When to efficiently rebalance a portfolio

Masaaki Fukasawa

Osaka University

April, 2023

A joint work with Masayuki Ando

Continuous-time finance

Denote by $S = (S^1, \ldots, S^d)$ the price process vector of d risky assets. Under a risk-free rate r, the wealth process V for an investment strategy $H = (H^1, \ldots, H^d)$ and a consumption plan c is described as

$$\mathrm{d}V_t = \sum_{i=0}^{d} H_t^i \mathrm{d}S_t^i - c_t \mathrm{d}t,$$

where

$$\frac{\mathrm{d}S_t^0}{S_t^0} = r_t \mathrm{d}t, \quad H^0 = \frac{1}{S^0}\left(V - \sum_{i=1}^{d} H^i S^i\right).$$

Under an admissibility condition $V > 0$, this can be rewritten as

$$\frac{\mathrm{d}V_t}{V_t} = \sum_{i=0}^{d} \pi_t^i \frac{\mathrm{d}S_t^i}{S_t^i} - \frac{c_t}{V_t}\mathrm{d}t, \quad \sum_{i=0}^{d} \pi^i = 1$$

in terms of $\pi^i = H^i S^i / V$

A constant weight asset allocation

Here, $\pi^i = H^i S^i / V$ represents the fraction of the wealth invested in the ith asset.

In this study we will assume $\pi = (\pi^1, \dots, \pi^d)$ is continuous and of finite variation. This includes the case it is a constant, that is, a constant weight asset allocation.

- Kelly criterion
- CRRA utility maximization (Merton strategy)
- Epstein-Zin utility maximization
- Relative performance criteria
- $1/N$ strategy

A concrete example

Assume

$$\frac{\mathrm{d}S_t^i}{S_t^i} = \mu_t^i \mathrm{d}t + \sigma_t^i \mathrm{d}W_t^i, \quad \sigma_t^i \sigma_t^j \mathrm{d}\langle W^i, W^j \rangle_t = \Sigma_t^{ij} \mathrm{d}t$$

for adapted processes μ^i and Σ^{ij} with $\Sigma = [\Sigma^{ij}]$ regular. Under

$$\frac{\mathrm{d}V_t}{V_t} = \sum_{i=0}^{d} \pi_t^i \frac{\mathrm{d}S_t^i}{S_t^i},$$

Itô's formula gives

$$\mathsf{E}\left[\log \frac{V_T}{V_0}\right] = \int_0^T \mathsf{E}\left[-\frac{1}{2}(\pi_t - \theta_t)^\top \Sigma_t (\pi_t - \theta_t) + \frac{1}{2}\theta_t^\top \Sigma_t \theta_t\right] \mathrm{d}t,$$

where

$$\theta := \Sigma^{-1}(\mu - r\mathbf{1}), \quad \mu = (\mu^1, \dots, \mu^d)^\top, \quad \mathbf{1} = (1, \dots, 1)^\top.$$

The growth optimal portfolio is therefore $\pi = \theta$.

Objective

Notice $H^i = V\pi^i/S^i$ is not of finite variation, needs to be approximated by a simple predictable process $H^{n,i}$. Denote by

$$Z_t^n = V_t - V_t^n = \sum_{i=0}^{n} \int_0^t (H_u^i - H_u^{n,i})\mathrm{d}S_u^i$$

the tracking error process,. We want to minimize

$$\mathrm{E}[\langle Z^n \rangle_T] = \sum_{i,j=1}^{d} \mathrm{E}\left[\int_0^T (H_u^i - H_u^{n,i})(H_u^j - H_u^{n,j})\mathrm{d}\langle S^i, S^j \rangle_u\right]$$

under a cost constraint

$$\mathrm{E}[C_T^n] \leq c_n,$$

where C^n is the counting process of the jumps of the d dimensional simple predictable process $(H^{n,1}, \ldots, H^{n,d})$.

Note on the objective

On the tracking error,

- $\mathsf{E}[\langle Z^n \rangle_T] = \mathsf{E}[|Z^n_T|^2]$ under any martingale measure.
- $\mathsf{E}[\langle Z^n \rangle_T]$ is time consistent, while $\mathsf{E}[|Z^n_T|^2]$ is not in general.
- it is the cumulative local mean squared error (c.f. local mean variance criteria):

$$\sum_{i=1}^{N} \mathsf{E}[|Z^n_{t_i} - Z^n_{t_{i-1}}|^2] \approx \mathsf{E}[\langle Z^n \rangle_T],$$

  where $0 = t_0 < \cdots < t_N = T$.

The cost is

- the expected number of transactions.
- the expected cumulative fixed transaction costs.

Variational inequality approach under the Black-Scholes model:

- Martini and Patry (1999) for mean squared error

Fixed transaction costs under the Black-Scholes model

- Schroder (1995), Whalley-Wilmott (1997), Korn (1998), Liu(2004), Lo et al. (2004) : for CARA utility
- Morton and Pliska (1995), Atkinson and Wilmott (1995) : growth optimal portfolio under quasi-fixed transaction costs
- Altrarovici, Muhle-Karbe and Soner (2015): asymptotics for CRRA; No-trade region is defined as

$$\left\{\pi; (\pi - \pi^*)^\top M(\pi - \pi^*) \leq h\right\},$$

where π^* is the frictionless Merton proportion and M is the solution of *"a matrix valued algebraic Riccati equation"*. Asymptoticaly optimal is *"the strategy that consumes at the frictionless Merton rate, does not trade while the current position lies in the above no-trade region, and jumps to the frictionless Merton proportion once its boundaries are breached."*

High-frequency asymptotic approach: one-dimensional case

Fukasawa (2011, 2014): For general one-dimensional continuous semimartingales H and S (corresponding to the case $d = 1$),

$$\varliminf_{n\to\infty} \mathsf{E}[C^n_T]\mathsf{E}[\langle Z^n\rangle_T] \geq \frac{1}{6}\mathsf{E}[\int_0^T K_t \mathrm{d}\langle H\rangle_t]^2$$

for "any" sequence of simple predictable processes with $\mathsf{E}[C^n_T] \to \infty$, under the condition

$$\mathrm{d}\langle S\rangle = K_t^2 \mathrm{d}\langle H\rangle_t$$

with continuous K (without assuming π is of finite variation). Equivalently,

$$\mathsf{E}[\langle Z^n\rangle_T] \geq \frac{1}{6n}\mathsf{E}[\int_0^T K_t \mathrm{d}\langle H\rangle_t]^2 + o(\frac{1}{n}), \quad n \to \infty$$

under $\mathsf{E}[C^n_T] \leq n$. The lower bound is attained by discretizations of H with

$$\tau^n_{j+1} = \inf\left\{t > \tau^n_j; K_{\tau^n_j}|H_t - H_{\tau^n_j}|^2 = h_n\right\}, \quad h_n \to 0.$$

Related works: Fukasawa (2011), Cai et al. (2016, 2018).

Multi-dimensional case

Gobet and Landon (2014), Gobet and Stazhynski (2018): extension to d dimensional cases when the strategy is of the form $H^i_t = f^i(t, S_t, R_t)$ with R of finite variation.

- in the context of hedging
- approximations by discretizations

Our contribution:

- investment strategies
- optimality among simple predictable strategies

Our structure condition

The strategy $H = (H^1, \dots, H^d)$ has a continuous drift and there exist regular matrix valued continuous adapted processes J and K such that

$$\mathrm{d}\langle H, H\rangle_t = J_t \mathrm{d}t, \quad \mathrm{d}\langle S, S\rangle_t = K_t^\top J_t K_t \mathrm{d}t.$$

On the structure condition

Theorem A

Let π^0 and $\pi = (\pi^1, \dots, \pi^d)$ be continuous adapted processes of finite variation with $\sum_{i=0}^d \pi^i = 1$ and assume

$$\frac{\mathrm{d}V_t}{V_t} = \sum_{i=0}^{d} \pi_t^i \frac{\mathrm{d}S_t^i}{S_t^i} - c_t \mathrm{d}t,$$

$$\mathrm{d}\langle \log S^i, \log S^j \rangle_t = \Sigma_t^{ij} \mathrm{d}t, \quad \mathrm{d} \log S_t^0 = r_t \mathrm{d}t$$

for continuous adapted processes c, r, μ^i (the drift of S^i) and Σ^{ij}.
If $\Sigma_t = [\Sigma_t^{ij}]$ is regular and $\pi_t^i \neq 0$ for all t and $i \geq 0$, then the structure condition is met with

$$J = G^\top G, \;\; G = \Sigma^{1/2}(H^\top \pi - \mathrm{diag}(H)), \;\; K = J^{-1/2}\Sigma^{1/2}\mathrm{diag}(S)$$

for $H = (H^1, \dots, H^d)$, $H^i = V\pi^i / S^i$.

Admissible sequence

For a given strategy $H = (H^1, \dots, H^d)$ with the structure condition, we say a sequence of d dimensional simple predictable processes

$$(H^{n,1}, \dots, H^{n,d}), \quad H^{n,i}_t = \sum_{j=0}^{\infty} H^{n,i}_j 1_{(\tau^n_j, \tau^n_{j+1}]}(t),$$

where $\{\tau^n_j\}$ is a sequence of $\{\mathcal{F}_t\}$-stopping times and $H^{n,i}_j$ is $\mathcal{F}_{\tau^n_j}$-measurable, is admissible for H if

1. $\sup_{t \in [0,T]} |H^i_t - H^{n,i}_t| \to 0$ in probability for each $i = 1, \dots, d$, and
2. $\mathrm{E}[C^n_T]\langle Z^n \rangle_T$ is uniformly integrable, where $C^n_T = \max\{j; \tau^n_j \le T\}$ and

$$\langle Z^n \rangle_T = \sum_{i,j=1}^{d} \int_0^T (H^i_u - H^{n,i}_u)(H^j_u - H^{n,j}_u) \mathrm{d}\langle S^i, S^j \rangle_u.$$

We do NOT assume a priori that $H^{n,i}_j = H^i_{\tau^n_j}$.

Main result

Theorem B

Under the structure condition, for any admissible sequence,

$$\varliminf_{n\to\infty} \mathsf{E}[C^n_T]\mathsf{E}[\langle Z^n\rangle_T] \geq \mathsf{E}\left[\int_0^T \mathrm{Tr}\left(U_t J_t\right)\mathrm{d}t\right]^2,$$

where $U_t = u(J_t, K_t)$ with a map u defined later. Further, if $H^{n,i}_j = H^i_{\tau^n_j}$ with

$$\tau^n_{j+1} = \inf\left\{t > \tau^n_j; \left(H_t - H_{\tau^n_j}\right)^\top U_{\tau^n_j}\left(H_t - H_{\tau^n_j}\right) \geq h_n\right\},$$

then there exists a sequence of stopping times $\sigma^m \uparrow T$ such that

$$\lim_{n\to\infty} \mathsf{E}[C^n_{\sigma^m}]\mathsf{E}[\langle Z^n\rangle_{\sigma^m}] = \mathsf{E}\left[\int_0^{\sigma^m} \mathrm{Tr}\left(U_t J_t\right)\mathrm{d}t\right]^2.$$

A matrix-valued algebraic Riccati equation

- Atkinson and Wilmott (1995)
- Gobet and Landon (2014)
- Altarovici, Muhle-Karbe and Soner (2015)

An extension of Lemma from Gobet and Landon (2014):

Lemma

Denote by $\mathcal{M}_d$ and $\mathcal{S}_d$ respectively the sets of $d \times d$ matrices and positive definite matrices. For any $J \in \mathcal{S}_d$ and $K \in \mathcal{M}_d$, there exists a unique nonnegative definite matrix $U = u(J, K)$ such that

$$2\mathrm{Tr}(JU)U + 4UJU = K^\top JK$$

and the map u is continuous on $\mathcal{M}_d \times \mathcal{S}_d$. Further, if $K^\top JK \in \mathcal{S}_d$, then $U = u(J, K) \in \mathcal{S}_d$.

Towards the proof: one dimensional case revisited

When $d = 1$,

$$\langle Z^n \rangle_T = \sum_{j \geq 0} \int_{\tau_j^n \wedge T}^{\tau_{j+1}^n \wedge T} (H_t - H_j^n)^2 K_t^2 \mathrm{d}\langle H \rangle_t$$

and by Itô's formula,

$$\int_{\tau_j^n}^{\tau_{j+1}^n} (H_t - H_j^n)^2 K_{\tau_j^n}^2 \mathrm{d}\langle H \rangle_t = \frac{1}{6} K_{\tau_j^n}^2 \left((H_{\tau_{j+1}^n} - H_j^n)^4 - (H_{\tau_j^n} - H_j^n)^4 \right) - \frac{2}{3} K_{\tau_j^n}^2 \int_{\tau_j^n}^{\tau_{j+1}^n} (H_t - H_j^n)^3 \mathrm{d}H_t.$$

We have

$$\mathsf{E}[\langle Z^n \rangle_T] \approx \frac{1}{6} \mathsf{E}\left[\sum K_{\tau_j^n}^2 \mathsf{E}[((\Delta_j + \delta_j)^4 - \delta_j^4)) | \mathcal{F}_{\tau_j^n}] \right]$$

for $\Delta_j = H_{\tau_{j+1}^n} - H_{\tau_j^n}$ and $\delta_j = H_{\tau_j^n} - H_j^n$.

Pearson's inequality

Let Δ be a centered random variable. Then,

$$\mathrm{E}[\Delta^4]\mathrm{E}[\Delta^2] \geq \mathrm{E}[\Delta^3]^2 + \mathrm{E}[\Delta^2]^3.$$

One way to prove this is to notice that for any (a, b),

$$\begin{aligned} 0 &\leq \mathrm{E}[(a(\Delta^2 - \mathrm{E}[\Delta^2]) + b\Delta)^2] \\ &= (a, b) \begin{pmatrix} \mathrm{E}[\Delta^4] - \mathrm{E}[\Delta^2]^2 & \mathrm{E}[\Delta^3] \\ \mathrm{E}[\Delta^3] & \mathrm{E}[\Delta^2] \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}. \end{aligned}$$

To apply this to our problem(assuming $\mathrm{E}[\Delta_j|\mathcal{F}_{\tau_j^n}] = 0$), note

$$\mathrm{E}[((\Delta_j + \delta_j)^4 - \delta_j^4))|\mathcal{F}_{\tau_j^n}] = \mathrm{E}[\Delta_j^4|\mathcal{F}_{\tau_j^n}] + 4\delta_j\mathrm{E}[\Delta_j^3|\mathcal{F}_{\tau_j^n}] + 6\delta_j^2\mathrm{E}[\Delta_j^2|\mathcal{F}_{\tau_j^n}]$$

$$= 6\mathrm{E}[\Delta_j^2|\mathcal{F}_{\tau_j^n}]\left(\delta_j + \frac{1}{3}\frac{\mathrm{E}[\Delta_j^3|\mathcal{F}_{\tau_j^n}]}{\mathrm{E}[\Delta_j^2|\mathcal{F}_{\tau_j^n}]}\right)^2 + \mathrm{E}[\Delta_j^4|\mathcal{F}_{\tau_j^n}] - \frac{2}{3}\frac{\mathrm{E}[\Delta_j^3|\mathcal{F}_{\tau_j^n}]^2}{\mathrm{E}[\Delta_j^2|\mathcal{F}_{\tau_j^n}]}.$$

Multidimensional case

For $U = u(J, K)$, the solution of the matrix-valued algebraic Riccati equation

$$2\mathrm{Tr}(JU)U + 4UJU = K^\top JK,$$

by Itô's formula applied to the C^2 function $f(x) = (x^\top Ux)^2$, we have

$$\begin{aligned} \mathsf{E}[\langle Z^n \rangle_T] &= \sum_{i,j=1}^{d} \mathsf{E}[\int_0^T (H^i_t - H^{n,i}_t)(H^j_t - H^{n,j}_t) K^i_t K^j_t \mathrm{d}\langle H^i, H^j \rangle_t] \\ &\approx \mathsf{E}\left[\sum \left((\Delta_j + \delta_j)^\top (\Delta_j + \delta_j)\right)^2 - \left(\delta_j^\top \delta_j\right)^2\right] \end{aligned}$$

for

$$\begin{aligned} \Delta_j &= U^{1/2}_{\tau^n_j}(H^1_{\tau^n_{j+1}} - H^1_{\tau^n_j}, \dots, H^d_{\tau^n_{j+1}} - H^d_{\tau^n_j})^\top, \\ \delta_j &= U^{1/2}_{\tau^n_j}(H^1_{\tau^n_j} - H^{n,1}_j, \dots, H^d_{\tau^n_j} - H^{n,d}_j)^\top. \end{aligned}$$

Let Δ be a d-dimensional L^4 random variable with $\mathrm{E}[\Delta]=0$ and $\delta \in \mathbb{R}^d$.

$$
\begin{aligned}
&\mathrm{E}[((\Delta+\delta)^T(\Delta+\delta))^2]-(\delta^T\delta)^2 \\
&= \mathrm{E}[(\Delta^T\Delta+2\delta^T\Delta+\delta^T\delta)^2]-(\delta^T\delta)^2 \\
&= \mathrm{E}[(\Delta^T\Delta)^2]+4\delta^T\mathrm{E}[\Delta\Delta^T]\delta+4\mathrm{E}[\delta^T\Delta(\Delta^T\Delta)]+2\delta^T\delta\mathrm{E}[\Delta^T\Delta].
\end{aligned}
$$

Taking the gradient with respect to δ,

$$
2(4\mathrm{E}[\Delta\Delta^T])+2\mathrm{E}[\Delta^T\Delta])\delta+4\mathrm{E}[\Delta(\Delta^T\Delta)]
$$

and so, the minimum is attained at

$$
\delta=-(2\mathrm{E}[\Delta\Delta^T]+\mathrm{E}[\Delta^T\Delta]I)^{-1}\mathrm{E}[\Delta(\Delta^T\Delta)].
$$

Substitute this to get

$$
\begin{aligned}
&\mathrm{E}[((\Delta+\delta)^T(\Delta+\delta))^2]-(\delta^T\delta)^2 \\
&\geq \mathrm{E}[(\Delta^T\Delta)^2]-\mathrm{E}[\Delta(\Delta^T\Delta)]\left(\mathrm{E}[\Delta\Delta^T]+\frac{1}{2}\mathrm{E}[\Delta^T\Delta]I\right)^{-1}\mathrm{E}[\Delta(\Delta^T\Delta)].
\end{aligned}
$$

A multi-dimensional extension of Pearson's inequality

Theorem C

Let Δ be a d-dimensional L^4 random variable with $\mathrm{E}[\Delta] = 0$ and $\delta \in \mathbb{R}^d$. Then, for any positive definite matrix A,

$$\mathrm{E}[(\Delta^T\Delta)^2] - \mathrm{E}[\Delta(\Delta^T\Delta)] \left(\mathrm{E}[\Delta\Delta^T] + A\right)^{-1} \mathrm{E}[\Delta(\Delta^T\Delta)] \geq \mathrm{E}[\Delta^T\Delta]^2.$$

Proof: for any $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}^d$,

$$\mathrm{E}[(\alpha(\Delta^T\Delta - \mathrm{E}[\Delta^T\Delta]) + \beta^T\Delta)^2] \geq 0$$

The left hand side is a quadratic form with respect to the symmetric matrix

$$\begin{pmatrix} \mathrm{E}[(\Delta^T\Delta - \mathrm{E}[\Delta^T\Delta])^2] & \mathrm{E}[\Delta^T(\Delta^T\Delta)] \\ \mathrm{E}[\Delta(\Delta^T\Delta)] & \mathrm{E}[\Delta\Delta^T] \end{pmatrix}$$

and the above nonnegativity implies that the matrix is nonnegative definite.

(continued) Therefore the matrix

$$\begin{pmatrix} \mathrm{E}[(\Delta^T\Delta - \mathrm{E}[\Delta^T\Delta])^2] & \mathrm{E}[\Delta^T(\Delta^T\Delta)] \\ \mathrm{E}[\Delta(\Delta^T\Delta)] & \mathrm{E}[\Delta\Delta^T] + A \end{pmatrix}$$

is a positive definite and so, has a positive determinant. By the determinant formula for block matrices, the determinant is computed as

$$\left|\mathrm{E}[\Delta\Delta^T] + A\right|$$
$$\times \left(\mathrm{E}[(\Delta^T\Delta - \mathrm{E}[\Delta^T\Delta])^2] - \mathrm{E}[\Delta(\Delta^T\Delta)] \left(\mathrm{E}[\Delta\Delta^T] + A\right)^{-1} \mathrm{E}[\Delta(\Delta^T\Delta)] \right).$$

Summary

- We can asymptotically efficiently discretize a rebalancing strategy.
- The rebalancing times should be hitting times of an ellipsoid.
- The ellipsoid is stochastic, the solution of a quadratic equation with the covariance matrices.
- Discretization is an efficient approximation, for which the proof requires a nontrivial multidimensional extension of Pearson's inequality.