~~€3,136.00~~ **€2,665.60** ex VAT

12 in stock

Add to basket **Add to quote**

ERNIE has been using it since 2019.

Is there any sense of reality here?

Two competing technologies:

- Superconducting qubits: each qubit can interact with its nearest neighbour, limited decoherence time, needs super-cooling; `IBM`, `Google`, `AWS`, `Alibaba`, `Rigetti`, `Intel`, `D-Wave`.
- Ion trapped: ions trapped in electric fields, that can be *perturbed* by laser beams. `Quantinuum`, `IonQ` , `Quantum Factory` , `Alpine Quantum Technologies`, `eleQtron`, `Oxford Ionics`.

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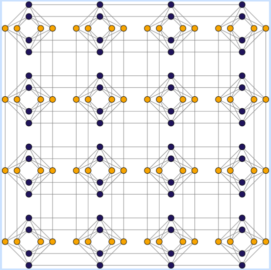
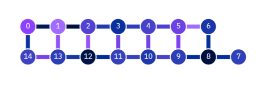
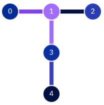
Reading between the *Quantum supremacy* lines:

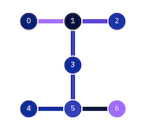
- Number of qubits;
- Connectivity;
- Coherence time.

Q Tech: interesting graph theoretic problems









Postulate 1 – Statics

Associated to any physical system is a complex inner product space (Hilbert space) known as the state space of the system. The system is completely described at any given point in time by its state vector, which is a unit vector in its state space.

Postulate 2 – Dynamics

The evolution of the closed Q system is described by the Schrödinger equation

$$\mathrm{i}\hbar\partial_t |\psi(t)\rangle = \mathcal{H} |\psi(t)\rangle ,$$

where $\hbar$ is Planck's constant and $\mathcal{H}$ is a time-independent Hermitian operator (Hamiltonian of the system).

Note that, for any $0 \leq t_1 \leq t_2$, Schrödinger's equation gives us

$$|\psi(t_2)\rangle = \mathcal{U}(t_1, t_2) |\psi(t_1)\rangle , \quad \mathcal{U}(t_1, t_2) = \exp\left\{\frac{-\mathrm{i}(t_2 - t_1)}{\hbar}\mathcal{H}\right\}.$$

Lemma: if $\mathcal{H}$ is Hermitian ($\mathcal{H}^\dagger := (\mathcal{H}^*)^\top = \mathcal{H}$) and $\alpha \in \mathbb{R}$, then $\exp\{i\alpha\mathcal{H}\}$ is unitary.

Unitary operators – Q logic gates

Unitary operators preserve the inner product and hence norms: given $|\mathrm{u}\rangle$ and $|\mathrm{v}\rangle$, and a unitary operator $\mathcal{U}$, then

$$(|\mathcal{U}\mathrm{u}\rangle)^\dagger \cdot |\mathcal{U}\mathrm{v}\rangle = \langle \mathrm{u}\mathcal{U}^\dagger| \, |\cdot|\mathcal{U}\mathrm{v}\rangle = \langle \mathrm{u}| \, \mathcal{U}^\dagger\mathcal{U} \, |\mathrm{v}\rangle = \langle \mathrm{u}|\mathrm{v}\rangle \, .$$

In Q mechanics, all physical transformations (rotations, translations, time evolution) correspond to (unitary) maps from Q states to Q states.

Unitary operators can then be viewed as *Q logic gates* implementing Q computations.

Since unitary operators are *invertible* ($\mathcal{U}^{-1} = \mathcal{U}^\dagger$), then Q computing is *reversible*.

Quantum logic gates

Quantum logic gates allow to transform qubits, i.e. to rotate them on the unit sphere. It generalises classical operations. It can be represented as a unitary matrix in $\mathbb{C}^2$ ($\mathtt{G}^\dagger\mathtt{G} = \mathtt{G}\mathtt{G}^\dagger = \mathtt{I}$).

Example: There is no Boolean function φ such that applied twice to a classical bit would result in a NOT gate: $\varphi(\varphi(0)) = 1$ and $\varphi(\varphi(1)) = 0$. In Q computing, let

$$\mathtt{G} := \frac{1}{2}\begin{pmatrix} 1+\mathrm{i} & 1-\mathrm{i} \\ 1-\mathrm{i} & 1+\mathrm{i} \end{pmatrix},$$

Then

$$\mathtt{G}^2 = \frac{1}{4}\begin{pmatrix} (1+\mathrm{i})^2 + (1-\mathrm{i})^2 & 2(1+\mathrm{i})(1-\mathrm{i}) \\ 2(1+\mathrm{i})(1-\mathrm{i}) & (1+\mathrm{i})^2 + (1-\mathrm{i})^2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

so that

$$\mathtt{G}^2 |0\rangle = \mathtt{G}^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle \quad \text{and} \quad \mathtt{G}^2 |1\rangle = \mathtt{G}^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle .$$

The Spectral Theorem

The eigenfunctions of an Hermitian operator form a complete set of basis functions.

Spectral Theorem: *If $\mathcal{A}$ is Hermitian, there exists an orthonormal basis consisting of eigenvectors of $\mathcal{A}$. Each eigenvalue is real.*

Therefore the state $|\Psi\rangle$ of the system can be written as a superposition of eigenfunctions $\{|\psi_i\rangle\}$ of $\mathcal{A}$:

$$|\Psi\rangle = \sum_i \alpha_i |\psi_i\rangle,$$

where again the coefficients $\{\alpha_i\}$ are complex *probability amplitudes* with $\sum_i |\alpha_i|^2 = 1$.

Postulate 3 – Measurement

If we measure the Hermitian operator $\mathcal{A}$ in the state $|\Psi\rangle$, the possible outcomes are the eigenvalues $\{\lambda_i\}$ of $\mathcal{A}$. The probability $\mathbb{P}_i$ to measure λ_i is given by

$$\mathbb{P}_i = |\alpha_i|^2.$$

After the outcome λ_i, the state of the system becomes

$$|\Psi\rangle = |\psi_i\rangle\,.$$

This can be understood as a projection onto the eigenstate $|\psi_i\rangle$: define $\Pi_i := |\psi_i\rangle\langle\psi_i|$, then the state evolves from $|\Psi\rangle$ to $\Pi_i|\Psi\rangle$, with

$$\Pi_i|\Psi\rangle = |\psi_i\rangle\langle\psi_i|\,|\Psi\rangle = |\psi_i\rangle\langle\psi_i|\left(\sum_j \alpha_j|\psi_j\rangle\right) = \sum_j \alpha_j|\psi_i\rangle\langle\psi_i|\,|\psi_j\rangle = \alpha_i|\psi_i\rangle\,.$$

We this need to perform measurement on the same $\mathtt{Q}$ state many times to generate sufficient statistics (akin to Monte Carlo).

Postulate 4 – Composite Systems

The state space of a composite physical system is the tensor product of the state spaces of the individual component physical systems.

If one component physical system is in state $|\psi_1\rangle$ and a second component physical system is in state $|\psi_2\rangle$, then the state of the combined system is

$$|\psi_1\rangle \otimes |\psi_2\rangle .$$

Not all combined systems can be split into a tensor product of states of individual components. When this is not the case, the components are called *entangled*.

The power of entanglement

Consider an n-qubit system, where (recall) an individual qubit can be found, after measurement, in $|0\rangle$ or $|1\rangle$, i.e. we need to specify 2 probability amplitudes to describe the state of the qubit.

If all the qubits are independent, the quantum state can be represented as

$$|\Psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \otimes \ldots \otimes |\psi_n\rangle ,$$

and we need to specify 2n probability amplitudes.

If all individual qubits are entangled (hence, there is no tensor product representation), we need to specify 2^n probability amplitudes.

Quantum Machine Learning

Classical GAN

A generator and a discriminator compete against each other to improve themselves:

- the generator improves by becoming better at generating good samples (i.e. close to real data) from random noise
- the discriminator improves by being able to recognise real data from 'fake' (namely generated) data.
- Both are built as neural networks with hyperparameters over which to optimise.

Given a generator $\mathfrak{G}(\cdot, \boldsymbol{\theta}^{\mathfrak{G}}) : \mathcal{X} \to (0,1)$ and a discriminator $\mathfrak{D}(\cdot, \boldsymbol{\theta}^{\mathfrak{D}}) : \mathcal{X} \to (0,1)$ ($\boldsymbol{\theta}^{\mathfrak{G}}, \boldsymbol{\theta}^{\mathfrak{D}}$: hyperparameters), the goal is

$$\min_{\boldsymbol{\theta}^{\mathfrak{G}}} \max_{\boldsymbol{\theta}^{\mathfrak{D}}} \left\{ \mathbb{E}_{\mathrm{x}\sim\mathbb{P}_{\mathrm{data}}} \left[\log(\mathfrak{D}\left(\mathrm{x}; \boldsymbol{\theta}^{\mathfrak{D}}\right) \right] + \mathbb{E}_{\mathrm{z}\sim\mathbb{P}_{\mathfrak{G}(\cdot,\boldsymbol{\theta}^{\mathfrak{G}})}} \left[\log\left(1 - \mathfrak{D}\left(\mathfrak{G}\left(\mathrm{z}; \boldsymbol{\theta}^{\mathfrak{G}}\right); \boldsymbol{\theta}^{\mathfrak{D}}\right)\right)\right] \right\},$$

where $\mathrm{x} \sim \mathbb{P}_{\mathrm{data}}$ means some sample x generated from the original, 'true' data, whereas $\mathrm{z} \sim \mathbb{P}_{\mathfrak{G}}$ refers to sample generated from the generator $\mathfrak{G}$.

Q GAN

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Probability that the measurement yields a positive answer given the true data:

$$\mathbb{P}(\mathrm{T}|\sigma) = \mathrm{Tr}(\mathrm{T}\sigma),$$

Probability that the measurement yields a positive answer given generated data:

$$\mathbb{P}(\mathrm{T}|\mathfrak{G}) = \mathrm{Tr}(\mathrm{T}\rho).$$

Q GAN

- Data represented by a density operator (positive semi-definite, Hermitian matrix of trace one) $\sigma = |\psi\rangle\langle\psi|$;
- Generator $\mathfrak{G}$ generates some output density matrix ρ;
- Discriminator: tries to identify the true data from the fake one, i.e. it makes a positive operator-valued measurement with outcomes T (True) or F (False).

Probability that the measurement yields a positive answer given the true data:

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Probability that the measurement yields a positive answer given generated data:

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Adversarial game:

$$\min_{\mathfrak{G}} \max_{\mathrm{T}} \Big\{ \mathrm{Tr}(\mathrm{T}\rho) - \mathrm{Tr}(\mathrm{T}\sigma) \Big\}.$$

Note: Both the set of positive measurement operators T *(with* 1*-norm less than one) and the set of density matrices* ρ *are convex.*

Variational circuit representations

- Both generator and discriminator represented as variational quantum circuits parameterised by a vector of parameters (e.g. rotation angles of all the gates);
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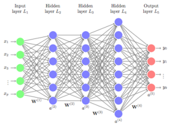
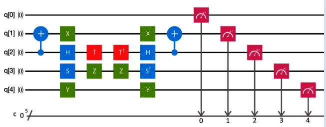
Questions:

- Optimal architecture of variational quantum circuit?
- Computing the gradient?

Variational Quantum Circuit





Architecture of the Q Generator

$$\mathfrak{G}\,|0\rangle^{\otimes n} := \prod_{l=L}^{1} \mathrm{U}_l(\boldsymbol{\theta}_l). \qquad (1)$$

For each layer $l \in \{1, \dots, L\}$, $\mathrm{U}_l(\boldsymbol{\theta}_l)$ acts on all n qubits, and $\boldsymbol{\theta}_l \in [0, 2\pi)^M$;
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General form of the L-layer neural network is (1), where, for each $l \in \{1, \dots, L\}$,

$$\mathrm{U}_l(\boldsymbol{\theta}_l) = \left\{ \bigotimes_{i=1}^{n} \mathtt{Q}^{1+(i \bmod n)}(\theta^i_{\mathrm{imp}}) \mathtt{R_X}(\theta^i_{\mathrm{ex}}) \right\} \left\{ \left(\bigotimes_{i=1}^{n} \mathtt{R_Z}(\theta^i_{\mathtt{Z},l}) \right) \left(\bigotimes_{i=1}^{n} \mathtt{R_X}(\theta^i_{\mathtt{X},l}) \right) \left(\bigotimes_{i=1}^{n} \mathtt{R_Y}(\theta^i_{\mathtt{Y},l}) \right) \right\},$$

where $\mathtt{Q}^i$ means that qubit i is the control qubit and the gate acts on qubit $(i+1)$.
Note that $1 + (i \bmod n) = 1 + i$ when $i \in \{1, \dots, n-1\}$ and is equal to 1 when $i = n$.
The total number of hyperparameters is therefore $5n$ per layer, thus $5nL$ in total.

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- Non-linearities? *from what to what?*
- Universal approximation theorem?
- Conjecture: higher expressive power, only limited results so far.

Q encoding

- Encoding classical data into quantum states. For $x_j \in [0,1]$ and $p \in \mathbb{N}$,

$$\frac{x_{j,1}}{2} + \frac{x_{j,2}}{2^2} + \ldots + \frac{x_{j,p}}{2^p} \qquad \text{(p-binary approximation of x_j)},$$

  where $x_{j,k} \in \{0,1\}$, for $k \in \{1,2,\ldots,p\}$.
- Q code for x_j:

$$|x_j\rangle := |x_{j,1}\rangle \otimes |x_{j,2}\rangle \otimes \ldots \otimes |x_{j,p}\rangle = |x_{j,1}x_{j,2}\ldots x_{j,p}\rangle\,,$$

- Encoding $\mathrm{x} \in [0,1]^n$:

$$|\mathrm{x}\rangle := |x_{0,1}x_{0,2}\ldots x_{0,p}\rangle \otimes \ldots \otimes |x_{n-1,1}\ldots x_{n-1,p}\rangle\,.$$

- Not so convenient.....

Simple example: Q Classifier

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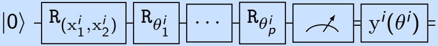

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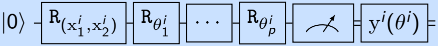

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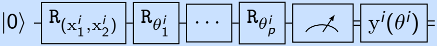

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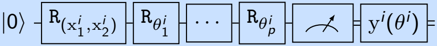

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Comments:

- Very similar to a classical NN;
- No activation function;
- No weights;
- No entanglement;
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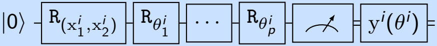

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Inner product? Construct $\mathrm{U}_{\mathbf{w}}$ such that
$\mathrm{U}_{\mathbf{w}}\left(\mathrm{H}^{\otimes m}|0\rangle^{\otimes m}|\mathrm{x}\rangle\right) = \left[\sum_{j=0}^{2^m-1} \mathrm{e}^{\frac{2\mathrm{i}\pi j}{2^m}\mathrm{x}^\top \mathbf{w}} |j\rangle\right] \otimes |\mathrm{x}\rangle \to$ inverse Q Fourier transform.
Activation function? Build U such that $\mathrm{U}|\mathrm{x}\rangle = \mathrm{e}^{2\mathrm{i}\pi\sigma(\mathrm{x})}|\mathrm{x}\rangle \to$ Q Phase estimation

Application: Generating SVI

(my first paper–with Jim–in QF ... 4 pages!!)

- $$p_T(k) = \left(\frac{\partial^2 \mathrm{C}_{\mathrm{BS}}(k, T, \sigma_{\mathrm{imp}}(k, T))}{\partial K^2}\right)_{K=S_0 \mathrm{e}^k}.$$

  SVI parameterisation proposed by Gatheral:

  $$w_{\mathrm{SVI}}(k, T) = \sigma^2_{\mathrm{imp}}(k, T) T = a + b\left(k - m + \rho\sqrt{(k-m)^2 + \xi^2}\right), \qquad \text{for any } k \in \mathbb{R},$$

  with the parameters $\rho \in [-1, 1]$, $a, b, \xi \geq 0$ and $m \in \mathbb{R}$.
- SVI Density:

  $$p_T(k) = \frac{g_{\mathrm{SVI}}(k, T)}{\sqrt{2\pi w_{\mathrm{SVI}}(k, T)}} \exp\left\{-\frac{d_-(k, w_{\mathrm{SVI}}(k, T))^2}{2}\right\},$$

  with (all derivatives with respect to k.)

  $$g_{\mathrm{SVI}}(k, T) := \left(1 - \frac{k w'_{\mathrm{SVI}}(k, T)}{2 w_{\mathrm{SVI}}(k, T)}\right)^2 - \frac{w'_{\mathrm{SVI}}(k, T)^2}{4}\left(\frac{1}{4} + \frac{1}{w_{\mathrm{SVI}}(k, T)}\right) + \frac{w''_{\mathrm{SVI}}(k, T)}{2},$$

SVI Example

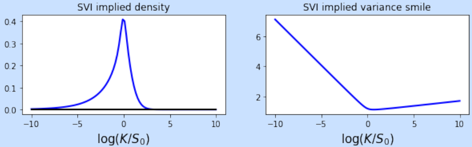

Figure: Density of $\log(S_T)$ in SVI

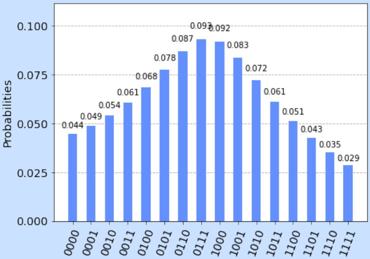

Figure: Discretised distribution of $\log(S_T)$ on $[-1, 1]$ with 2^4 points

SVI Numerics

Target wave function:

$$|\psi_{\text{target}}\rangle = \sum_{i=0}^{2^n-1} \sqrt{p_i}\,|i\rangle\,,$$

where, for each $i \in \{0, \dots, 2^n-1\}$, $p_i = \mathbb{P}\left(\log(S_T) \in \left[-1+\frac{2i}{2^n}, -1+\frac{2(i+1)}{2^n}\right)\right)$.

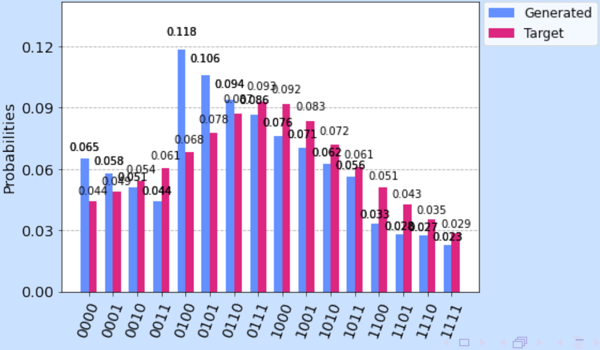

Figure: Comparison between the target and the generated distributions at the end of the training

Q Food for thought

- Barren plateaux?
  - Consider a Q circuit $\mathrm{U}(\boldsymbol{\theta}) = \prod_{l=L}^{1} \mathrm{U}_l(\theta_l)$, with $\mathrm{U}_l(\theta_l) = \mathrm{e}^{-\mathrm{i}\theta_l V_l}$.
  - Objective function of a variational problem: $\mathcal{E}(\boldsymbol{\theta}) := \langle 0| \mathrm{U}(\boldsymbol{\theta})^\dagger \mathcal{A} \mathrm{U}(\boldsymbol{\theta}) |0\rangle$.
  - We can show that
$$\mathbb{V}\left[\nabla \mathcal{E}(\boldsymbol{\theta})\right] = \frac{\cdots}{N^2 - 1}.$$

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- Q Wasserstein GAN
- Other types of NN?

Quantum Wasserstein GAN

- Classical Wasserstein distance:

$$\min_{\pi \in \Pi(p,q)} \int_{\mathcal{X}} \int_{\mathcal{Y}} \pi(x, y) c(x, y) \mathrm{d}x \mathrm{d}y$$

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- This is only a **semi-metric** (no triangle inequality)

Optimisation, Q annealing, Hamiltonians, noise...

An optimisation problem

$$\textbf{Problem: } \text{Given } f : \{0,1\}^n \to \mathbb{R}, \qquad \min_{z\in\{0,1\}^n} f(z). \tag{2}$$

- Hamiltonian formulation: $\mathcal{H}_F := \sum_{z\in\{0,1\}^n} f(z) |z\rangle \langle z|$.
- If $(|z_i\rangle)$ are eigenvectors of $\mathcal{H}_F$, then

$$\begin{aligned}
\mathcal{H} |z_i\rangle &= \left(\sum_{z\in\{0,1\}^n} f(z) |z\rangle \langle z|\right) |z_i\rangle \\
&= \left(\sum_{z\in\{0,1\}^n\setminus\{z_i\}} f(z) |z\rangle \langle z|\right) |z_i\rangle \quad + \Big(f(z_i) |z_i\rangle \langle z_i| \Big) |z_i\rangle \\
&= 0 \quad + f(z_i) |z_i\rangle \langle z_i|z_i\rangle \\
&= f(z_i) |z_i\rangle,
\end{aligned}$$

so that $(f(z_i))$ are eigenvalues of $\mathcal{H}_F$.
- Solving (2) amounts to finding the smallest eigenvalues (minimum energy) of $\mathcal{H}_F$.
- Problem: it is often difficult to find them.

Constant Hamiltonian simulation

Schrödinger equation (normalised with $\hbar = 1$):

$$i\hbar \frac{d |\psi(t)\rangle}{dt} = \mathcal{H} |\psi(t)\rangle \qquad \text{(Schrödinger equation)}.$$

is solved as

$$|\psi(t)\rangle = e^{-i\mathcal{H}t} |\psi(0)\rangle$$

at time $t \geq 0$. If $\mathcal{H} |\psi_0\rangle = \lambda_0 |\psi_0\rangle$, then

$$|\psi(t)\rangle = e^{-i\mathcal{H}t} |\psi_0\rangle = e^{-i\lambda_0 t} |\psi_0\rangle ,$$

i.e. no transition over time between different eigenstates!!

Time-dependent Hamiltonian simulation $\mathcal{H}(\cdot)$

Schrödinger equation over $[0, \tau]$; time change $t(\cdot)$ with $t(0) = 1$ and $t(1) = \tau$:

$$\mathrm{i}\frac{\mathrm{d}\,|\psi(s)\rangle}{\mathrm{d}s} = t'(s)\mathcal{H}(s)\,|\psi(s)\rangle\,, \qquad \text{on } [0,1]. \tag{3}$$

Consider $\mathcal{H}(s) = r(s)\mathcal{H}_0 + (1 - r(s))\mathcal{H}_F$, for two Hamiltonians $\mathcal{H}_0$ and $\mathcal{H}_F$, where $r(\cdot)$ is a continuous adiabatic evolution path decreasing from $r(0) = 1$ to $r(1) = 0$.
Let $|\psi(\cdot)\rangle$ be the solution to the Schrödinger equation, so that

$$|\psi(s)\rangle = \mathcal{U}(s)\,|\psi(0)\rangle\,, \qquad \text{for some unitary operator } \mathcal{U}.$$

Consider (3) with $t(s) = s\tau$, hence

$$\mathrm{i}\frac{\mathrm{d}\,|\psi(t)\rangle}{\mathrm{d}t} = \tau\mathcal{H}(t)\,|\psi(t)\rangle\,, \qquad \text{on } [0,1].$$

Q Adiabatic Theorem

Let $|\phi(t)\rangle$ be the ground state of $\mathcal{H}_t$ and the adiabatic schedule $r(s) = 1 - s$, so that

$$\mathcal{H}(s) = (1-s)\mathcal{H}_0 + s\mathcal{H}_F.$$

Theorem[...]. If there exists $\delta > 0$ such that

$$\tau \geq \frac{2}{\delta}\left\{c_0 \frac{\|\mathcal{H}_F - \mathcal{H}_0\|}{\overline{\Delta}^2} + \left(3c_1^2 + c_1 + c_3\right)\frac{\|\mathcal{H}_F - \mathcal{H}_0\|^2}{\overline{\Delta}^3}\right\},$$

with $\overline{\Delta} := \min_{s\in[0,1]} \Delta_s$, then, starting the system in the state $|\psi(0)\rangle = |\phi(0)\rangle$, the Schrödinger evolution yields at time 1 a state $|\psi(1)\rangle$ satisfying $\||\phi(1)\rangle - |\psi(1)\rangle\| \leq \delta$.

The 1-bit Disagree problem

The 1-bit Disagree problem reads

$$f(z) := \begin{cases} 1, & \text{if } z = 1, \\ 0, & \text{if } z = 0. \end{cases}$$

$$\mathcal{H}_F := \frac{1+\sigma^z}{2} = \frac{1}{2}\left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right) = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} = |0\rangle\langle 0|,$$

so that

$$\begin{aligned} \mathcal{H}_F|0\rangle &= |0\rangle\langle 0|\,|0\rangle = |0\rangle = 1\cdot|0\rangle, \\ \mathcal{H}_F|1\rangle &= |0\rangle\langle 0|\,|1\rangle = 0 = 0\cdot|1\rangle, \qquad \text{(ground state).} \end{aligned}$$

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Define now

$$\mathcal{H}_0 := \frac{1-\sigma^x}{2} = \frac{1}{2}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix} = \frac{1}{2}\Big(|0\rangle\langle 0| + |1\rangle\langle 1| - |1\rangle\langle 0| - |0\rangle\langle 1|\Big).$$

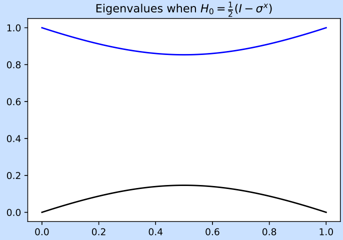
One can check that

$$\begin{aligned} \mathcal{H}_0|+\rangle &= |+\rangle = 1\cdot|+\rangle, \\ \mathcal{H}_0|-\rangle &= 0 = 0\cdot|-\rangle, \qquad \text{(ground state)}. \end{aligned}$$

Interpolating Hamiltonian:

$$\mathcal{H}(t) := (1-t)\,\mathcal{H}_0 + t\mathcal{H}_F, \qquad t \in [0,1].$$

Eigenvalues: $\lambda_{\pm}(t) = \frac{1}{2}\left(1 \pm \sqrt{1-2t(1-t)}\right)$.



The Q adiabatic theorem applies!!

The commuting issue for the 1-bit Disagree problem

Consider instead

$$\mathcal{H}_0 := \frac{1-\sigma^z}{2} = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = |1\rangle\langle 1|,$$

$$\mathcal{H}_0|0\rangle = |1\rangle\langle 1||0\rangle = 0 \qquad \text{and} \qquad \mathcal{H}_0|1\rangle = |1\rangle\langle 1||1\rangle = |1\rangle.$$

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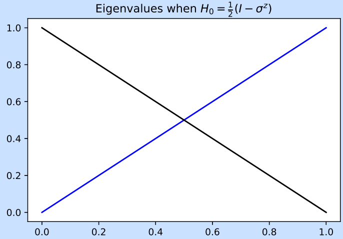
Interpolating Hamiltonian:

$$\mathcal{H}(t) := (1-t)\,\mathcal{H}_0 + t\mathcal{H}_F = \begin{pmatrix} t & 0 \\ 0 & 1-t \end{pmatrix}, \qquad \text{for } t \in [0,1].$$

Eigenvalues: $\lambda(t) \in \{t, 1-t\}$: $\mathcal{H}(t)|0\rangle = t|0\rangle$ and $\mathcal{H}(t)|1\rangle = (1-t)|1\rangle$.

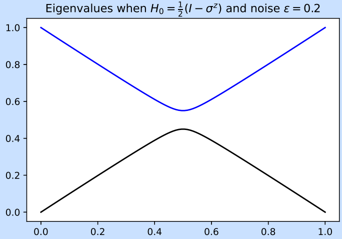


Adding noise.....

Consider a noisy version of the interpolating Hamiltonian:

$$\mathcal{H}^{\varepsilon}(t) := \mathcal{H}(t) + \varepsilon \begin{pmatrix} 0 & t(1-t) \\ t(1-t) & 0 \end{pmatrix} = \begin{pmatrix} t & \varepsilon t(1-t) \\ \varepsilon t(1-t) & 1-t \end{pmatrix}, \qquad \text{for } t \in [0,1].$$

The two eigenvalues (say for $\varepsilon = 0.2$ behave as follows:



And the spectral gap is restored!

The 2-bit Disagree problem

$$f(x) := \begin{cases} 0, & \text{if } x_1 \neq x_2, \\ 1, & \text{otherwise.} \end{cases}$$

$$\mathcal{H}_F := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \frac{1}{2}\left\{\mathtt{I}\otimes\mathtt{I} + (\mathtt{Z}\otimes\mathtt{I})(\mathtt{I}\otimes\mathtt{Z})\right\}$$

with $\mathtt{I}$ the identity matrix in $\mathcal{M}_2(\mathbb{R})$, $\mathtt{Z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\otimes$ the Kronecker product.

Eigenvalues:

$$e_1^F = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \qquad e_2^F = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad e_3^F = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad e_4^F = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

with eigenvalues $0, 0, 1, 1$, so that the ground states are $\{e_1^F, e_2^F\}$.

- Initial Hamiltonian:

$$\mathcal{H}_0 = \frac{1}{2}\left\{(\mathtt{I}\otimes\mathtt{I} - \mathtt{X}\otimes\mathtt{I}) + (\mathtt{I}\otimes\mathtt{I} - \mathtt{I}\otimes\mathtt{X})\right\} = \frac{1}{2}\begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix};$$

- Eigenvalues $\{0,1,1,2\}$ and ground state $e_1^0 = (1,1,1,1)^\top = 2\,|{+}{+}\rangle$;
- Take $\mathcal{H}_t := (1-r(t))\mathcal{H}_0 + r(t)\mathcal{H}_F$;
- Apply the Q Adiabatic theorem;

Questions

- How to find $\mathcal{H}_0$ in general? Idea: PQC.
- Reality has noise: $\mathcal{H}_t \longrightarrow \mathcal{H}_t^\varepsilon$ for all $t\in(0,1)$ (or noise-induced algorithm);
- Question: understand $\mathcal{H}^\varepsilon$ (pathwise) as $\varepsilon \downarrow 0$.

Quantum Monte Carlo

Classical Monte Carlo

X: random variable, with $\mu := \mathbb{E}[\nu(X)]$ and $\sigma^2 := \mathbb{V}[\nu(X)]$.
(Φ: given nice enough map, both μ and σ^2 are finite).

$$\widehat{\mu}_N := \frac{1}{N}\sum_{i=1}^{N} X_i.$$

- *Law of large numbers*: $\widehat{\mu}_N$ converges to μ almost surely as $N \uparrow \infty$;
- *Central Limit Theorem*:

$$\lim_{N\uparrow\infty} \frac{\widehat{\mu}_N - \mu}{\sigma/\sqrt{N}} = \mathcal{N}(0,1), \quad \text{in distribution.}$$

This implies that

$$\mathbb{P}\left(|\widehat{\mu}_N - \mu| \leq \varepsilon\right) = \mathbb{P}\left(\left|\frac{\widehat{\mu}_N - \mu}{\sigma/\sqrt{N}}\right| \leq \frac{\varepsilon\sqrt{N}}{\sigma}\right) = \mathbb{P}\left(|\mathcal{N}(0,1)| \leq \frac{\varepsilon\sqrt{N}}{\sigma}\right).$$

If we want $\mathbb{P}(|\mathcal{N}(0,1)| \geq z) = 1 - \delta$, we require $z = \frac{\varepsilon\sqrt{N}}{\sigma}$, i.e. $N = \mathcal{O}\left(\frac{1}{\varepsilon^2}\right)$.

One may replace σ^2 by its unbiased estimator $s^2 := \frac{1}{N-1}\sum_{i=1}^{N}(X_i - \widehat{\mu}_N)^2$.

Amplitude estimation

[Brassard, Høyer, Mosca, Tapp, 2002] and [Montanaro, 2015]

- Inputs:
  - a quantum state $|\psi\rangle$ and a projector $\mathtt{P}$;
  - Unitary $\mathrm{U} := 2|\psi\rangle\langle\psi| - \mathtt{I}$ and $\mathtt{V} := \mathtt{I} - 2\mathtt{P}$;
  - $\mathtt{N} \in \mathbb{N}$
- Output: Estimate $\widehat{\mathbf{a}}$ of $\mathbf{a} = \langle\psi|\mathtt{P}|\psi\rangle$ such that

$$|\widehat{\mathbf{a}} - \mathbf{a}| \leq 2\pi \frac{\sqrt{\mathbf{a}(1-\mathbf{a})}}{\mathtt{N}} + \frac{\pi^2}{\mathtt{N}^2},$$

  with probability at least $\frac{8}{\pi^2}$, using U and $\mathtt{V}$, $\mathtt{N}$ times each.

Note: the probability can be improved to $1-\delta$ (for any $\delta > 0$) using the *Powering Lemma*, at the cost of a $\mathcal{O}(\log(1/\delta))$ multiplicative factor.
Fix $\varepsilon > 0$ and let $\mathtt{N} := \frac{2\pi}{\varepsilon\sqrt{\mathbf{a}}}$. Then (for $|\mathbf{a}| < 1$)

$$|\widehat{\mathbf{a}} - \langle\psi|\mathtt{P}|\psi\rangle| \leq \mathbf{a}\sqrt{1-\mathbf{a}}\varepsilon + \frac{\mathbf{a}}{4}\varepsilon^2 \leq \varepsilon\mathbf{a},$$

Powering Lemma

[Jerrum, Valiant, Vazirani, 1986]
Let $\mathcal{A}$ be a (quantum or classical) algorithm aiming at estimating μ and whose output satisfies

$$|\widehat{\mu} - \mu| \leq \varepsilon,$$

except with probability less than $\frac{1}{2}$.
Then, for any $\delta > 0$, it suffices to repeat $\mathcal{A}$ $\log(1/\delta)$ times and take the median to obtain $\widehat{\mu}$ with

$$|\widehat{\mu} - \mu| < \varepsilon,$$

with probability at least $1 - \delta$.

Quantum Monte Carlo [Montanaro, 2015]

Algorithm

- Inputs:
  - Algorithm $\mathcal{A}$ with random output $\nu(\mathcal{A}) \in [0,1]$; $\mathtt{N} \in \mathbb{N}$; $\delta > 0$; n qubits;
  - $k < n$ qubits are measured. The outcome of the measurement of $x \in \{0,1\}^k$ is mapped into $\nu(x) \in [0,1]$;
  - $\mathtt{W} |x\rangle_k |0\rangle := |x\rangle_k \left(\sqrt{1-\nu(x)}\,|0\rangle + \sqrt{\nu(x)}\,|1\rangle\right)$;
- Steps:
  - Apply $\mathtt{N}$ iterations of Amplitude Estimation with

$$|\psi\rangle := (\mathtt{I} \otimes \mathtt{W})(\mathcal{A} \otimes \mathtt{I})\,|0\rangle^{\otimes(n+1)} \qquad \text{and} \qquad \mathtt{P} := \mathtt{I} \otimes |1\rangle\langle 1|\,. \tag{4}$$

  - Repeat (4) $\mathcal{O}(\log(1/\delta))$ times and output the median.

Theorem

The algorithm outputs $\widetilde{\mu}$ such that, with probability at least $1-\delta$,

$$|\widetilde{\mu} - \mathbb{E}[\nu(\mathcal{A})]| \leq C\left(\frac{\sqrt{\mathbb{E}[\nu(\mathcal{A})]}}{\mathtt{N}} + \frac{1}{\mathtt{N}^2}\right),$$

To get $|\widetilde{\mu} - \mathbb{E}[\nu(\mathcal{A})]| \leq \varepsilon$, one then needs $\mathtt{N} = \mathcal{O}(1/\varepsilon)$

Cost of the QMC algorithm

- The circuit U is used $\mathcal{O}(\mathtt{N}\log(1/\delta))$ times:
  - $\mathtt{N}$ times for Quantum Amplitude Estimation;
  - $\log(1/\delta)$ times for the Powering Lemma;

Refinements:

- output bounded in l^2 [Montanaro, 2015];
- output with bounded variance;
- multilevel....
- variance reduction....

Application to option pricing

Goal: $\Pi := \mathbb{E}[\nu(W_T)]$, for some Brownian motion W.

- Discretise (*quantisation*) the support $\mathbb{R} \to [\underline{w}, \overline{w}]$ with 2^n points, and assume that

$$\mathcal{A} |0\rangle^{\otimes n} = \sum_{j=0}^{2^n-1} \sqrt{p_j} |j\rangle, \qquad \text{with } p_j := \frac{\mathbb{P}(w_j)}{\sum_k \mathbb{P}(w_k)},$$

  and we identify w_j with $|j\rangle$.
- In particular, take $\nu(w) = \left(S_0 \exp\left\{\sigma w - \frac{\sigma^2 T}{2}\right\} - K\right)_+$

$$\mathtt{B} : |j\rangle |0\rangle \mapsto |j\rangle |\widehat{\nu}_j\rangle, \qquad \widehat{\nu}_j : \text{binary approximation of } \nu(w_j).$$

- $\mathtt{W} |j\rangle |\widehat{\nu}_j\rangle \mapsto |j\rangle |\widehat{\nu}_j\rangle \left(\sqrt{1-\widehat{\nu}_j} |0\rangle + \sqrt{\widehat{\nu}_j} |1\rangle\right)$ (as in QMC)
- Inverting $\mathtt{B}$ yields $|j\rangle |0\rangle^{\otimes n} \left(\sqrt{1-\widehat{\nu}_j} |0\rangle + \sqrt{\widehat{\nu}_j} |1\rangle\right)$.
- Ignoring $|0\rangle^{\otimes n}$, we can now use QMC to obtain an estimate of $\mathbb{E}[\nu(W_T)]$.

Quantum simulation (different...)

Schrödinger: the evolution of a quantum system satisfies (ignoring Planck):

$$\mathrm{i}\partial_t |\psi(t)\rangle = \mathcal{H} |\psi(t)\rangle, \qquad |\psi(0)\rangle \in \dots$$

with solution $|\psi(t)\rangle = \mathrm{e}^{-\mathrm{i}\mathcal{H}t} |\psi(0)\rangle$. The Hamiltonian $\mathcal{H}$ is usually large and $\mathrm{e}^{-\mathrm{i}\mathcal{H}t}$ is hard to compute. First-order approximation $\mathrm{e}^{-\mathrm{i}\mathcal{H}t} \approx 1 - \mathrm{i}\mathcal{H}t$ unsatisfactory.

Assumptions

- $\mathcal{H} = \sum_{l=1}^{L} \mathcal{H}_l$, where each $\mathcal{H}_l$ acts on a 'small' subsystem (such that $\mathrm{e}^{-\mathrm{i}\mathcal{H}_l t}$ is easy to compute); note that $\mathcal{H}_l$ and $\mathcal{H}_k$ do not commute, but $\mathrm{e}^{-\mathrm{i}\mathcal{H}t}$ can be approximated with the Suzuki-Lie-Trotter formula.
- $T = m\delta$ (m represents the number of time steps in the Suzuki-Lie-Trotter discretisation);
- Measurement operator $\mathtt{M}$ and $\mu := \mathbb{E}[\mathtt{M}] = \mathrm{Tr}(\mathtt{M}\rho)$, with $\rho = |\psi\rangle\langle\psi|$;
- $\widehat{\mu} := \frac{1}{\mathtt{N}} \sum_{j=1}^{\mathtt{N}} X_j$;

Theorem [Wang, 2011] There exist $C_1, C_2 > 0$ such that, for all n, m,

$$\mathbb{E}\left[(\widehat{\mu} - \mu)^2\right] \leq \frac{C_1}{\mathtt{N}} + \frac{C_2}{m^4}.$$

Q for PDEs

Option Pricing in the Black-Scholes model

- Black-Scholes SDE:

$$\frac{\mathrm{d}S_t}{S_t} = r\,\mathrm{d}t + \sigma\,\mathrm{d}W_t, \qquad \text{for } t \geq 0.$$

- European Call option with payoff $V(T, S_T) = \max(S_T - K, 0)$

- Feynman-Kac:

$$\left(\partial_t + \frac{\sigma^2 s^2}{2}\partial_{ss} + rS\partial_s - r\right) V(t,s) = 0, \qquad \text{for } s > 0,\ t \in [0, T),$$

with terminal condition $V(T, s)$. This is equivalent to the heat equation

$$\partial_\tau u(\tau, x) = \frac{1}{2}\partial_{xx} u(\tau, x),$$

where the boundary condition is now at time zero ($\tau = \sigma^2(T - t)$).

From Black-Scholes to Schrödinger

- The Wick rotation $\xi = -\mathrm{i}\tau$ turns the heat PDE into $-\mathrm{i}\partial_{\xi} u(\xi, x) = \frac{\partial_{xx} u(\xi,x)}{2}$, or

$$-\mathrm{i}\partial_{\xi} |\psi\rangle = \mathcal{H} |\psi\rangle \qquad \text{(Schrödinger equation)},$$

  where $|\psi\rangle$ plays the role of the $u(\cdot,\cdot)$, and $\mathcal{H} = \frac{1}{2}\partial_{xx}$.

- **Explicit solution:**

$$|\psi(\xi)\rangle = \mathrm{e}^{\mathrm{i}\mathcal{H}\xi} |\psi(0)\rangle,$$

  where $\mathrm{e}^{\mathrm{i}\mathcal{H}\xi}$ is the time evolution operator and $|\psi(0)\rangle$ an initial state with $\langle\psi(0)|\psi(0)\rangle = 1$.

Possible algorithms:

- HHL algorithm: to solve (high-dimensional) linear systems;
- Variational algorithms: Zhao, Sun, Cohen, Stokes, Veerapaneni [2022], Fontanela, Jacquier, Oumgari [2021]

A hybrid quantum algorithm

- Problem: normalised imaginary time evolution

$$|\psi(\tau)\rangle = \gamma(\tau)\, \mathrm{e}^{-\mathcal{H}\tau} |\psi(0)\rangle .$$

- Approximate $|\psi(\tau)\rangle$ by a Q circuit composed of parameterised gates such that $|\psi(\tau)\rangle \approx |\phi(\boldsymbol{\theta}_\tau)\rangle$, for some time-dependent parameters $\boldsymbol{\theta}_\tau = (\theta_\tau^1, \cdots, \theta_\tau^N) \in \mathbb{R}^N$.

- Assuming an initial state $|\psi_0\rangle$, so that the ansatz is $|\phi(\tau)\rangle = \Phi(\boldsymbol{\theta}_\tau)|\psi_0\rangle$ at time τ, where $\Phi(\boldsymbol{\theta}_\tau)$ is sequence of unitary gates $\Phi(\boldsymbol{\theta}_\tau) = \mathcal{S}\left(\mathrm{U}_N(\theta_\tau^N), \ldots, \mathrm{U}_k(\theta_\tau^k), \ldots, \mathrm{U}_1(\theta_\tau^1)\right)$.

$$\boldsymbol{\theta}_\tau^* := \operatorname*{arg\,min}_{\boldsymbol{\theta}\in\mathbb{R}^N} \left\| |\psi(\tau)\rangle - \Phi(\boldsymbol{\theta}_\tau)|\psi_0\rangle \right\| .$$

At time τ

The optimisation problem reduces to the system of ODEs

$$\mathrm{A}(\tau)\dot{\boldsymbol{\theta}}_\tau = \mathrm{C}(\tau),$$

for all τ, where $\dot{\boldsymbol{\theta}}_\tau := \partial_\tau \boldsymbol{\theta}_\tau$, and

$$\mathrm{A}(\tau) = \left(\Re\left(\frac{\partial \langle\phi(\tau)|}{\partial\theta^i}\frac{\partial |\phi(\tau)\rangle}{\partial\theta^j}\right)\right)_{i,j=1,\ldots,N}, \qquad \mathrm{C}(\tau) = \left(\Re\left(\frac{\partial \langle\phi(\tau)|}{\partial\theta^i}\mathcal{H}\,|\phi(\tau)\rangle\right)\right)_{i=1,\ldots,N}.$$

In this setting, both A and C can be measured efficiently using a quantum computer. In order to build the hybrid classical-quantum scheme, we assume:

- (i) Every unitary gate in the algorithm depends on a single parameter.
- (ii) $\mathcal{H} = \sum_{i=1}^N \lambda_i h_i$, for $\lambda \in \mathbb{R}^N$ and tensor products h_i of Pauli matrices.

Simulation from τ to Δ_τ

- Once $\mathrm{A}(\tau)$ and $\mathrm{C}(\tau)$ are obtained, the time evolution can be computed numerically using a classical computer.
- Euler scheme:

$$\boldsymbol{\theta}_{\tau+\Delta_\tau} = \boldsymbol{\theta}_\tau + \Delta_\tau \dot{\boldsymbol{\theta}}_\tau = \boldsymbol{\theta}_\tau + \Delta_\tau \mathrm{A}(\tau)^{-1}\mathrm{C}(\tau), \tag{5}$$

  for some small time step Δ_τ.
- ... and so on until time $\tau = T$.

European Call option

- **Model:** Black-Scholes $\mathrm{d}S_t = \sigma S_t \mathrm{d}W_t$, with $\sigma = 20\%$, $S_0 = K = 100$, $T = 1$.
- **Goal:** Compute $\mathbb{E}[\max(S_T - K, 0)]$.
- Discretise the state space on logarithmic scale on an equidistant grid $[S_{\min}, S_{\max}] = [50, 150]$.
- With four qubits, the discretisation represents $|\psi\rangle$ using $2^4 = 16$ points, where $|\psi_{\mathrm{F}}\rangle = |0000\rangle$ and $|\psi_{\mathrm{F}}\rangle = |1111\rangle$ represent the solution at $S_{\min}$ and $S_{\max}$.
- The Hamiltonian $\mathcal{H} = \frac{1}{2}\partial_{xx}$ is discretised by second-order finite differences

$$\frac{1}{2\Delta_x^2}\begin{bmatrix} -2b\Delta_x^2 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -2b\Delta_x^2 \end{bmatrix},$$

  where Δ_x is the discretisation step in space.
- We split $[0, T]$ into n_T steps.
- We compute A and C as above.
- The evolution of $\boldsymbol{\theta}_\tau$ is obtained from the Euler scheme.

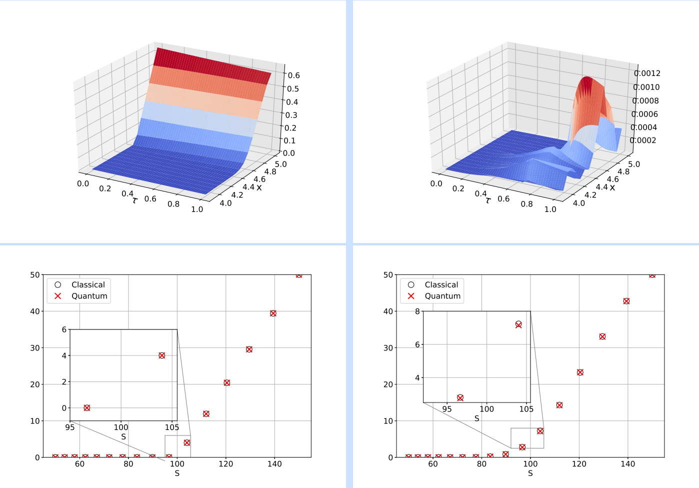


Figure: Top: European prices (left) and errors (right) $\| |\psi(\tau)\rangle - |\phi(\boldsymbol{\theta}_\tau)\rangle \|$. Bottom: Comparison with closed-form formula at maturity (left) and at inception (right).

Quantum Physics Inspired Neural Networks

(In progress, with J. Dees and S. Laizet)

Problem: Find u solution to

$$\mathcal{L}u(x) = 0, \qquad x \in \Omega.$$

Given: samples $(x_i, u(x_i))$.

- Classical NN: $u(x) \approx u_{NN}(x, \Theta)$.

$$\min_{\theta} \frac{1}{N} \sum_{i=1}^{N} \Big[u(x_i) - u_{NN}(x_i, \Theta)\Big]^2.$$

  Usually not very efficient at generalising outside training sample.
- PINN: $u(x) \approx u_{NN}(x, \Theta)$.

$$\min_{\theta} \frac{1}{N} \sum_{i=1}^{N} \Big[u(x_i) - u_{NN}(x_i, \Theta)\Big]^2 + \frac{1}{M} \sum_{j=1}^{M} \mathcal{L}u_{NN}(x_j, \Theta)^2.$$

- QPINN: $u(x) \approx u_{\mathrm{Q}}(x, \Theta)$.

Wrapping up

IBM roadmap

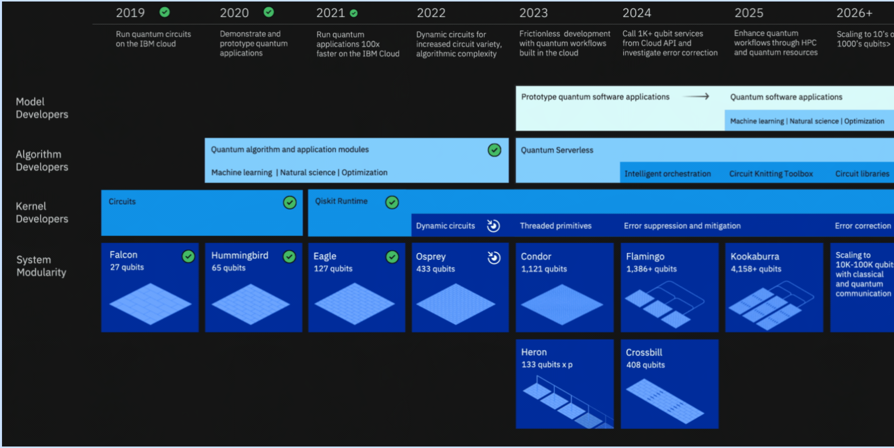


Figure: Source: ibm.com

Take-away message

- Initial entering cost (new language, new culture)
- New tools and methods, to learn
- Get *inspired*
- More maths are needed: numerical analysis, stochastic
- Q hardware is advancing fast
- Future: Hybrid Q / Classical
- Q solvers for (non-?) linear systems
- Q Monte Carlo

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