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Example sheet 1 - Lent 2012

Problem 1 (Allais paradox). (a) Consider these two measures on $(\mathbb{R}, \mathcal{B})$:

$$\lambda_1 = 0.33\ \delta_{101} + 0.66\ \delta_{100} + 0.01\ \delta_0$$
$$\mu_1 = \delta_{100}$$

That is, λ_1 corresponds to the random payout of £101 with probability 33%, of £100 with probability 66% and of zero with probability 1%, while μ_1 corresponds to a certain payout of £100. Which would you prefer?

(b) How about

$$\lambda_2 = 0.34\ \delta_{100} + 0.66\ \delta_0$$
$$\mu_2 = 0.33\ \delta_{101} + 0.67\ \delta_0?$$

(c) Suppose you answered μ_1 in part (a) and μ_2 in part (b). Show that your preferences do not conform to the independence axiom.

[Hint: $\frac{1}{2}(\mu_1 + \mu_2) = \frac{1}{2}(\lambda_1 + \lambda_2)$.]

Problem 2. Suppose preference relation $\succ$ satisfies the von Neumann–Morgenstern axioms as described in lectures.

(a) Show that if $\lambda \succ \mu$ and $1 > p > q > 0$, then

$$p\lambda + (1-p)\mu \succ q\lambda + (1-q)\mu.$$

(b) Show that if $\lambda \succ \mu \succ \nu$ there exists a unique $p \in (0,1)$ such that $\mu \sim p\lambda + (1-p)\nu$.

Problem 3. Given the preference relation $\succ$ satisfying the axioms from the lectures, and two distinguished probability measures $\lambda_1 \succ \lambda_0$, let U_0 be the function as defined in lectures. Let μ and ν be such that

$$\lambda_1 \succ \mu \succ \nu \succ \lambda_0.$$

(a) Show that $U_0(\mu) > U_0(\nu)$.

(b) Show that then for any $0 < p < 1$

$$U_0(p\mu + (1-p)\nu) = pU_0(\mu) + (1-p)U_0(\nu).$$

Problem 4. Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set. Suppose is $f : \mathcal{X} \to \mathbb{R}$ concave and $g : \mathcal{X} \to \mathbb{R}^m$ linear. Show that the function ϕ defined by

$$\phi(b) = \sup\{f(x) : g(x) = b, x \in \mathcal{X}\}$$

is concave.

Problem 5. The market model consists of the initial prices $P_0 \in \mathbb{R}^n$ and terminal prices $P_1 \sim N_n(\mu, V)$ where $\mu \in \mathbb{R}^n$ and V is non-negative definite $n \times n$ matrix.

(a) Show that there is no arbitrage if V is non-singular. In general, what are the precise conditions on the kernel of V and the vectors P_0 and μ such that the market is arbitrage free?

(b) From now on, assume V is non-singular. Let $U(x) = -e^{-\gamma x}$ be the CARA utility function, where $\gamma > 0$ is constant. For an initial wealth X_0, consider the problem

$$\text{maximise } \mathbb{E}\ U(H \cdot P_1) \text{ subject to } H \cdot P_0 = X_0$$

Find the optimiser H^* explicitly.
(c) Find the value function $V(X_0)$ explicitly. Is V an increasing function?
(d) Suppose $X_0 < P_0 \cdot V^{-1}\mu/\gamma$. Verify that $U'(H \cdot P_1) = \lambda Z$ where $\lambda = V'(X_0)$ and Z is a state price density.

Problem 6. Consider the one period utility maximisation problem. Show that if there exists a real $\lambda > 0$, a state price density Z and portfolio $H \in \mathbb{R}^n$ such that

$$X_0 = H \cdot P_0 \text{ and } I(\lambda Z) = H \cdot P_1$$

then H is an optimal solution.

Problem 7. Consider a single period market with two assets. The first asset is a riskless bond with prices $B_0 = 1$ and $B_1 = 1 + r$ for a constant r. The second asset is a stock with prices $(S_t)_{t\in\{0,1\}}$.

Let (ϕ^*, π^*) be the optimal solution to the problem

$$\text{maximise } \mathbb{E}U(\phi B_1 + \pi S_1) \text{ subject to } \phi B_0 + \pi S_0 = X_0$$

for a given concave increasing utility function U. Prove that the investor is holds a non-negative number of shares of the stock if

$$\mathbb{E}\ S_1 > (1 + r)S_0$$

Does this agree with your intuition?

Problem 8. Let A be a $m \times n$ matrix. Prove that exactly one of the following statements is true:

- There exists an $x \in \mathbb{R}^n$ with $x_i > 0$ for all $i = 1, \ldots, n$ such that $Ax = 0$.
- There exists a $y \in \mathbb{R}^m$ with $(A^T y)_i \geq 0$ for all $i = 1, \ldots, n$ such that $A^T y \neq 0$.

What does this have to do with finance?