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Example sheet 3 - Lent 2012

Problem 1. Let W be a scalar Brownian motion, let K and T be positive constants. Find a predictable process α such that

$$\int_0^T \alpha_s dW_s = K \text{ a.s.}$$

Explain why we must introduce some notion of admissibility to talk sensibly about continuous-time investment.

Problem 2. Consider a continuous time, n-asset market with prices $(P_t)_{t\geq 0}$, and let $(Z_t)_{t\geq 0}$ be a state price density. For each $t \geq 0$, let $U(t,\cdot)$ be a utility function satisfying the Inada condition, and let $I(t,\cdot)$ be the corresponding inverse marginal utility. Suppose that for every $y > 0$, the positive random variable $\int_0^\infty Z_s I(s, yZ_s)ds$ is integrable.

(a) Show that for any $x > 0$, there exists a $y > 0$ such that

$$\mathbb{E}\left(\int_0^\infty Z_s I(s, yZ_s)ds\right) = x.$$

(b) Now fix x and y satisfying the above equation. Suppose there exists a predictable process H^* such that

$$\int_0^t H_s^* \cdot d(Z_s P_s)$$

defines a uniformly integrable martingale and such that

$$x + \int_0^\infty H_s^* \cdot d(Z_s P_s) = \int_0^\infty Z_s c_s^* ds$$

where $c_t^* = I(t, yZ_t)$. Show that the wealth process X^* corresponding to holdings H^* and consumption rate c^* is given by

$$X_t^* = \mathbb{E}\left(\int_t^\infty Z_t^{-1} Z_s c_s^* ds | \mathcal{F}_t\right)$$

and in particular, $X_0^* = x$ and the controls (H^*, c^*) are admissible in the sense that the wealth stays non-negative.

(c) Show that for any admissible (H, c), the stochastic integral

$$\int_0^t H_s \cdot d(Z_s P_s)$$

defines a supermartingale.

(d) Show that (H^*, c^*) is optimal for the infinite horizon optimal consumption problem

$$\text{maximise } \mathbb{E}\left(\int_0^\infty U(s, c_s)ds\right) \text{ subject to } X_0^{(H,c)} = x \text{ and admissibility}$$

[As suggested in lecture, you could consider the Lagrangian

$$L(H, c; \lambda) = \mathbb{E}\left(\int_0^\infty U(s, c_s)ds + \int_0^\infty \lambda_s(H_s \cdot dP_s - c_s ds - dX_s^{(H,c)})\right)$$

with Lagrange multiplier $\lambda_t = yZ_t$.]

Problem 3. Suppose the market consists of $d + 1$ assets with prices

$$\begin{aligned} dB_t &= B_t r dt \\ dS_t &= \mathrm{diag}(S_t)(\mu\, dt + \sigma\, dW_t) \end{aligned}$$

where W is a d-dimensional Brownian motion, and $r \in \mathbb{R}$, $\mu \in \mathbb{R}^d$ and $\sigma \in \mathbb{R}^{d\times d}$ are constant. Assume σ is non-singular and that the filtration is generated by W. What is the unique state price density process Z?

We are interested in the infinite-horizon optimal consumption problem

$$\text{maximise } \mathbb{E}\left(\int_0^\infty U(s, c_s) ds\right) \text{ subject to } X_0^{(H,c)} = x \text{ and admissibility}$$

(a) Using the previous problem, find explicitly the optimal solution when the utility functions are of the form

$$U(t, c) = \frac{c^{1-R}}{1-R} e^{-\beta t}.$$

for a risk aversion parameter $R > 0, R \neq 1$ and subjective discount rate $\beta > 0$. What restrictions on the parameters β and R must you assume?

(b) Use the martingale representation theorem to prove a mutual fund theorem for the problem for general utility functions $U(t, \cdot)$.

Problem 4. Let $P = (B, S)$ be the market model as in the previous question. Consider the pure investment problem

$$\text{maximise } \mathbb{E}\ U(X_T^\theta) \text{ subject to } X_0^\theta = x.$$

What is the HJB equation for this problem?

Problem 5. Let $V : [0, T] \times (0, \infty) \to \mathbb{R}$ satisfy the non-linear PDE

$$\frac{\partial V}{\partial t} = \frac{\left(\frac{\partial V}{\partial x}\right)^2}{\frac{\partial^2 V}{\partial x^2}}.$$

(a) Assuming that $x \mapsto V(t, x)$ satisfies the Inada conditions for each t, show that the convex dual function $\hat{V}$ satisfies the *linear* PDE

$$\frac{\partial \hat{V}}{\partial t} + y^2 \frac{\partial^2 \hat{V}}{\partial y^2} = 0$$

where $\hat{V}(t, y) = \sup_{x>0}(V(t, x) - xy)$. Why is this not surprising?

(b) Show that the function

$$-\sum_{i=1}^n a_i \frac{y^{1-r_i}}{1-r_i} e^{(1-r_i)r_i t}$$

satisfies the linear PDE in part (a) for any constants $a_1, \ldots, a_n$ and $r_1, \ldots, r_n \neq 1$. Show that if $a_i > 0$ and $r_i > 0$ for all i, then the function above satisfies the dual Inada conditions.

Problem 6. Consider[1] a two-asset market $P = (B, S)$ where

$$dB_t = B_t r(Y_t)dt$$
$$dS_t = S_t(\mu(Y_t)dt + \sigma(Y_t)dW_t)$$
$$dY_t = \beta(Y_t)dt + \alpha(Y_t)d\tilde{W}_t$$

where W and $\tilde{W}$ are Brownian motions with correlation ρ.
(a) Write down the HJB equation for the problem

$$\text{maximise } \mathbb{E}\, U(X_T^\theta) \text{ subject to } X_0^\theta = x.$$

(b) Now, assume that $U(x) = \frac{x^{1-R}}{1-R}$. Make the subsitution $V(t,x,y) = U(x)f(t,y)$ in the HJB equation from part (a) to derive the non-linear PDE

$$\frac{\partial f}{\partial t} + \frac{\alpha^2}{2}\frac{\partial^2 f}{\partial y^2} + \beta\frac{\partial f}{\partial y} + r(1-R)f + \frac{1-R}{2Rf}\left(\alpha\rho\frac{\partial f}{\partial y} + \lambda f\right)^2 = 0$$

where $\lambda(y) = (\mu(y) - r(y))/\sigma(y)$.
(c) Make the substitution $f(t,y) = g(t,y)^\delta$ and derive the PDE

$$\frac{\partial g}{\partial t} + \frac{\alpha^2}{2}\left[\frac{\partial^2 g}{\partial y^2} + (\delta - 1)\frac{\left(\frac{\partial g}{\partial y}\right)^2}{g}\right] + \beta\frac{\partial g}{\partial y} + \frac{r(1-R)}{\delta}g + \frac{1-R}{2\delta R g}\left(\alpha\rho\delta\frac{\partial g}{\partial y} + \lambda g\right)^2 = 0.$$

Show that this PDE is actually linear when

$$\delta = \frac{R}{R + \rho^2(1-R)}.$$

(d) Let δ be given by the above formula. Let

$$d\hat{Y}_t = \left(\beta(\hat{Y}_t) - (1 - 1/R)\alpha(\hat{Y}_t)\lambda(\hat{Y}_t)\rho\right)dt + \alpha(\hat{Y}_t)d\hat{W}_t$$

for a Brownian motion $\hat{W}$. Show that

$$M_t = g(t, \hat{Y}_t)e^{\int_0^t k(\hat{Y}_s)ds}$$

defines a local martingale, where

$$k(y) = \frac{(1-R)}{\delta}\left[r(y) + \frac{1}{2R}\lambda(y)^2\right].$$

Assuming M is a true martingale, and using the Markov property of $\hat{Y}$, show that

$$g(t,y) = \mathbb{E}\left(e^{\int_t^T k(\hat{Y}_s)ds} | \hat{Y}_t = y\right).$$

Problem 7. Let P be a general market model, and let X^H be the wealth generated by using the trading strategy H.
(a) Use Hölder's inequality to show that for any state price density Z,

$$\mathbb{E}[U(X_T^H)] \leq U(X_0)\left(\mathbb{E}[Z_T^{1-1/R}]\right)^R$$

[1] This problem is taken from the paper Th. Zariphopoulou. A solution approach to valuation with unhedgeable risks. *Finance and Stochastics* **5**: 61–82 (2001).

where $U(x) = \frac{x^{1-R}}{1-R}$ for $0 < R < 1$. When is there equality?
(b) Use part (a) to solve the pure investment problem

$$\text{maximise } \mathbb{E}[U(X_T^H)] \text{ subject to } X_0 = x \text{ and admissibility}$$

for the market model $P = (B, S)$ in Problem 3, where U is as in part (a).