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Lecture 5 - Lent 2012

**The dual problem**

In the previous lecture we saw that in order to find an optimal portfolio, it is sufficient to replicate $I(\lambda Z^*)$ where $Z^*$ was a state price density. We now want to characterise those state price densities $Z^*$ such that the claim $I(\lambda Z^*)$ is replicable.

Let $U$ be a utility function satisfying the Inada conditions. Define a new function $\hat{U} : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ by

$$\hat{U}(y) = \sup_{x>0} U(x) - xy.$$

This function is called the *convex dual* function. You should check that $\hat{U}$ is decreasing and convex and satisfies the dual Inada conditions

$$\lim_{y\downarrow 0} U'(y) = -\infty \text{ and } \lim_{y\uparrow\infty} U'(y) = 0.$$

Note that for each $y$, the $x$ that maximises $U(x) - xy$ satisfies

$$U'(x) = y \Leftrightarrow y = I(x)$$

so we could also write

$$\hat{U}(y) = U \circ I(y) - yI(y).$$

Differentiating the above yields the useful identity

$$\hat{U}'(y) = -I(y).$$

Now, for fixed $\lambda > 0$, consider the problem

$$D: \text{ minimise } \mathbb{E}[\hat{U}(\lambda Z)] \text{ subject to } Z \text{ a state price density}$$

The following relates the optimal solution to this problem and investment problem considered before:

**Theorem.** Suppose $Z^*$ is the optimal solution of the problem $D$, and $\mathbb{E}|\hat{U}(YZ^*)| < \infty$ for all bounded positive random variables $Y$. Then

$$\mathbb{E}[ZI(\lambda Z^*) = \mathbb{E}[Z^* I(\lambda Z^*)]$$

for all state price densities $Z$. In particular, $I(\lambda Z^*)$ is attainable.

*Proof.* Suppose that the sample space $\Omega = \{\omega_1, \ldots, \omega_N\}$ is finite[1]. In particular, every scalar random variables $Y : \Omega \to \mathbb{R}$ can be identified with the a vector in $y \in \mathbb{R}^N$ in by $Y(\omega_j) = y_j$.

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[1]This assumption is needed to get the argument presented in lecture to work. However, it is not necessary. Here is a proof in the general case. Let

$$\phi(b) = \inf\{\mathbb{E}[\hat{U}(\lambda Z)] : Z > 0 \text{ a.s.}, \mathbb{E}(Z|P_1|) < \infty \text{ and } \mathbb{E}(ZP_1) = b\}$$

Note that $\phi$ is convex, so that by the Lagrangian necessity theorem there exists a Lagrange multiplier $-H^* \in \mathbb{R}^n$ such that

$$\mathbb{E}[\hat{U}(\lambda Z^*) - H^* \cdot [P_0 - \mathbb{E}(Z^* P_1)] \leq \mathbb{E}[\hat{U}(\lambda Z) - H^* \cdot [P_0 - \mathbb{E}(ZP_1)]$$

Now note that the set of state price densities is a convex and open subset of $\mathbb{R}^N$ . Hence if $Z$ is another state price density, then the random variable $Z_\varepsilon = (1-\varepsilon)Z^* + \varepsilon Z$ is one too if $|\varepsilon|$ is small enough. Note that the map

$$\varepsilon \mapsto \mathbb{E}\, \hat{U}(\lambda Z_\varepsilon)$$

is minimised at $\varepsilon = 0$. Hence, Fermat's theorem says that the derivative of the map vanishes at $\varepsilon = 0$ assuming it is differentiable. Since $\Omega$ is finite, interchanging expectation and differentiability is not an issue

$$\mathbb{E}\left( \frac{\hat{U}(\lambda Z_\varepsilon) - \hat{U}(\lambda Z^*)}{\varepsilon} \right) \to \mathbb{E}[(Z - Z^*)\hat{U}'(\lambda Z^*)]$$

Noting that $\hat{U}'(y) = -I(y)$ completes the proof in this case. □

**Putting it all together**

Suppose you have a market model $(P_t^1, \cdots, P_t^n)$ and you want to introduce a contingent claim with time-1 payout $\xi_1$. What time-0 price $\xi_0$ should you assign it?

Well, if the original market has no arbitrage, then there would exist a state price density $Z$. Then if you set $\xi_0 = \mathbb{E}(Z\xi_1)$, then the augmented market with prices $(P, \xi)$ is also free of arbitrage. Indeed, $Z$ is a state price density for the augmented market as well.

If $\xi_1$ is attainable, then for any choice of state price density $Z$, we would get the same initial price $\xi_0$. Otherwise, what should we do?

We could go back to our preference ideas. The *buyer's indifference price* (also called *reservation bid price*) $p_B(\xi_1)$ is defined by

$$\sup\{\mathbb{E}\, U(H \cdot P_1) : H \cdot P_0 = X_0\} = \sup\{\mathbb{E}\, U(H \cdot P_1 + \xi_1) : H \cdot P_0 = X_0 - p_B(\xi_1)\}$$

Similarly, the *seller's indifference price* (also called *reservation ask price*) $p_A(\xi_1)$ is defined by

$$\sup\{\mathbb{E}\, U(H \cdot P_1) : H \cdot P_0 = X_0\} = \sup\{\mathbb{E}\, U(H \cdot P_1 - \xi_1) : H \cdot P_0 = X_0 + p_A(\xi_1)\}$$

Clearly, $p_B(\xi_1) = -p_A(-\xi_1)$, so it is sufficient to consider just buyer's prices.

At this point, you should

**Exercise.** show that if the market is complete, then

$$p_B(\xi_1) = p_A(\xi_1) = \mathbb{E}(Z\xi_1)$$

where $Z$ is the unique state price density.

Consider the buyer's unit price $p_B(\varepsilon\xi_1)/\varepsilon$ of a small quantity of the claim. Doing Taylor expansions, we arrive at the linear pricing rule

$$\mathbb{E}[\xi_1 U'(H^* \cdot P_1)] = \lambda \xi_0$$

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for all $Z > 0$ such that $\mathbb{E}(Z|P_1|) < \infty$. Let $Z_\varepsilon = Z^*(1 + \varepsilon Y)$ where $|Y| < 1$ a.s. and $0 < \varepsilon < 1$. Then

$$\begin{aligned} \mathbb{E}(Z^* Y H^* \cdot P_1) &\geq \mathbb{E}\left( \frac{\hat{U}(\lambda Z^*) - \hat{U}(\lambda Z_\varepsilon)}{\varepsilon} \right) \\ &\to -\mathbb{E}[Z^* Y \hat{U}'(\lambda Z^*)] \end{aligned}$$

by the convexity of $\hat{U}$ and the monotone convergence theorem. But since $Y$ is arbitrary and $\hat{U}' = -I$, we can conclude that $H^* P_1 = I(\lambda Z^*)$ as claimed. □

where we let $\xi_0 = \lim_{\varepsilon \downarrow 0} p_B(\varepsilon \xi_1)/\varepsilon$.

That is to say, in order to find the time-0 indifference price $\xi_0$ of a small quantity of a contingent claim with payout $\xi_1$, we can set

$$\xi_0 = \mathbb{E}(Z^* \xi_1)$$

where $Z^* = U'(H^* \cdot P_1)/\lambda$. But the conclusion of the last section is that $Z^*$ can also be identified with an optimal solution of the problem

$$D: \text{ minimise } : \mathbb{E}\, \hat{U}(\lambda Z) \text{ subject to } Z \text{ a state price density.}$$

*Example.* If

$$U(x) = \frac{x^{1-R}}{1-R}$$

is the CRRA utility function with relative risk aversion parameter $R > 0, R \neq 1$, then the marginal utility is

$$U'(x) = x^{-R} \Leftrightarrow I(y) = y^{-1/R}$$

so the convex dual function is

$$\hat{U}(y) = -\frac{y^{1-1/R}}{1-1/R}.$$

If you read papers[2] on financial mathematics, you may come across the so-called $q$-optimal equivalent martingale measure, defined as the equivalent martingale measure that minimises the $q$-th moment of its density:

$$\mathbb{E}\left[\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)^q\right].$$

Why is this measure important? Well, letting $q = 1 - 1/R > 0$, we see that the state price density that the agent with this utility function would use to compute indifference prices corresponds to the $q$-optimal measure.

*Example.* If

$$U(x) = \log y$$

(corresponding to $R = 1$ above) then the marginal utility is

$$U'(x) = \frac{1}{x} \Leftrightarrow I(y) = \frac{1}{y}$$

so the convex dual function is

$$\hat{U}(y) = -\log y - 1.$$

The dual optimiser for this objective function corresponds to the *Föllmer–Schweizer minimal martingale measure*, in the case when the asset prices are continuous semimartingales. We will come back to this point later in the course.

*Example.* If

$$U(x) = -e^{-\gamma x}$$

[2] For instance: D. Hobson. Stocastic volatility models, correlation and the $q$-optimal measure. *Mathematical Finance.* **14**(4): 537–556. (2004)

then $U$ is not Inada according the definition given last time. So far, we have considered utility functions which are finite the positive half-line $(0, \infty)$. But without too much effort we can define the right concepts for utility functions finite on the whole $(-\infty, \infty)$. For instance, the correct Inada condition in this case is

$$\lim_{x \downarrow -\infty} U'(x) = \infty \text{ and } \lim_{x \uparrow +\infty} U'(x) = 0.$$

which the CARA utility function clearly satisfies. Now

$$U'(x) = \gamma e^{-\gamma x} \Leftrightarrow I(y) = -\frac{1}{\gamma} \log(y/\gamma)$$

and hence

$$\hat{U}(y) = \frac{y}{\gamma}(\log(y/\gamma) - 1).$$

The dual optimiser in this case corresponds to the *minimal entropy martingale measure.* [3] This measure arises in many papers on indifference pricing since, among other reasons, the calculations of indifference prices in the CARA case are often somewhat tractable.

[3] See, for instance, Delbaen, Grandits, Rheinlander, Samperi, Schweizer, Stricker. Exponential hedging and entropic penalties. *Mathematical Finance* **12**: 99–12. (2002)