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An 'explicit' solution to an optimal investment problem

Consider a financial market with two assets, whose prices (B, S) evolve as

$$dB_t = B_t r_t dt$$
$$dS_t = S_t(\mu_t\ dt + \sigma_t\ dW_t)$$

where W is a Brownian motion. Recall that a self-financing investor's wealth X_t^θ can be written as

$$X_t^\theta = X_0 \exp\left\{ \int_0^t [r_s + \theta_s(\mu_s - r_s) - \theta_s^2 \sigma_s^2/2] ds + \int_0^t \theta_s \sigma_s dW_s \right\}$$

where θ_t is the fraction of the investor's wealth held in the risky asset. We will assume that $\sigma_t > 0$ and use the notation $\lambda_t = (\mu_t - r_t)/\sigma_t$.

As usual, all the processes are adapted to a filtration $(\mathcal{F}_t)_{t\geq 0}$. We will now make more assumptions on the structure of this filtration, and then exploit this additional structure to solve a utility maximisation problem.

Let $\tilde{W}$ be another Brownian motion which is correlated with W, such that

$$\langle W, \tilde{W}\rangle_t = \rho t$$

for a fixed correlation $-1 \leq \rho \leq 1, \rho \neq 0$. Let $(\mathcal{F}_t)_{t\geq 0}$ be the filtration generated by the processes $(W, \tilde{W})$.

Let $(\tilde{\mathcal{F}}_t)_{t\geq 0}$ be the filtration generated by $\tilde{W}$. This filtration is strictly smaller than $(\mathcal{F}_t)_{t\geq 0}$ if $|\rho| < 1$.

Assumption. The processes (r, μ, σ) are predictable with respect to the smaller filtration $(\tilde{\mathcal{F}}_t)_{t\geq 0}$.

Check that the model considered by Zariphopoulou in 2001 fits into this framework.

Now consider the problem

$$\text{maximise } \mathbb{E}[U(X_T^\theta)]$$

where $U(x) = \frac{x^{1-R}}{1-R}$ where the coefficient of relative risk aversion is $0 < R < 1$.

Theorem.[1] Define a positive random variable ζ by

$$\zeta = \exp\left\{ \int_0^T \left((1-R)[1 + (\tfrac{1-R}{R})\rho^2] r_s + (\tfrac{1-R}{R})\lambda_s^2/2\right) ds + \int_0^T \lambda_s \rho (\tfrac{1-R}{R}) d\tilde{W}_s \right\}$$

and assume it is integrable. Then for all θ we have

$$\mathbb{E}[U(X_T^\theta)] \leq U(X_0)[\mathbb{E}(\zeta)]^{R/[R+(1-R)\rho^2]}$$

with equality if and only if

$$\theta_t = \frac{\rho\gamma_t + \lambda_t}{\sigma_t(R + (1-R)\rho^2)}$$

[1]This is one of the main results of my paper, (2004) Explicit solutions to utility maximization problems in incomplete markets. *Stochastic Processes and Their Applications* **114**(1): 109–125.

where γ is the process (guaranteed to exist by Itô's martingale representation theorem) such that

$$\zeta = \mathbb{E}(\zeta) \exp\left\{ -\frac{1}{2} \int_0^T \gamma_s^2 ds + \int_0^T \gamma_s d\tilde{W}_s \right\}.$$

It will be notationally easier to prove this theorem via two lemmas:

Lemma. Let ρ and q be constants such that $-1 \le \rho \le 1$ and $q \ge 1$. Let $(\alpha)_{t\ge0}$ be be predictable with respect to the larger filtration $(\mathcal{F}_t)_{t\ge0}$ and $(\beta)_{t\ge0}$ be be predictable with respect to the smaller filtration $(\tilde{\mathcal{F}}_t)_{t\ge0}$. Let

$$\eta_t = \exp\left\{ -\int_0^t \left(\frac{1}{2}[(q-1)\rho^2 + 1]\alpha_s^2 + \rho\alpha_s\beta_s \right) ds + \int_0^t \alpha_s dW_s \right\}$$

$$Z_t = \exp\left\{ -\int_0^t \frac{1}{2}\beta_s^2 ds + \int_0^t \beta_s d\tilde{W}_s \right\}$$

Then

$$\mathbb{E}\{Z_T[\mathbb{E}(\eta_T|\tilde{\mathcal{F}}_T)^q]\} \le 1$$

with equality if and only if α is predictable with respect to the smaller filtration $(\tilde{\mathcal{F}}_t)_{t\ge0}$ and the local martingale defined by

$$\exp\left\{ -\int_0^t \frac{1}{2}(\beta_s + \rho q\alpha_s)^2 ds + \int_0^t (\beta_s + \rho q \alpha_s) d\tilde{W}_s \right\}$$

is a true martingale.

Proof. Itô's formula says

$$d\eta_t = -\left(\frac{1}{2}(q-1)\rho^2\eta_t\alpha_t^2 + \rho\alpha_t\beta_t\eta_t \right) dt + \eta_t\alpha_t dW_t.$$

Recall that the above differential notation actually denotes an equality for stochastic integrals.

Now, with a little work we can show

$$\mathbb{E}\left(\int_0^t k_s dW_s | \tilde{\mathcal{F}}_T \right) = \rho \int_0^t \mathbb{E}(k_s|\tilde{\mathcal{F}}_T) d\tilde{W}_s$$

so that

$$d\tilde{\mathbb{E}}(\eta_t) = -\left(\frac{1}{2}(q-1)\rho^2\tilde{\mathbb{E}}(\eta_t\alpha_t^2) + \rho\beta_t\tilde{\mathbb{E}}(\alpha_t\eta_t) \right) dt + \rho\tilde{\mathbb{E}}(\eta_t\alpha_t) d\tilde{W}_t.$$

where we're now using the notation $\tilde{\mathbb{E}}(\cdot) = \mathbb{E}(\cdot|\tilde{\mathcal{F}}_T)$.

Again, Itô's formula says (after some calculation...)

$$\begin{aligned} d\{Z_t[\tilde{\mathbb{E}}(\eta_t)]^q\} =& Z_t[\tilde{\mathbb{E}}(\eta_t)]^{q-2}\frac{q(q-1)}{2}\rho^2 \left([\tilde{\mathbb{E}}(\eta_t\alpha_t)]^2 - \tilde{\mathbb{E}}(\eta_t)\tilde{\mathbb{E}}(\eta_t\alpha_t^2) \right) dt \\ &+ Z_t[\tilde{\mathbb{E}}(\eta_t)]^{q-1}[\mathbb{E}(\eta_t)\beta_t + q\rho\tilde{\mathbb{E}}(\eta_t\alpha_t)]d\tilde{W}_t. \end{aligned}$$

Note that the drift term is non-positive by the Cauchy–Schwarz inquality, and hence the expression above defines a positive supermartingale. This proves

$$\mathbb{E}\{Z_t[\tilde{\mathbb{E}}(\eta_t)]^q\} \le 1.$$

Note that the supermartingale is a local martingale if and only if α_t is $\tilde{\mathcal{F}}_T$-measurable.

$\square$

The following lemma uses the same notation as above:

Lemma. Let ξ be an positive $\tilde{\mathcal{F}}_T$-measurable random variable. Then

$$\mathbb{E}[\xi\eta_T] \leq [\mathbb{E}(Z_T^{-1/(q-1)}\xi^{q/(q-1)})^{(q-1)/q}$$

with equality if and only if

$$Z_T^{-1/(q-1)}\xi^{q/(q-1)} = C\exp\left\{-\int_0^T \frac{1}{2}(\beta_s+\rho q\alpha_s)^2 ds + \int_0^T (\beta_s+\rho q\alpha_s)d\tilde{W}_s\right\}$$

where $C = \mathbb{E}(Z_T^{-1/(q-1)}\xi^{q/(q-1)})$.

Proof. This is just Hölder's inequality:

$$\begin{aligned}\mathbb{E}[\xi\eta_T] &= \mathbb{E}[\xi\tilde{\mathbb{E}}(\eta_T)] \\ &\leq [\mathbb{E}(Z_T^{-1/(q-1)}\xi^{q/(q-1)})^{(q-1)/q}[\mathbb{E}(Z_T\tilde{\mathbb{E}}(\eta_T)^q)]^{1/q}\end{aligned}$$

and the result follows from the previous lemma and the condition for equality in Hölder's inequality. $\square$

Proof of the theorem. Note that

$$U(X_T^\theta) = U(X_0)\exp\left\{\int_0^T [(1-R)r_s + (1-R)\theta_s\sigma_s\lambda_s - (1-R)\theta_s^2\sigma_s^2/2]ds + \int_0^T (1-R)\theta_s\sigma_s dW_s\right\}$$

Now we can apply the second lemma to our problem with $\beta_t = \lambda_t/\rho$, $\alpha_t = (1-R)\sigma_t\theta_t$ and

$$q = \frac{R+(1-R)\rho^2}{(1-R)\rho^2}$$

and

$$\xi = e^{\int_0^T (1-R)r_s ds}.$$

$\square$