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Motivation

Last time we considered the problem to

$$\text{maximise } \mathbb{E}\, U(H \cdot P_1) \quad \text{subject to } H \cdot P_0 = X_0.$$

If U is concave and satisfies a regularity condition, we saw that there exists a λ such that

$$\mathbb{E}[U'(H^* \cdot P_1)P_1] = \lambda P_0$$

where H^* is the optimal portfolio. Furthermore, introducing the indirect utility (or value function)

$$V(X_0) = \sup\{\mathbb{E}\, U(H \cdot P_1) : H \cdot P_0 = X_0, H \in \mathbb{R}^n\}$$

we saw that the Lagrange multiplier λ is just the marginal indirect utility $V'(X_0)$.

We usually assume that the utility function U is increasing. Unfortunately, this assumption alone does not necessarily imply that V is increasing also. But if we further assume that V is increasing (which seems reasonable, since this says we prefer having more initial wealth to less), we can conclude the Lagrange multiplier for the investment problem is positive.

Now, if we define a random variable Z by

$$Z = \frac{U'(H^* \cdot P_1)}{\lambda}$$

then Z has two properties:

(1) $Z > 0$ a.s.
(2) $\mathbb{E}[ZP_1] = P_0$

Crashcourse on one-period financial mathematics

The previous section was merely motivation. We now make a definition which is independent of any expected utility maximisation problem.

Definition. Given a market model $(P_t^1, \ldots, P_t^n)_{t\in\{0,1\}}$ a *state price density* is a random variable Z such that

(1) $Z > 0$ a.s.
(2) $\mathbb{E}(Z\|P_1\|) < \infty$ and $\mathbb{E}[ZP_1] = P_0$.

Remark. Synonyms for state price density include *pricing kernel* and *stochastic discount factor.*

In many situations, we assume that one of our n given assets is a numéraire, i.e. has always has positive price. Then we can decompose our market model as

$$P = (B, S)$$

where $B = (B_t)_{t\in\{0,1\}}$ is a positive scalar processes and $S = (S_t)_{t\in\{0,1\}}$ is an $n-1$-dimensional process. Let Z be a state price density, and introduce a new measure $\mathbb{Q}$ by the density

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = Z\frac{B_1}{B_0}.$$

First, note that $\mathbb{Q}$ is a probability measure since

$$\mathbb{E}^{\mathbb{P}}\left(Z\frac{B_1}{B_0}\right) = 1$$

Similarly, we have

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\left(\frac{S_1}{B_1}\right) &= \mathbb{E}^{\mathbb{P}}\left(Z\frac{S_1}{B_0}\right) \\ &= \frac{S_0}{B_0}.\end{aligned}$$

The measure $\mathbb{Q}$ is called an *equivalent martingale measure* relative to the numéraire modelled by B, since the discounted process $(S_t/B_t)_{t\in\{0,1\}}$ is a martingale under the equivalent measure $\mathbb{Q}$. If we assume that B_1 is not random (for insance, if the numéraire is a government bond[1]), then we say the numéraire is *risk-free*, in which case the equivalent martingale measure is called a *risk neutral measure*. Finally, if the numéraire is unambiguous in a given context, we might call $\mathbb{Q}$ the *pricing measure.*

Now recall a fundamental notion in market modelling:

Definition. An *arbitrage* is a portfolio $H \in \mathbb{R}^n$ such that
(1) $H \cdot P_0 \leq 0 \leq H \cdot P_1$ a.s.
(2) $\mathbb{P}(H \cdot P_0 < H \cdot P_1) > 0$.

The most important structural theorem is then:

Theorem. (The first fundamental theorem of asset pricing) A market is free of arbitrage if and only if there exists a state price density.

Proof.[2] One direction is easy. Suppose there exists a a state price density Z. Let H be a strategy such that

$$H \cdot P_0 \leq 0 \leq H \cdot P_1.$$

Since $Z > 0$ we must have

$$\begin{aligned}0 &\leq \mathbb{E}(ZH \cdot P_1) \\ &= H \cdot \mathbb{E}(ZP_1) \\ &= H \cdot P_0 \\ &\leq 0\end{aligned}$$

and hence $H \cdot P_0 = 0 = H \cdot P_1$ a.s. In particular, $\mathbb{P}(H \cdot P_0 < H \cdot P_1) = 0$ and H is not an arbitrage.

The other direction is more difficult. One possibility is to suppose that there is no arbitrage and show that a certain utility maximisation problem always has an optimal solution. Then

[1]Nowadays, it might be hard to argue that the payout of Euro zone bonds are certain.
[2]Not lectured.

by the arguments outlined above, the marginal utility is proportional to a state price density. This approach was presented in the Advanced Financial Models course.

Another approach, due to Douglas Kennedy, is a follows. First define a set of candidate state price densities by

$$\mathcal{Z} = \{Z > 0 \text{ a.s. and } \mathbb{E}(Z\|P_1\|) < \infty\}$$

and the consider the set

$$\mathcal{C} = \{\mathbb{E}(ZP_1) : Z \in \mathcal{Z}\} \subseteq \mathbb{R}^n.$$

First note that $\mathcal{Z}$ is non-empty, since $Z_0 = e^{-\|P_1\|}$ is certainly an element. In particular, the set $\mathcal{C}$ is non-empty. Also, it is easy to check that $\mathcal{C}$ is convex.

If $P_0 \in \mathcal{C}$ then there would exist a state price density. So suppose $P_0 \notin \mathcal{C}$. We will show that there exists an arbitrage. Now, by the separating hyperplane theorem, there exists a vector $H \in \mathbb{R}^n$ such that

(1) $H \cdot (y - P_0) \geq 0$ for all $y \in \mathcal{C}$

(2) there exists $y^* \in \mathcal{C}$ such that $H \cdot (y^* - P_0) > 0$.

But since every $y \in \mathcal{C}$ can be written $y = \mathbb{E}(ZP_1)$ for some $Z \in \mathcal{Z}$, we can rewrite the above, using the notation $X_t = H \cdot P_t$. as

(1) $\mathbb{E}(ZX_1) \geq X_0$ for all $Z \in \mathcal{Z}$

(2) there exists $Z^* \in \mathcal{Z}$ such that $\mathbb{E}(ZX_1) > X_0$.

Letting $Z = \varepsilon Z_0$ for $\epsilon > 0$ in (1) and sending $\varepsilon \downarrow 0$ yields

$$X_0 \leq 0.$$

Now let $Z = (\frac{1}{\varepsilon}\mathbb{1}_{\{X_1<0\}} + 1)Z_0$ for $\varepsilon > 0$ in (1) and multiplying by ε yields

$$\mathbb{E}(Z_0 X_1 \mathbb{1}_{\{X_1<0\}}) \geq \varepsilon(X_0 - \mathbb{E}\ Z_0 X_1) \to 0$$

But since the left-hand side is non-negative, we must conclude that $\mathbb{P}(X_1 < 0) = 0$, i.e.

$$X_1 \geq 0 \text{ a.s.}$$

Finally, by (2) we see that $\mathbb{P}(X_1 > X_0) > 0$ and hence H is an arbitrage. □

Given our n assets with prices P, we ask what happens if we introduce a new asset. We will call this new asset a contingent claim (since its payout may depends on the realisations of the other asset prices) and we will call its time-1 payout ξ_1. Note that ξ_1 is simply modelled a scalar random variable.

A certain class of contingent claims are easy to handle–those whose payouts are simply linear combinations of the payouts of the existing assets. We give them a name:

Definition. A claim is *attainable* (or *replicable*) if there exists a portfolio H such that $\xi_1 = H \cdot P_1$.

The following classification of attainable claims was proven in Advanced Financial Models:

Theorem. A claim with payout ξ_1 is attainable in an arbitrage-free market if and only if there is a real number ξ_0 such that $\mathbb{E}(Z\xi_1) = \xi_0$ for all state price densities Z such that $\mathbb{E}(Z|\xi_1|) < \infty$.

Definition. A market model is *complete* if every contingent claim is attainable.

Theorem. (The second fundamental theorem of asset pricing) An arbitrage-free market is complete if and only if there exists a *unique* equivalent martingale measure.

A sufficient condition for optimality.

We are now returning to the investor's optimisation problem. We already have a characterisation of the solution, so if we were only interested in one-period models, we would be done. But since we are interested in multi-period models in both discrete and continuous time, we now rewrite some of the results we have in terms of the notions introduced in the previous section since they carry forward to more complicated models.

First, let us refine some assumptions on the utility function:

Definition. A utility function $U : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ satisfies *Inada conditions* if

(1) U is increasing, continuously differentiable and concave,
(2) $\lim_{x\downarrow 0} U'(x) = \infty$ and $\lim_{x\uparrow\infty} U'(x) = 0$.

If U is Inada, then the marginal utility U' is a decreasing bijection from $(0, \infty)$ to $(0, \infty)$. In particular, there exists an inverse function $I = (U')^{-1}$.

Again, consider the problem

$$P: \text{ maximise } \mathbb{E}\, U(H \cdot P_1) \quad \text{subject to } H \cdot P_0 = X_0.$$

where U is Inada and $X_0 > 0$. Our main result is the following:

Theorem. Suppose there exists a $\lambda > 0$ and a state price density Z^* such that

$$\mathbb{E}[ZI(\lambda Z^*)] = X_0$$

for all state price densities Z. Then there exists a portfolio $H^* \in \mathbb{R}^n$ such that

$$H^* \cdot P_1 = I(\lambda Z^*)$$

and H^* is an optimal solution to the investment problem P.

Proof. This is an exercise. □

Note that in complete markets, there exists only one state price density Z. Hence we need only find such that

$$\mathbb{E}[ZI(\lambda Z)] = X_0$$

to apply the theorem. However, the left-hand side is a decreasing, and most cases continuous[3] function of λ so finding λ amounts to inverting $\lambda \mapsto \mathbb{E}[ZI(\lambda Z)]$. In this sense, we get the following meta-theorem:

Meta-theorem. Optimal investment complete markets is trivial.

[3] If there exists a $\lambda^* > 0$ such that $\mathbb{E}[ZI(\lambda^* Z)] < \infty$, then by the monotone convergence theorem $\lambda \mapsto \mathbb{E}[ZI(\lambda Z)]$ is continuous and decreasing on $[\lambda^*, \infty)$. In the one-period case, complete models are necessarily defined on finite probability spaces, so integrability is not an issue. However, in the continuous-time models to come, we should be more careful.