Markov properties of stationary Gaussian term structure models

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Abstract

We consider a Gaussian model for the term structure of interest rates. We assume that the forward rate surface is stationary under a martingale measure, and examine the implications of a variety of Markov-type conditions on the form of the covariance structure. We also show that any such model arising from an injective transformation of a Brownian Sheet must be one of the models introduced in Kennedy (1997); with the covariance structure determined by three parameters.

Keywords: Term structure model, Gaussian random field.

1 Introduction

In this paper we are concerned with Gaussian random field models for the term structure of interest rates. This is model of the form $P_{s,t} = \exp(-\int_s^t F_{s,u}\, du)$, $t \geq s$, where $P_{s,t}$ denotes the time-s value of receiving £1 at the future time t; and where $\{F_{s,t} : t \geq s\}$, the surface of instantaneous forward rates, is a continuous Gaussian random field under a measure $\mathbb{Q}$. It is convenient to set $F_{s,t} = F_{s,s}$ for $s > t$ so that F exists on all of $\mathbb{R}^2$. The measure $\mathbb{Q}$ is assumed to be an 'equivalent martingale measure' in that: (i) $\mathbb{Q} \sim \mathbb{P}$ where $\mathbb{P}$ denotes the 'objective probability'; which describes our beliefs about how observed bond prices evolve, and (ii) for each t, the discounted bond prices, $Z_{s,t} = \exp(-\int_0^s F_{u,u}\, du - \int_s^t F_{s,u}\, du)$, are $\mathbb{Q}$-martingales with respect to the filtration $\mathcal{F}_s = \sigma\{F_{u,v} : u \leq s\}$, the information contained in the forward rate surface up to time s.

The reason for working under a martingale measure is due to the application to arbitrage-free option pricing. In a complete arbitrage-free market the current value of a option which pays the random amount X at a future time, is the expected discounted value of X under the martingale measure (see Harrison & Kreps (1979) and Harrison & Pliska (1981)). In this paper we are not concerned with the question of option pricing. Instead we investigate the consequences of various structural assumptions about F such as Markov-type properties and stationarity.

It is proved in Kennedy (1997) that the martingale measure assumption implies that F satisfies an 'independent increments' property: for $s \le s' \le t$, $F_{s',t} - F_{s,t}$ is independent of $\mathcal{F}_s$, which we denote

$$F_{s',t} - F_{s,t} \perp \mathcal{F}_s. \tag{1.1}$$

We will use $s \wedge s'$ to denote $\min(s, s')$ and $s \vee s'$ to denote $\max(s, s')$. Defining $\Gamma(s,t,s',t') = \mathbb{C}\mathrm{ov}(F_{s,t}, F_{s',t'})$ to be the covariance structure of F, it is straightforward to show that the independent increments property implies the existence of a function $c(\cdot,\cdot,\cdot)$ such that $\Gamma(s,t,s',t') = c(s \wedge s', t, t')$. In Kennedy (1997) the effect on a Gaussian term structure model of Markov-type properties is also considered. It is shown that under the strongest formulation, $\mathbb{C}\mathrm{ov}(F_{s,t})$ is determined by just three parameters and has the form $\mathbb{C}\mathrm{ov}(F_{s,t}, F_{s',t'}) = \sigma^2 \exp(-\lambda(t \wedge t' - s \wedge s') - \mu|t \vee t' - s \wedge s'|)$, $0 \le \lambda \le 2\mu$.

Throughout this paper F will denote a general continuous Gaussian random field on $\mathbb{R}^2$ with the independent increments property above. We will also assume that we are only dealing with the stochastic part of the forward rate surface, so $\mathbb{E}F_{s,t} = 0$, and that $F_{s,t}$ is stationary under $\mathbb{Q}$ in the sense that

$$(F_{\alpha_1}, \dots, F_{\alpha_n}) =_{\mathcal{D}} (F_{S(x)\alpha_1}, \dots, F_{S(x)\alpha_n}), \quad x \in \mathbb{R},\ n \in \mathbb{N},\ \alpha_i \in \mathbb{R}^2.$$

Here $S(x)(s,t) = (s+x, t+x)$ is the translation parallel to the line $s = t$.

Many people have raised such objections to Gaussian term structure models as that they give rise to a positive probability of negative rates, and one might also frown on the assumption that interest rates are stationary. However, as we are only working under a measure equivalent to $\mathbb{P}$, we are not, in fact restricting ourselves to models which are stationary under $\mathbb{P}$, nor are we suggesting that under $\mathbb{P}$ forward rates are necessarily Gaussian. There will be a positive probability of negative rates under $\mathbb{P}$, but this can be arbitrarily small and is not related to the probability of negative rates under $\mathbb{Q}$.

The layout of this paper is as follows: in Section 2 we introduce some convenient notation and prove some basic results; in particular that the covariance structure of the infinitesimal increments is well defined and closely related to the stationary distribution of the forward rate curve. We also prove the intuitive result that if a long rate ($\lim_{t\to\infty} F_{s,t}$) exists for all s with probability one then it is also constant with probability one. (Note that we use 'with probability one' to mean with probability one under $\mathbb{Q}$, but since $\mathbb{Q} \sim \mathbb{P}$, this is equivalent to holding with probability one under $\mathbb{P}$.)

In Section 3 we describe some of the Markov-type properties discussed in Kennedy (1997) and introduce a few new properties. We show how they interrelate and investigate the implications for the covariance structure of F.

Taking as a starting point the attention given in Kennedy (1994) to constructing Gaussian term-structure models via a continuous transformation of a Brownian Sheet, in Section 3.1 we show that it is not very fruitful to impose the extra restriction of stationarity in this case. We will show that the three parameter model is the only Gaussian field with the properties of both stationarity and independent increments to arise from an injective transformation of a Brownian sheet.

2 Notation and preliminary results

In this section we will introduce some notation and prove a few elementary results about stationary Gaussian random fields with independent increments. In particular we will show that if a long rate ($\lim_{t\to\infty} F_{s,t}$) exists then it is constant, and that the whole distribution of the field is determined by the distribution of $\{F_{0,x} : x \in \mathbb{R}\}$. We shall also prove that there is a well defined notion of the 'covariance structure of the infinitesimal increments'.

In the introduction two fundamental assumptions about the forward rate surface $\{F_{s,t} : s \geq t\}$ were introduced: stationarity,

$$(F_{\alpha_1}, \dots, F_{\alpha_n}) =_{\mathcal{D}} (F_{S(x)\alpha_1}, \dots, F_{S(x)\alpha_n}), \quad x \in \mathbb{R},\ n \in \mathbb{N},\ \alpha_i \in \mathbb{R}^2,$$

where $S(x)(s,t) = (s+x, t+x)$ is the translation parallel to the line $s = t$, and independent increments,

$$\sigma(F_{s',t} - F_{s,t} : s' \geq s,\ t \in \mathbb{R}) \ \perp\ \mathcal{F}_s,$$

where $\mathcal{F}_s = \sigma(F_{s',t} : s' \leq s,\ t \in \mathbb{R})$. We recall also that since bond prices (and hence forward rates) initially exist only on the set $H = \{(s,t) \in \mathbb{R}^2 : t \geq s\}$, we extend the range of definition of F from H to $\mathbb{R}^2$ by setting $F_{s,t} = F_{s,s}$ for $s \geq t$.

Remark If F is stationary, the distribution of F is determined by the stationary distribution of the forward-rate curve $\{F_{0,t} : t \in \mathbb{R}\}$. To see this, recall that the independent increments property implies that $\Gamma(s,t,s',t') = c(s \wedge s', t, t')$ for some function c; by stationarity we have

$$\begin{aligned} c(s \wedge s', t, t') &= \operatorname{cov}\left(F_{s\wedge s',t}, F_{s\wedge s',t'}\right) \\ &= \operatorname{cov}\left(F_{0,t-s\wedge s'}, F_{0,t'-s\wedge s'}\right) \\ &= f(t - s\wedge s',\ t' - s \wedge s'), \end{aligned} \tag{2.1}$$

where we define $f(t,t') := \mathbb{C}\mathrm{ov}(F_{0,t}, F_{0,t'})$.

Define the maps $p_1, p_2 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by

$$p_1(s,t,s',t') = t \wedge t' - s \wedge s', \tag{2.2}$$

$$p_2(s,t,s',t') = t \vee t' - s \wedge s', \tag{2.3}$$

and the map $p : \mathbb{R}^2 \times \mathbb{R}^2 \to H$ by $p(s,t,s',t') = (p_1, p_2)$; thus if Γ is the covariance structure of a stationary random field model, then $\Gamma \circ p^{-1}$ is well defined and equals f. We say that a symmetric function $\Phi : X \times X \to \mathbb{R}$ *non-negative definite* on $X \times X$ if for all $\{(a_i, x_i) : i = 1 \dots n\}$, $a_i \in \mathbb{R}$, $x_i \in X$, we have $\sum_{i,j} a_i a_j \Phi(x_i, x_j) \geq 0$. It is well known that any symmetric non-negative definite function on $X \times X$ is the covariance structure of some Gaussian process on X. Thus to fit a stationary Gaussian model to some interest rate data, we might consider the problem of finding a symmetric non-negative definite function $\Gamma : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ and a symmetric non-negative definite function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that $\Gamma = f \circ p$. There is normally a much easier way of selecting a model based on the covariance function of the infinitesimal incerments (see Remark 2 below), but the functions Γ and f will be convenient for investigating Markov properties. A simple result that we will need later is that since F is continuous, Γ, and hence also c and f, are continuous functions.

The usual approach to fitting an interest rate model to data makes use of the covariance structure of the infinitesimal increments, $\lim_{\delta \downarrow 0} \frac{1}{\delta} \mathbb{C}\mathrm{ov}(F_{s+\delta,u} - F_{s,u}, F_{s+\delta,v} - F_{s,v})$, so we will begin by showing that this quantity exists for (Lebesgue) almost all s. Let $(\mathrm{D}f)(x,y)$ be the directional derivative defined by $(\mathrm{D}f)(x,y) = \lim_{\delta \downarrow 0} \frac{1}{\delta}(f(x+\delta, y+\delta) - f(x,y))$,

Proposition 2.1 *For each $t, t' \in \mathbb{R}$,*

$$\tau_s(t,t') \stackrel{\text{def}}{=} \frac{\partial}{\partial s} c(s,t,t') \tag{2.4}$$

exists for (Lebesgue) almost all $s \in \mathbb{R}$. Moreover, if we suppose $s, u, v \in \mathbb{R}$ are such that either $(\partial/\partial s)c(s,u,v)$ or $(\mathrm{D}f)(u-s, v-s)$ exists, then both exist and

$$\tau_s(u,v) = -(\mathrm{D}f)(u-s, v-s) = \lim_{\delta \downarrow 0} \frac{1}{\delta} \mathbb{C}\mathrm{ov}(F_{s+\delta,u} - F_{s,u}, F_{s+\delta,v} - F_{s,v}).$$

Proof For the first part, let $s \leq s'$ and $t, t' \in \mathbb{R}$. From the independent increments property we have

$$\begin{aligned}
\mathbb{V}\mathrm{ar}(F_{s',t} + F_{s',t'}) &= \mathbb{V}\mathrm{ar}(F_{s,t} + F_{s,t'} + (F_{s',t} - F_{s,t}) + (F_{s',t'} - F_{s,t'})) \\
&= \mathbb{V}\mathrm{ar}(F_{s,t} + F_{s,t'}) + \mathbb{V}\mathrm{ar}(F_{s',t} - F_{s,t} + F_{s',t'} - F_{s,t'}) \\
&\geq \mathbb{V}\mathrm{ar}(F_{s,t} + F_{s,t'})
\end{aligned}$$

so we see that $\mathbb{V}\mathrm{ar}(F_{s,t}+F_{s,t'})$ is non-decreasing in s. Similarly we can show that $\mathbb{V}\mathrm{ar}(F_{s,t}-F_{s,t'})$ is also non-decreasing in s. Since $\mathbb{C}\mathrm{ov}(F_{s,t},F_{s,t'})=\frac{1}{4}[\mathbb{V}\mathrm{ar}(F_{s,t}+F_{s,t'})-\mathbb{V}\mathrm{ar}(F_{s,t}-F_{s,t'}]$, $c(s,t,t')$ is of finite variation in s, and hence $(\partial/\partial s)c(s,t,t')$ exists for (Lebesgue) almost all s (Dudley 1989, Section 7.2.7).

Secondly, let $\delta>0$ and consider the covariance structure of the increment between times s and $s+\delta$: by independent increments we have

$$\begin{aligned}\mathbb{C}\mathrm{ov}(F_{s+\delta,u}-F_{s,u},F_{s+\delta,v}-F_{s,v}) &= \mathbb{C}\mathrm{ov}(F_{s+\delta,u},F_{s+\delta,v})-\mathbb{C}\mathrm{ov}(F_{s,u},F_{s,v})\\ &= c(s+\delta,u,v)-c(s,u,v)\end{aligned}$$

and by stationarity

$$\mathbb{C}\mathrm{ov}(F_{s+\delta,u},F_{s+\delta,v})-\mathbb{C}\mathrm{ov}(F_{s,u},F_{s,v})=\mathbb{C}\mathrm{ov}(F_{0,u-s-\delta},F_{0,v-s-\delta})-\mathbb{C}\mathrm{ov}(F_{0,u-s},F_{0,v-s}). \tag{2.5}$$

Dividing throughout by δ and letting $\delta\downarrow 0$ completes the proof.

□

From (2.5) we see that

$$f(u-s-\delta,u-s-\delta)-f(u-s,u-s)=\mathbb{V}\mathrm{ar}(F_{s+\delta,u}-F_{s,u}). \tag{2.6}$$

A straightforward consequence of this is that 'long rates are constant'.

Corollary 2.2 *Suppose $F_{s,\infty}=\lim_{t\to\infty}F_{s,t}$ exists for all s with probability one, then $F_{s,\infty}$ is constant in s with probability one.*

Proof Define $F^*_{s,t}=\sup_{x\in[s,s+1]}(F_{x,t}-F_{s,t})^2$ and $F^*_{s,\infty}=\sup_{x\in[s,s+1]}(F_{x,\infty}-F_{s,\infty})^2$. If $(t_n)_{n=1}^{\infty}$ is any sequence of times tending to infinity, the event $\{F^*_{s,\infty}>\epsilon\}$ is certainly contained in $\{F^*_{s,t_n}\geq\epsilon \text{ i.o.}\}$ by the continuity of F. Now consider the sequence $t_n=n$. Note that $F_{u,n}:u\in[s,s+1]$ is a martingale since it has independent increments and mean zero. By Doob's submartingale inequality applied to the martingale $F_{u,n}:u\in[s,s+1]$ and (2.6),

$$\begin{aligned}\mathbb{Q}[F^*_{s,n}\geq\epsilon] &\leq \epsilon^{-1}\mathbb{V}\mathrm{ar}(F_{s+1,n}-F_{s,n})\\ &= \epsilon^{-1}(f(n-s-1,n-s-1)-f(n-s,n-s)).\end{aligned}$$

As $f(-s,-s)\geq f(m-s,m-s)\geq 0$ for $m\geq 0$, we have $\sum_n\mathbb{Q}[F^*_{s,n}\geq\epsilon]<\infty$. Thus by the first Borel-Cantelli Lemma, $\mathbb{Q}[F^*_{s,n}\geq\epsilon \text{ i.o.}]=0$. Hence $\mathbb{Q}[F^*_{s,\infty}\geq\epsilon]=0$, and with probability one, $F_{s,\infty}$ is constant on $[s,s+1]$. As $\mathbb{R}$ is a countable union of unit intervals, it follows that with probability one, $F_{s,\infty}$ is constant on $\mathbb{R}$. □

Remarks

1. An immediate consequence of (2.6) is that $\mathbb{V}\mathrm{ar}(F_{0,t})$ is non-increasing in t.

2. If the derivative $\mathrm{D}f$ exists, we can write $c(0,x,y) = c(-s,x,y) + \int_{-s}^{0} \tau_u(x,y)\,du$. Then if, for all x and y, the limit $\lim_{s\to\infty} f(x+s,y+s)$ also exists, we can let $s \uparrow \infty$ and deduce $f(x,y) = \lim_{s\to\infty} f(x+s,y+s) + \int_0^\infty \tau_0(x+u,y+u)\,du$. This is of the form

$$f(x,y) = \kappa(x,y) + \int_0^\infty \tau(x+u,y+u)\,du, \tag{2.7}$$

where $\tau : \mathbb{R}^2 \to \mathbb{R}$ is non-negative definite satisfying

$$\int_x^\infty \tau(u,u)\,du < \infty \quad \text{for all } x \in \mathbb{R}, \tag{2.8}$$

and $\kappa : \mathbb{R}^2 \to \mathbb{R}$ is the covariance structure of a stationary Gaussian process on $\mathbb{R}$. Conversely, given any such τ and κ, the definition of f via equation (2.7) gives the covariance structure of a stationary field with independent increments and the Cauchy-Schwarz inequality combined with (2.8) ensure that f is finite.

Thus F is the sum of a field which is constant in s and a field that tends to zero as $t \to \infty$. Note that it is considerably easier to generate a model by specifying τ and κ, than by specifying the pair of symmetric non-negative definite functions Γ and f.

3 Markov Properties

Recall the definition $\mathcal{F}_s = \sigma(F_{u,t} : u \le s, t \in \mathbb{R})$, of the σ-algebra generated by the forward-rates up to time t, and define

$$\mathcal{G}_s := \sigma(F_{s,t} : t \in \mathbb{R})$$
$$\mathcal{H}_s := \sigma(F_{u,t} : u \ge s, t \in \mathbb{R}).$$

The natural interpretation of the Markov condition is that $\mathcal{F}_s$ and $\mathcal{H}_s$ be conditionally independent given $\mathcal{G}_s$. Another Markov property, which is a common feature of one-factor models, is for the short rate, $F_{s,s}$ to be Markov as a one-dimensional process. It is easy to show that the independent increments property (1.1) automatically implies the former, but the latter requires something stronger. In this section, we will consider stronger forms of the Markov property and investigate their consequences for the covariance structure of F. Kennedy (1997) also considers this problem; under his strongest formulation, the covariance structure of a stationary model is determined by just three parameters, and has the form

$$\mathbb{C}\mathrm{ov}(F_{s_1,t_1}, F_{s_2,t_2}) = \sigma^2 \exp\bigl(-\lambda(t_1 \wedge t_2 - s_1 \wedge s_2) - \mu|t_1 \vee t_2 - s_1 \wedge s_2|\bigr), \tag{3.1}$$

Recall that the Brownian Sheet is the zero-mean, continuous Gaussian field on $[0,\infty)^2$ with covariance structure

$$\mathbb{C}\mathrm{ov}(X_{s_1,t_1}, X_{s_2,t_2}) = (s_1 \wedge s_2)(t_1 \wedge t_2).$$

This field satisfies the severe Markov property that for all $s_1 \le s_2 \le s_3$, $t_1 \le t_2 \le t_3$ and all $s_1 \ge s_2 \ge s_3$, $t_1 \ge t_2 \ge t_3$, we have $F_{s_1,t_1} \perp F_{s_3,t_3} \,|\, F_{s_2,t_2}$. Kennedy (1994) demonstrates how continuous injective transformations of the Brownian Sheet can be used to construct interesting random field models; here we will show that any stationary model which arises from a continuous injective transformation of a Brownian Sheet must be the three parameter model with covariance structure given by (3.1).

Before we define our Markov properties, we must introduce some more σ-algebras; define

$$\mathcal{F}^-_{s,t} := \sigma(F_{u,v} : u \le s, v \le t)$$
$$\mathcal{F}^+_{s,t} := \sigma(F_{u,v} : u \le s, v \ge t)$$

and define $\mathcal{G}^\pm_{s,t}$ and $\mathcal{H}^\pm_{s,t}$ similarly.

Definition We say that F has the *SWNE-Markov property* if for all s and t we have $\mathcal{F}^-_{s,t} \perp \mathcal{H}^+_{s,t} \,|\, \sigma(F_{s,t})$. Similarly, we say that F has the *NWSE-Markov property* if for all s and t we have $\mathcal{F}^+_{s,t} \perp \mathcal{H}^-_{s,t} \,|\, \sigma(F_{s,t})$.

Remarks

1. Using the independent increments property, we can replace $\mathcal{H}^+_{s,t}$ with $\mathcal{G}^+_{s,t}$ in the definition of SWNE-Markov and $\mathcal{H}^-_{s,t}$ with $\mathcal{G}^-_{s,t}$ in the definition of NWSE-Markov. An immediately consequence of the SWNE-Markov property is that forward-rates of fixed maturity are Ornstein-Ulhenbeck processes; in particular, the spot rate is Markov.

2. We can think of the NWSE-Markov property as imposing a form of independence between long and short rates; for example, suppose W_t and W'_t are independent Brownian motions and let $B_t = W_t$ for $t \ge 0$, $B_t = W'_{-t}$ for $t \le 0$. The field F, defined by

$$F_{s,t} = (B_s - B_t)\,\mathrm{I}(s \ge t), \tag{3.2}$$

is NWSE-Markov, is zero on $s \le t$ and non-deterministic on $s > t$.

Two related Markov properties are introduced in Kennedy (1997):

Definition If for all $s_1 \le s_2 \le s_3$ and $t_1, t_2 \in \mathbb{R}$,

$$F_{s_1,t_1} \perp F_{s_3,t_2} \,|\, F_{s_2,t_1}, \tag{3.3}$$

we say that F has the *second Markov property (MP2)*. If $F_{0,t}$ is Markov as a one-dimensional process indexed by t, we say that F is *t-Markov*.

Remark From the independent increments property, it is enough to consider the case $s_3 = s_2$ in (3.3).

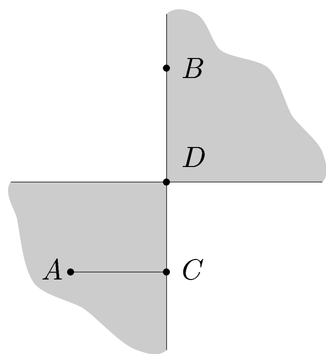
We will split MP2 into the *upper second (MP2.1)* and *lower second (MP2.2)* Markov properties by restricting (3.3) to $t_2 \geq t_1$ and $t_2 \leq t_1$ respectively.

Proposition 3.1

(i) The random field F is SWNE-Markov iff F is t-Markov and satisfies MP2.1.

(ii) The random field F is NWSE-Markov iff F is t-Markov and satisfies MP2.2.

Proof We will only prove the first statement; the proof of the second is very similar. The implication that if F is SWNE-Markov, then it is t-Markov and satisfies MP2.1 is immediate.



Conversely, let $s_1 \leq s_2$, $t_1 \leq t_2 \leq t_3$ and set $A = F_{s_1,t_1}$, $B = F_{s_2,t_3}$, $C = F_{s_2,t_1}$ and $D = F_{s_2,t_2}$. From MP2.1, we have $A \perp B \,|\, C$ and $A \perp D \,|\, C$ so provided $\mathbb{V}\mathrm{ar}(D \,|\, C) > 0$ we have

$$\begin{aligned}\mathbb{C}\mathrm{ov}(A, B \,|\, C, D) &= \mathbb{C}\mathrm{ov}(A, B \,|\, C) - \mathbb{V}\mathrm{ar}(D \,|\, C)^{-1}\mathbb{C}\mathrm{ov}(A, D \,|\, C)\mathbb{C}\mathrm{ov}(B, D \,|\, C) \\ &= 0.\end{aligned}$$

If $\mathbb{V}\mathrm{ar}(D \,|\, C) = 0$, then D is a.s. a function of C and $\mathbb{C}\mathrm{ov}(A, B \,|\, C, D) = \mathbb{C}\mathrm{ov}(A, B \,|\, C) = 0$. We also have

$$\mathbb{C}\mathrm{ov}(A, B \,|\, C, D) = \mathbb{C}\mathrm{ov}(A, B \,|\, D) - \mathbb{V}\mathrm{ar}(C \,|\, D)^{-1}\mathbb{C}\mathrm{ov}(A, C \,|\, D)\,\mathbb{C}\mathrm{ov}(B, C \,|\, D),$$

and $\mathbb{C}\mathrm{ov}(B, C \,|\, D) = 0$ by t-Markovness, implying $\mathbb{C}\mathrm{ov}(A, B \,|\, D) = 0$. (If $\mathbb{V}\mathrm{ar}(C \,|\, D) = 0$, then C is a.s. a function of D; thus $\mathbb{C}\mathrm{ov}(A, B \,|\, C, D) = \mathbb{C}\mathrm{ov}(A, B \,|\, D)$ and $\mathbb{C}\mathrm{ov}(A, B \,|\, D) = 0$ again.) Thus $A \perp B \,|\, D$ and the result follows. □

Recall the definitions of $H = \{(x,y) : y \geq x\}$, and of the map $p : \mathbb{R}^2 \times \mathbb{R}^2 \to H$, given by $p(s_1,t_1,s_2,t_2) = (p_1,p_2)$ where p_1 and p_2 are defined by (2.2) and (2.3), and the fact that if F is stationary with covariance structure Γ, then $\Gamma \circ p^{-1}$ is well defined and equal to f, the covariance structure of $\{F_{0,t} : t \in \mathbb{R}\}$. We now introduce a property which will be central to the discussion of transformations of a Brownian sheet.

Definition A random field F is said to *SWNE-factorise* if for all $(x,y) \in \mathbb{R}^2$, there exists $(s_1,t_1,s_2,t_2) \in p^{-1}(x,y)$ with $s_1 < s_2$, $t_1 < t_2$ and open neighbourhoods, U_1, U_2 of (s_1,t_1), (s_2,t_2) respectively such that for all $\alpha \in U_1$, $\beta \in U_2$, we have

$$\Gamma(\alpha,\beta) = \Gamma_1(\alpha)\Gamma_2(\beta)$$

for some functions $\Gamma_1 : U_1 \to \mathbb{R}$, $\Gamma_2 : U_2 \to \mathbb{R}$. A field is said to *NWSE-factorise* if it SWNE-factorises after reflection in the horizontal axis.

Theorem 3.2 *Suppose that F is a stationary random field model which is not deterministic everywhere. If F SWNE-factorises then for some constant μ,*

$$f(x,y) = \exp\bigl(g(x \wedge y) - \mu|x-y|\bigr), \qquad -2\mu \leq \frac{g(y)-g(x)}{y-x} \leq 0. \tag{3.4}$$

while if F NWSE-factorises then

$$f(x,y) = \exp\bigl(g(x \vee y) - \mu|x-y|\bigr), \qquad \frac{g(y)-g(x)}{y-x} \leq 2\mu \wedge 0, \tag{3.5}$$

where we allow g to take the value $-\infty$ in this case.

Proof We will break down the proof into several steps, but first observe that the restrictions on g and μ follow from (i) the fact that $f(x,x)$, and hence $g(x)$ is non-increasing in x, and (ii) the Cauchy-Schwarz inequality applied to $f(x,y)$.

Step 1 Consider the case $\Gamma > 0$. We will first show that f has the correct form locally. Suppose F SWNE-factorises (the NWSE case will be very similar). Let $x < y$ and $(s_1,t_1,s_2,t_2) \in p^{-1}(x,y)$ with $s_1 < s_2$, $t_1 < t_2$. Let U_1 and U_2 be open discs with centres $(s_1,t_1),(s_2,t_2)$ respectively, such that for all $\alpha \in U_1$, $\beta \in U_2$

$$\alpha_x < \beta_x, \quad \alpha_y < \beta_y. \tag{3.6}$$

Now let $\alpha \in U_1$, $\beta \in U_2$ be arbitrary. By SWNE-factorisation and Remark 2 we have

$$\begin{aligned}\Gamma(\alpha,\beta) &= \Gamma_1(\alpha)\Gamma_2(\beta) \\ &= f(\alpha_y - \alpha_x, \beta_y - \alpha_x),\end{aligned} \tag{3.7}$$

so $\Gamma_2(\beta)$ cannot depend on β_x. Since $\Gamma > 0$, w.l.o.g. Γ_1 and Γ_2 are both positive. Setting $s(\alpha) = \alpha_y - \alpha_x$ and writing $\Gamma_2(\beta_y)$ for $\Gamma_2(\beta)$, we have

$$\begin{aligned}\log\Gamma(\alpha,\beta) &= \log f(s, \beta_y - \alpha_x) \\ &= \log\Gamma_1(\alpha_x, \alpha_x + s) + \log\Gamma_2(\beta_y).\end{aligned} \tag{3.8}$$

Using that fact that for any continuous functions a, b and c satisfying $a(x)+b(y) = c(x-y)$ on a connected open subset of $\mathbb{R}^2$, c must be linear, condition (3.8) implies that $\log f(s, \beta_y - \alpha_x)$ is linear in $\beta_y - \alpha_x$ for each s. Now let r be the radius of U_1 and V be the open disc with centre (s_1, t_1) and radius $\frac{1}{\sqrt{2}}r$. Let $\beta^* = (s_2, t_2)$, and for $\alpha \in V$, let $\alpha^* = (s_1, s_1 + s(\alpha))$. Restricting attention to $\alpha \in V$, we have $\alpha^* \in U_1$ and $s(\alpha) = s(\alpha^*)$ so

$$\log\Gamma_1(\alpha) + \log\Gamma_2(\beta) = \log f(\alpha_y - \alpha_x, \beta^*_y - \alpha^*_x) - \mu(s)[(\beta_y - \alpha_x) - (\beta^*_y - \alpha^*_x)]$$

for some function $\mu(s)$. Considering the dependence of both sides on β_y, and noting that s depends on α but not β, we see that $\mu(s)$ must independent of s. Thus we have

$$\begin{aligned}\log\Gamma(\alpha,\beta) =\ & \log f(\alpha_y - \alpha_x, \beta^*_y - \alpha^*_x) + \mu(\beta^*_y - \alpha^*_x) \\ & - \mu(\alpha_y - \alpha_x) - \mu(\beta_y - \alpha_y),\end{aligned} \tag{3.9}$$

for some constant μ. Defining

$$g(s) := \log f(s, \beta^*_y - \alpha^*_x) + \mu(\beta^*_y - \alpha^*_x) - \mu s$$

and noting that $\alpha_y - \alpha_x < \beta_y - \alpha_x$ in (3.7), we see from (3.9) that f has the correct form on $p(V \times U_2)$.

Since μ is uniquely determined by $f|_{p(V\times U_2)}$, so too is the function $g|_{p_1(V\times U_2)}$. As the set $\{(x,y) : x < y\}$ is connected, f must take the required form on the whole of $\{(x,y) : x < y\}$, for some constant μ and function g. As f is continuous and symmetric, it has the required form on all of $\mathbb{R}^2$. The proof in the NWSE case is identical, except that we replace (3.6) with $\alpha_x < \beta_x$, $\alpha_y > \beta_y$ and now have $\alpha_y - \alpha_x > \beta_y - \alpha_x$.

Step 2 Suppose that $\Gamma(\alpha,\alpha) = 0$ for some $\alpha \in \mathbb{R}^2$. It follows that $F_{s,t}$ is deterministic for $t - s \geq \alpha_y - \alpha_x$. To see this, first note that stationarity implies that $F_{S(x)\alpha}$ is deterministic for all $x \in \mathbb{R}$. Now let $t - s \geq \alpha_y - \alpha_x$. By the independent increments property, $F_{S(t-\alpha_y)\alpha} - F_{s,t} \perp F_{s,t}$, thus $F_{s,t}$ is deterministic.

Step 3 Now consider the case of general Γ. Let $r \in [-\infty, \infty]$ be the unique r such that F is deterministic on the set $Z = \{(s,t) : t - s \geq r\}$ and $\mathbb{V}\mathrm{ar}(F) > 0$ on $Z^c = \{(s,t) : t - s < r\}$. If $r = -\infty$ then F is deterministic everywhere, a case excluded in the statement of the theorem, so suppose $r > -\infty$. We know that $f = 0$ on $p(Z \times \mathbb{R}^2)$ and $p(\mathbb{R}^2 \times Z)$, so we now consider the form of f on $p(Z^c \times Z^c)$. Let $A = \{(\gamma, \gamma') \in Z^c \times Z^c : \Gamma(\gamma, \gamma') > 0\}$, which is open, and non-empty since $r > -\infty$. We note that $\{(\alpha, \alpha) : \alpha \in Z^c\}$ is a connected subset of A, and let B be the connected component of A containing $\{(\alpha, \alpha) : \alpha \in Z^c\}$. It will turn out that $p(B) = p(Z^c \times Z^c)$. As p is a continuous open mapping, $p(B)$ is open and connected. Our aim is to apply Step 1 to $f|_{p(B)}$, but we must check that we can choose U_1 and U_2 such that $U_1 \times U_2 \subseteq B$. If $(x, y) \in p(B)$, say $p(s_1, t_1, s_2, t_2) = (x, y)$, then $(s_1 \wedge s_2, t_1, s_1 \wedge s_2, t_2) \in B$. As B is open, we can find neighbourhoods U_1, U_2 such that $U_1 \times U_2 \subseteq B$ as required. Applying Step 1 to $f|_{p(B)}$, we deduce that f takes the required form on $p(B)$. Finally let $(\alpha, \beta) \in \partial B$, which cannot be empty unless $B = \mathbb{R}^2 \times \mathbb{R}^2$ when $\Gamma > 0$ and we are done by Step 1. We will show that $p(\alpha, \beta) \notin p(Z^c \times Z^c)$. Pick a sequence $\{(\alpha_x^n, \alpha_y^n, \beta_x^n, \beta_y^n)\}$ of points in B such that $(\alpha^n, \beta^n) \to (\alpha, \beta)$. If we are dealing with the SWNE case set

$$y_n = \alpha_y^n \wedge \beta_y^n - \alpha_x^n \wedge \beta_x^n,$$

and otherwise set

$$y_n = \alpha_y^n \vee \beta_y^n - \alpha_x^n \wedge \beta_x^n,$$

so that

$$\Gamma(\alpha^n, \beta^n) = \exp\bigl(g(y_n) - \mu|\beta_y^n - \alpha_y^n|\bigr).$$

As $(\alpha, \beta) \notin A$, Γ is continuous and $\mu > -\infty$ is constant throughout B, it follows that as $n \to \infty$, $g(y_n) \to -\infty$. Thus we must be dealing with NWSE rather than SWNE factorisation. Since $p_2(\alpha^n, \beta^n) = y_n$ and $\Gamma(\alpha^n, \beta^n) > 0$, we have $y_n < r$, so $(0, y_n, 0, y_n) \in B$. Now

$$\begin{aligned} \mathbb{V}\mathrm{ar}\,(F_{0,\alpha_y \vee \beta_y - \alpha_x \wedge \beta_x}) &= \lim_{n \to \infty} \mathbb{V}\mathrm{ar}\,(F_{0,y_n}) \\ &= \lim_{n \to \infty} \exp\bigl(g(y_n)\bigr) \\ &= 0. \end{aligned}$$

Thus we must have $\alpha_y \vee \beta_y - \alpha_x \wedge \beta_x \geq r$, so $p(\alpha, \beta) \notin p(Z^c \times Z^c)$, and so $p(\partial B) \subseteq H - p(Z^c \times Z^c)$. Since $p(B)$ is open and connected and $p(B) \subseteq p(Z^c \times Z^c)$ (which is also open and connected) we must have $p(B) = p(Z^c \times Z^c)$. Thus we have established that f has the correct form on all of $p(Z^c \times Z^c) = \{(x, y) : x \leq y < r\}$. Defining $g(x) = -\infty$ for $x \geq r$ gives the correct form for f on the whole of $\mathbb{R}^2$. □

Remarks

1. If F has either factorisation property, Γ is non-negative and interest-rates are positively correlated, a property usually observed in real interest-rates. In the SWNE-Markov case, Γ is strictly positive.

2. When F has both factorisation properties, f has the equivalent forms

$$\begin{aligned} f(x,y) &= \sigma^2 \exp\bigl(-\lambda x \wedge y - \mu|x-y|\bigr) \\ &= \sigma^2 \exp\bigl(-\lambda x \vee y - (\mu-\lambda)|x-y|\bigr) \end{aligned}$$

where $0 \leq \lambda \leq 2\mu$. This is the three parameter model of Kennedy (1997).

A consequence of the form of f given by Theorem 3.2 is that each factorisation property is equivalent to the corresponding Markov property:

Corollary 3.3 *Let F be a stationary random field model.*
(i) The field F is SWNE-Markov iff F SWNE-factorises.
(ii) The field F is NWSE-Markov iff F NWSE-factorises.

Proof We will only prove the first statement; the proof of the second is virtually identical. Let $s_1 < s_2 < s_3$, $t_1 < t_2 < t_3$ be arbitrary. To show F is SWNE-Markov, we must show that $F_{s_1,t_1} \perp F_{s_3,t_3} \,|\, F_{s_2,t_2}$, i.e., that

$$\Gamma\bigl((s_1,t_1),(s_2,t_2)\bigr)\Gamma\bigl((s_2,t_2),(s_3,t_3)\bigr) = \Gamma\bigl((s_1,t_1),(s_3,t_3)\bigr)\Gamma\bigl((s_2,t_2),(s_2,t_2)\bigr).$$

Using the form of f given by Theorem 3.2 we see both sides reduce to

$$\exp\bigl(g(t_1-s_1) + g(t_2-s_2) + \mu(t_3-t_1)\bigr).$$

Conversely, suppose $x < y$; set $r = (y-x)/4$ and choose U_1, U_2 as the open discs, radii r, with centres $(0,x), (y-x,y)$ respectively. Let γ be the point $(3r, x+r)$. For $\alpha \in U_1, \beta \in U_2$ we have $F_\alpha \perp F_\beta \,|\, F_\gamma$ by the SWNE-Markov property. Thus

$$\Gamma(\alpha,\beta) = \mathbb{V}\mathrm{ar}(F_\gamma)^{-1}\,\Gamma(\alpha,F_\gamma)\,\Gamma(\beta,F_\gamma),$$

so F SWNE-factorises. (If $\mathbb{V}\mathrm{ar}(F_\gamma) = 0$, it is easy to show that $\Gamma(\alpha,\beta) = 0$ for all $\alpha \in U_1$, $\beta \in U_2$ (see Step 2 of the proof of Theorem 3.2) so F trivially SWNE-factorises.) $\square$

Remark When F has both Markov properties, f has the form

$$f(x,y) = \sigma^2 \exp\bigl(-\lambda x \wedge y - \mu|x-y|\bigr).$$

Kennedy (1997) proves the equivalent result that a stationary t-Markov random field model satisfying MP2 has a covariance structure of this form. He also observes that this covariance structure arises as a continuous transformation of the Brownian Sheet

$$F_{s,t} = \sigma e^{-\mu t} X_{e^{\lambda s}, e^{(2\mu-\lambda)t}}.$$

In the next section we will show that this is the only continuous injective transformation of a Brownian Sheet to give rise to a stationary random field model.

3.1 Transformations of a Brownian sheet

Recall that the Brownian Sheet is the continuous Gaussian field on $[0,\infty)^2$ with mean zero and covariance structure

$$\mathbb{C}\mathrm{ov}(X_{s_1,t_1}, X_{s_2,t_2}) = (s_1 \wedge s_2)(t_1 \wedge t_2)$$

(see Adler (1981) or Rogers & Williams (1994)). In this section, we will show that the only stationary random field model which arises from an injective transformation of a Brownian Sheet, is the three parameter model shown in (3.1). Throughout this section, F will denote a stationary random field model of the form

$$F_{s,t} = K_{s,t} W_{\phi(s,t)} \tag{3.10}$$

where $K : \mathbb{R}^2 \to \mathbb{R}$ and $\phi : \mathbb{R}^2 \to [0,\infty)^2$ are continuous functions, and ϕ is injective. We let ϕ_x and ϕ_y denote the coordinate projections of ϕ, so

$$\phi(s,t) = (\phi_x(s,t), \phi_y(s,t)).$$

Lemma 3.4 *If $\Gamma(\alpha,\beta) = 0$ for some α, β, then $F(\gamma) = 0$ a.s. for all $\gamma \in \mathbb{R}^2$.*

Proof If $\Gamma(\alpha,\beta) = 0$ then either at least one of K_α and K_β is zero, or at least one of $\phi(\alpha)$ and $\phi(\beta)$ lies on the coordinate axes. Therefore, w.l.o.g., we may assume that $\mathbb{V}\mathrm{ar}(F_\alpha) = 0$ implying that the field is zero a.s. on the diagonal upper half plane through α (see Step 2 in the proof of Theorem 3.2). Now suppose that there exists $\gamma \in \mathbb{R}^2$ with $\mathbb{V}\mathrm{ar}(F_\gamma) \neq 0$. Let η be the point on $\{S(y)(\alpha) : y \in \mathbb{R}\}$ closest to γ, let $\gamma' = S(2(\eta_y - \gamma_y))\gamma$ and $\zeta = S(\eta_y - \gamma_y)\eta$. Since $F_\zeta = 0$ a.s. (by stationarity) and $F_\gamma \perp (F_{\gamma'} - F_\zeta)$, we have $F_\gamma \perp F_{\gamma'}$. Thus $\Gamma(\gamma,\gamma') = 0$, so by the previous argument, at least one of $\mathbb{V}\mathrm{ar}(F_\gamma)$ and $\mathbb{V}\mathrm{ar}(F_{\gamma'})$ is 0. Stationarity implies that both these variances are equal, and we conclude $\mathbb{V}\mathrm{ar}(F_{\gamma'}) = \mathbb{V}\mathrm{ar}(F_\gamma) = 0$, a contradiction. □

To exclude this trivial case, we will assume from now on that $\Gamma > 0$. We will now exploit the special Markov structure of the Brownian Sheet to show that F both SWNE and NWSE-factorises. Introduce the notation $T(d)$ for the translation parallel through a distance d parallel to the x-axis, $T(d)(s,t) = (s+d,t)$.

Theorem 3.5 *The field F both SWNE and NWSE-factorises.*

Proof We break up the proof in to several pieces.
Step 1 We first show that F SWNE-factorises. Let $x < y$, $\epsilon > 0$, and define

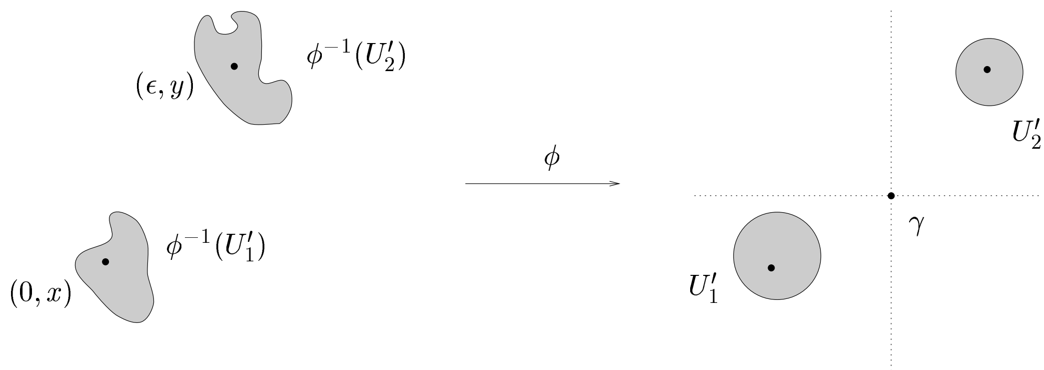
$$\mathrm{cro}(u,v) := \{(s,t) : s = u \text{ or } t = v\}$$

to be the 'cross' formed by the union of the horizontal and vertical lines through (u, v). Let $\gamma = \frac{1}{2}(\phi(0,x) + \phi(\epsilon, y))$. Note that either both $\phi(0,x)$ and $\phi(\epsilon, y)$ are contained in $\mathrm{cro}(\gamma)$ or neither is, so we have two cases to consider.

Case 1a If $\phi(0,x), \phi(\epsilon,y) \notin \mathrm{cro}(\gamma)$, we consider the situation when

$$\phi_x(0,x) < \phi_x(\epsilon,y), \qquad \phi_y(0,x) < \phi_y(\epsilon,y),$$

(the other cases can be handled in a similar way). Let U_1', and U_2' be open neighbourhoods of $\phi(0,x)$, $\phi(\epsilon,y)$ respectively which do not intersect $\mathrm{cro}(\gamma)$.



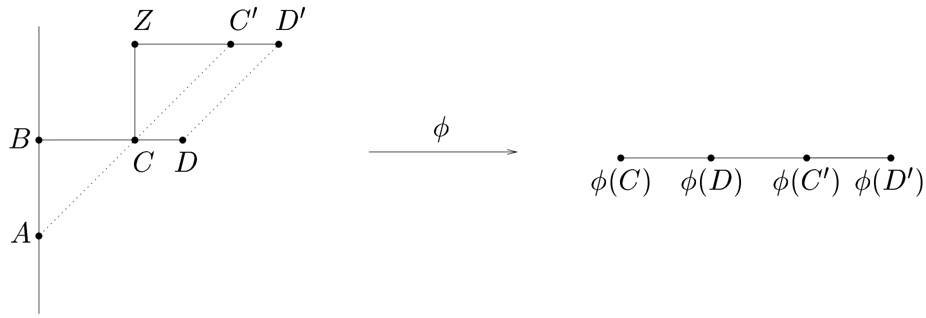
Now set $U_1 = \phi^{-1}(U_1')$, $U_2 = \phi^{-1}(U_2')$. By the Markov properties of the Brownian Sheet, we have

$$F_\alpha \perp F_\beta \,|\, X_\gamma \qquad \text{for } \alpha \in U_1,\, \beta \in U_2.$$

Thus F SWNE-factorises in a neighbourhood of (x, y).

Case 1bi Suppose $\phi(0,x), \phi(\epsilon,y) \in \mathrm{cro}(\gamma)$, but $\phi(T(t)(0,y)) \notin \mathrm{cro}(\phi(0,x))$ for some $t > 0$. Since $p((0,x), T(t)(0,y)) = (x,y)$ for all $t > 0$, a similar argument to the one used in Case 1a shows that F SWNE-factorises near (x, y).

Case 1bii Now suppose that $\phi(\{T(t)(0,y) : t > 0\}) \subseteq \mathrm{cro}(\phi(0,x))$. Let $A = (0,x)$, $B = (0,y)$, $C = (y-x, y)$ and $D = (\frac{3}{2}(y-x), y)$. Since ϕ is injective, $\phi(C) \neq \phi(A)$, and by continuity, $\phi(C)$ and $\phi(D)$ lie in the same 'branch' of $\mathrm{cro}(\phi(A)\big)$. For all $s, t \geq 0$, $p(S(s)A, T(t)S(s)B) = (x,y)$. Thus, either F SWNE-factorises by an argument similar to Case 1bi, or $\phi(\{T(t)S(s)B : t \geq 0\}) \subseteq \mathrm{cro}(\phi(S(s)A))$ for all $s \geq 0$. As $S(s)A$, $S(s)C$ and $S(s)D$ are distinct and ϕ is injective, $\phi(S(s)A)$, $\phi(S(s)C)$ and $\phi(S(s)D)$ are also distinct. From the continuity of ϕ, we deduce that $\phi(S(s)C)$ and $\phi(S(s)D)$ lie in the same branch of $\mathrm{cro}(\phi(S(s)A))$ for every $s \geq 0$.



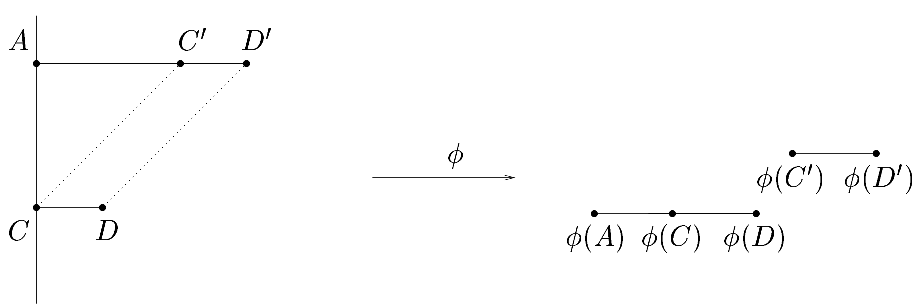
Now choose $s = y - x$, and let $C' = S(y-x)C$ and $D' = S(y-x)D$. As $S(y-x)A = C$, continuity of ϕ implies that $\phi(C)$, $\phi(D)$, $\phi(C')$ and $\phi(D')$ are collinear; in addition $\overrightarrow{\phi(C)\phi(D)}$ points in the same direction as $\overrightarrow{\phi(C')\phi(D')}$. Finally let $Z = (C_x, C_y + y - x)$. We have $F_D \perp F_{D'} \,|\, F_{C'}$, so $\phi(C') \in [\phi(D), \phi(D')]$, $F_C \perp F_{C'} \,|\, F_Z$, so $\phi(Z) \in [\phi(C), \phi(C')]$, and $F_Z \perp F_D \,|\, F_C$, so $\phi(C) \in [\phi(Z), \phi(D)]$, which imply $\phi(C) = \phi(Z)$. Hence ϕ is not injective, a contradiction.

Step 2 We now show NWSE-factorisation (the first two cases are very similar to Cases 1a and 1bi above). Let $\epsilon > 0$ and consider the points $\phi(\epsilon, x)$ and $\phi(0, y)$. Let $\gamma = \frac{1}{2}(\phi(\epsilon, x) + \phi(0, y))$.

Case 2a If $\phi(\epsilon, x), \phi(0, y) \notin \mathrm{cro}(\gamma)$ we can prove that F NWSE-factorises using a very similar argument to Case 1a.

Case 2bi Suppose $\phi(0, x), \phi(\epsilon, y) \in \mathrm{cro}(\gamma)$, but $\phi(T(t)(0, x)) \notin \mathrm{cro}(\phi(0, y))$ for some $t > 0$. Since $p(T(t)(0, x), (0, y)) = (x, y)$ for all $t > 0$, the argument of Case 1bi shows that F NWSE-factorises near (x, y).

Case 2bii Now suppose $\phi(\{T(t)(0, x) : t > 0\}) \subseteq \mathrm{cro}(\phi(0, y))$. Let $A = (0, y)$, $C = (0, x)$ and $D = (\frac{1}{2}(y - x), x)$ and note that $\phi(C)$ and $\phi(D)$ lie in the same branch of $\mathrm{cro}(\phi(A))$.



Since $p(S(s)A, T(t)S(s)C) = (x, y)$ for all $s, t > 0$, either F NWSE-factorises by the argument of Case 2bi or $\phi(\{T(t)S(s)C : t \geq 0\}) \subseteq \mathrm{cro}(\phi(S(s)A))$ for all $s > 0$. As ϕ is continuous and injective, $\phi(A)$, $\phi(C)$ and $\phi(D)$ are distinct and collinear. Let $C' = S(y-x)C$ and $D' = S(y-x)D$. As ϕ is continuous, $\overrightarrow{\phi(C)\phi(D)}$ is parallel to and points in the same direction as $\overrightarrow{\phi(C')\phi(D')}$. Finally, observe that $F_D \perp F_{D'} \,|\, F_{C'}$, $F_A \perp F_D \,|\, F_C$ and $F_C \perp F_{C'} \,|\, F_A$. Together these imply that $\phi(C) = \phi(A)$. Hence ϕ is not injective, giving a contradiction. □

Corollary 3.6 *The field F has the same law as*

$$\sigma e^{-\mu t} W_{e^{\lambda s}, e^{(2\mu-\lambda)t}} \tag{3.11}$$

where $0 \le \lambda \le 2\mu$ and W is a Brownian Sheet.

Proof Since F is NWSE-Markov it NWSE-factorises, by Corollary 3.3, and similarly, since it is SWNE-Markov, it SWNE-factorises. Thus by Remark (ii), on page 12, it has a covariance structure of the form

$$f(x,y) = \sigma^2 \exp\bigl(-\lambda(t_1 \wedge t_2 - s_1 \wedge s_2) - \mu|t_1 \vee t_2 - s_1 \wedge s_2|\bigr), \qquad 0 \le \lambda \le 2\mu,$$

where $f(x,y) = \mathbb{C}\mathrm{ov}(F_{0,x}, F_{0,y})$. But this is also the covariance structure of the field given by (3.11), so the result follows. □

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