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Example sheet 2 - Lent 2012

Problem 1. Find an example of a discrete-time local martingale that is not a true martingale.

Problem 2. Consider a discrete-time market model $P = (B, S)$ with $n = d + 1$ assets. The first asset is a numéraire, with price $B_t > 0$ a.s. for all $t \geq 0$. The other d assets have price $(S_t)_{t\geq 0}$, a d-dimensional adapted process. Assume that there are no dividends.

Writing the agents holdings as $H = (\phi, \pi)$, show that the budget constraint and self-financing conditions combine to become

$$\frac{X_{t+1}^{(\phi,\pi);c}}{B_{t+1}} = \frac{X_t^{(\phi,\pi);c}}{B_t} + \pi_{t+1} \cdot \left(\frac{S_{t+1}}{B_{t+1}} - \frac{S_t}{B_t} \right) - \frac{c_{t+1}}{B_{t+1}}.$$

Problem 3. Consider the set-up of problem 2. Now suppose the numéraire is cash, so that $B_t = 1$ a.s. for all $t \geq 0$, and that the other d assests have independent Gaussian increments

$$S_{t+1} - S_t \sim N(\mu, V).$$

where S_0 is a given constant vector, and V is non-singular.
(a) Use the dynamic programming principle to find the strategy $(\phi_t, \pi_t)_{1\leq t\leq T}$ that

$$\text{maximises } \mathbb{E}[U(X_T^{(\phi,\pi)})] \text{ subject to } X_0 = x$$

where $U(x) = -e^{-\gamma x}$. (Note that there is no consumption, and we do *not* insist that the strategy is admissible. Indeed, the optimal strategy you will find is not admissible.)
(b) Show that there exists a non-random vector $u \in \mathbb{R}^d$ that does not depend on the time horizon T, the initial wealth x or the coefficient of absolute risk aversion γ, and scalar predictable process $(k_t)_{1\leq t\leq T}$ such that $\pi_t^* = k_t u$ a.s. for all $t \geq 0$.
(c) Let V be the value function. Show that

$$V(0, x) = U(x)e^{-H(\mathbb{Q}|\mathbb{P})}$$

where

$$H(\mathbb{Q}|\mathbb{P}) = \mathbb{E}^{\mathbb{P}} \left[\frac{d\mathbb{Q}}{d\mathbb{P}} \log \frac{d\mathbb{Q}}{d\mathbb{P}} \right]$$

is the relative entropy and $\mathbb{Q}$ is an equivalent martingale measure. Why is this not surprising?

Problem 4. Consider the utility maximisation appearing in problem 3, but this time with an arbitrary[1] utility function U for all feasible H. Again show that there is non-random vector $u \in \mathbb{R}^d$ independent of T, x and U, and a scalar predictable process $(k_t)_{1\leq t\leq T}$ such that $\pi_t^* = k_t u$. [Hint: You'll probably need the integration by parts formula for Gaussian measure:

$$\mathbb{E}[Zf(Z)] = \mathbb{E}[\nabla f(Z)]$$

where $Z \sim N(0, I)$ and $f : \mathbb{R}^d \to \mathbb{R}$ is sufficiently well-behaved.]

[1]... but suitably well-behaved so that all formal manipulations can be justified...

Problem 5 (Martingale representation)**.** Let $\zeta_1, \zeta_2, \ldots$ be a sequence of independent Bernoulli random variables such that

$$\mathbb{P}(\zeta_t = 1) = p = 1 - \mathbb{P}(\zeta_t = 0)$$

Suppose that the filtration is $\mathcal{F}_t = \sigma(\zeta_1, \ldots \zeta_t)$. Show that for every martingale M there exists a predictable process $(\theta_t)_{t\geq 1}$ such that

$$M_t = M_0 + \sum_{s=1}^{t} \theta_s(\zeta_s - p).$$

Problem 6. Suppose N is a bounded (but non-constant) martingale and θ is a bounded predictable process. Let M be the process defined by

$$M_t = M_0 + \sum_{s=1}^{t} \theta_s(N_s - N_{s-1}).$$

Show that M is a martingale and that θ can be recovered from M and N by the formula

$$\theta_t = \frac{\mathbb{E}(M_t N_t|\mathcal{F}_{t-1}) - M_{t-1}N_{t-1}}{\mathbb{E}(N_t^2|\mathcal{F}_{t-1}) - N_{t-1}^2}.$$

Problem 7. Let $\xi_1, \xi_2, \ldots$ be a sequence of independent random variables such that

$$\mathbb{P}(\xi_i = u) = p = 1 - \mathbb{P}(\xi_i = d)$$

where $0 < d < 1 + r < u$ are constants. Let

$$S_t = S_0\xi_1 \cdots \xi_t \text{ and } B_t = (1 + r)^t.$$

Consider the two asset market with price process $P = (B, S)$. Suppose that the filtration is $\mathcal{F}_t = \sigma(\xi_1, \ldots \xi_t)$.
(a) Find the unique state price density process Z with $Z_0 = 1$.
(b) Solve the problem

$$\text{maximise } \mathbb{E}\log(X_T^H) \text{ subject to } X_0^H = x$$

in two ways:(1) by finding the Lagrange multiplier λ such that $U'(\lambda Z_T)$ is attainable from initial wealth x and applying problem 5 and 6, and (2) by solving the Bellman equation.

Problem 8. Consider a discrete time model with an asset with non-negative prices $(P_t)_{t\geq 0}$ and non-negative dividends $(\delta_t)_{t\geq 1}$. Let Z be a state price density process.
(a) By considering the martingale defined by $Z_t P_t + \sum_{s=1}^{t} Z_s\delta_s$, show that there exists a finite-valued non-negative random variable S such that

$$\sum_{s=1}^{t} Z_s\delta_s \to S \text{ a.s.}$$

[Hint: use the martingale convergence theorem.]
(b) Show that $\sum_{s=1}^{t} Z_s\delta_s \to S$ in L^1.
(c) Show that $P_t \geq \mathbb{E}\left(\sum_{u=t+1}^{\infty} Z_t^{-1}Z_u\delta_u|\mathcal{F}_t\right)$ for all $t \geq 0$. Can you give this inequality a financial interpretation?
(d) What condition must you assume on the process $(Z_tP_t)_{t\geq 0}$ to assert equality in part (c)?

Problem 9. Consider a discrete time model with an asset with *positive* prices $(P_t)_{t\geq 0}$ and non-negative dividends $(\delta_t)_{t\geq 1}$. Show that there is a self-financing trading strategy with corresponding wealth process $Q_t = P_t \prod_{s=1}^t \left(1 + \frac{\delta_s}{P_s}\right)$. What is the financial significance of this process? Let Z be a positive adapted process. Show that the process $\left(Z_t P_t + \sum_{s=1}^t Z_s \delta_s\right)_{t\geq 0}$ is a martingale if and only if $(Z_t Q_t)_{t\geq 0}$ is.