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# Risk Profiling Defined Benefit Pension Schemes

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# Abstract

This paper describes a dynamic stochastic optimization model of strategic asset liability management useful for advising under-funded defined benefit pension schemes on best practice for returning the fund to solvency and long term stability. We present an overview of the dynamic stochastic programming techniques involved and briefly describe the nature of Pioneer Investment's proprietary CASM simulator from which the asset class returns and pension scheme liabilities are generated. The stochastic optimization model is set out precisely and its solution using linear programming discussed. To illustrate the approach, two examples of defined benefit schemes using simple conservative fund liability models are presented. The optimal dynamic asset allocations of these examples reflect the motivation of second generation liability driven investment schemes. Although the final salary scheme models used in our examples are simple, more complex models can be incorporated into the system described with little extra effort. Most actuarial assessments used in practice can be modelled for this purpose.

# Risk Profiling Defined Benefit Pension Schemes

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Recent market turbulence, falling valuations of equity and property and lower long term bond yields have re-highlighted the problems of pension fund deficits. Since the start of 2008 some of the worst affected UK pension schemes have lost considerable amounts of capital and the substantial falls in interest rates have lowered discount factors to result in increased liabilities across the board. The total funding situation of UK defined benefit (DB) plans swung from a surplus of £130.4 billion in June 2007 to a £194.5 billion deficit in December 2008 (ONS, 2008b). Amongst the worst affected companies have been BT, Northern Foods and AGA Foodservice.

The international situation is varied, owing to different regulatory environments and historical backgrounds in different countries. Countries which have had mandatory pension funds for many years exhibit the largest pension fund totals with respect to the size of their economies (Clark *et al*., 2006). This is most evident in the Netherlands and Sweden where the ratios of assets to liabilities (coverage ratios) are highest. The UK, US, Ireland, Germany and Austria are lagging in terms of coverage and a third group, including France, Italy, Turkey, China and India, have very low coverage ratios. Accounting standards also affect the picture; standards such as FTK in Holland, FRS 17 in the UK and the international IAS 19 all require liabilities to be discounted on a corporate bond yield equivalent, but they differ in terms of practice, and changing the discount rate to a risk free rate—which would increase liabilities even more—has been mooted in a number of regions.

In considering the risks pension funds face we must look at the fund in its entirety. On the liability side funds are exposed to interest rates, inflation and mortality (Babel *et al*., 2008). On the asset side they face exposure to risk in different markets such as fixed income, equity and credit (Clark *et al.*, 2006). Hence they operate in a multi factor environment, exposed to a wide range of asset classes and liabilities which also have varying degrees of correlation amongst themselves. Because of this, the worst situation for pension funds is falling asset prices and lower bond yields as low yields lead to less discounting of future liabilities.

Approaches to providing solutions to the pension fund problem have developed over the last three to five years (see e.g. Deutsche Bank, 2005). In so-called 'first generation' products, pension funds attempted to hedge the interest rate and inflation components of their exposure using a mixture of bonds, interest rate swaps and inflation swaps. Funds invest in a variety of solutions, mostly individually on a bespoke basis (see e.g. Goldman Sachs, 2008) but also for some smaller schemes on a pooled fund basis, both of which involve cashflow matching and multi-year buckets with a single instrument in each bucket.

Whilst this has had the effect of hedging unwanted interest rate and inflation volatility, it also hedged 'good' volatility (see Dempster *et al*., 2007a, 2008) by locking in the funding

ratio. The main consequence of single factor liability hedging, therefore, is that it crystallises any existing deficit (Waring & Siegel, 2007) and immunizes the scheme from good as well as bad volatility. This may be acceptable in schemes where the funding ratio is already very high but it can be a problem for under-funded schemes. The most serious consideration for any liability hedging solution is the level of current long term yields. When real bond yields are at a historic low locking-in at this point in the yield cycle may be particularly undesirable.

Recent approaches to the problem are more holistic and attempt to manage liability exposure whilst seeking to add value within a given risk budget. Second generation solutions tend to focus on maximising the excess return above a liability benchmark at some specified level of risk (Babel *et al*., 2008).

In this paper we will present a new type of solution for defined benefit pension schemes. The approach we adopt here uses Monte Carlo generation of asset returns and prices and the liabilities of a defined benefit pension scheme over a wide range of economic conditions. We employ dynamic stochastic optimization to determine the optimal asset allocation and employer contribution rates which will enable the scheme to achieve a desired funding ratio within a give time horizon while respecting the trustees' risk appetite.

Our approach is *dynamic* and is based on an active asset allocation strategy with periodic portfolio rebalancing. We view this as superior to a sequential *static* Markowitz surplus-type strategy for several reasons, such as the dynamic nature of the liabilities and the fact that dynamic asset allocation can cope with the flow of capital in and out of the scheme in the form of contributions and benefit payments. Also static asset allocation relies on Gaussian return distributions for assets whereas dynamic asset allocation based on Monte Carlo is free to adopt any return distributions appropriate.

The paper is organised as follows. Section 2 briefly discusses the concepts behind the stochastic optimization techniques employed in the paper. An overview of asset return and liability generation using CASM, Pioneer Investments' proprietary corporate simulator, are summarised in Sections 3 and 4. In Section 5 the multiobjective dynamic stochastic optimization problem of restoring a DB pension scheme to a target coverage ratio is formulated and solved. Sections 6 give results for two simplified example schemes – one under-funded and the other well-funded. Section 7 concludes.

**OVERVIEW OF DYNAMIC STOCHASTIC PROGRAMMING**

Our approach to pension fund management uses dynamic stochastic programming to select allocations that are optimal with respect to fund liabilities and suitable measures of underfunding risk (Dempster *et al.*, 2003, 2006, 2007b). *Dynamic stochastic programming* (DSP) involves simulating economic factors, asset returns and liabilities forward over a number of *scenarios*. Using simulated market scenarios allows any distribution to be used for each of the asset returns or liabilities according to the model for the dynamics of these quantities.

The scenarios are arranged in a tree structure (shown in Exhibit 1) and decisions are made at points where the tree branches. Each of these decisions is optimal with respect to the all the simulated evolutions of the asset returns and liabilities that could occur after

the decision point. As the *root node* corresponds to the present moment, the decision made there is the one that is to be implemented. This means that the root node decision is *robust* with respect to all generated future scenarios representing future market conditions.
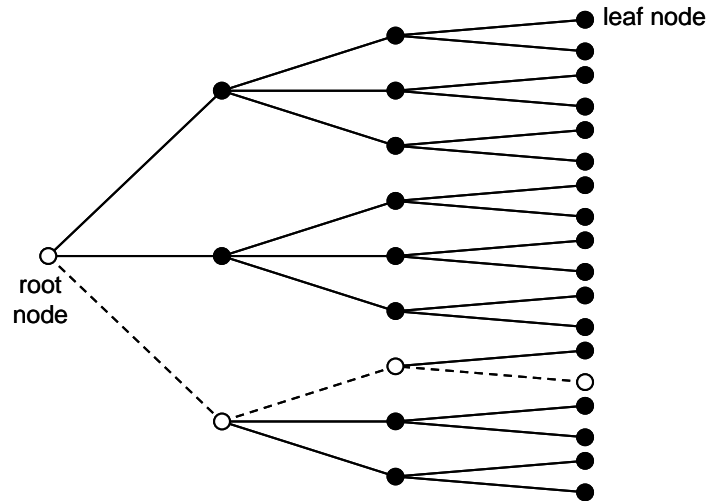


**EXHIBIT 1  A Scenario Tree with a Single Scenario Shown Dashed**

In DSP scenarios are used as a way of representing current uncertainty about the future. Since only the decision at the root node is to be implemented, other decisions further in the (simulated) future can be viewed simply as a "what-if" model of rational behaviour at those future points in time. In real applications, future decisions are made by re-running the simulation and optimization in the future after incorporating the further information that has become available through the passage of time. In the DSP Monte Carlo approach used here all scenarios are equally weighted, meaning that the probability of any particular one is inversely proportional to the total number of scenarios at the particular point in time being considered. To enhance the computational efficiency of solution it is possible to generate the tree based on the *expected value of perfect information* at each of the decision points (Dempster, 2006), increasing the branching of the tree to the planning horizon specifically in areas that have more influence on the objective function.

Clearly, a crucial step in the DSP approach is the forward simulation of the quantities of interest, here economic factors, asset returns and liabilities. Thus a requirement of the scenario generator in the present context is that it is possible to generate scenarios corresponding to a wide range of market conditions for both asset returns and liabilities. In the subsequent sections we describe the simulation of the quantities used in our model.

**ASSET RETURN SIMULATION**

Asset simulation is based on Pioneer Investments' proprietary corporate simulator CASM (Cascade Asset Simulation Model). Cascade simulators for asset price return simulation were introduced by Mulvey (1996). Another implementation for Pioneer Investments is given in Dempster *et al.* (2003) and Arbileche & Dempster (2005). In this implementation the complex nonlinear relationship between parameters of the major

currency region models is approximated by analytical expressions describing the drift parameters and the currency regions are linked by exchange rates.

CASM is a modification and extension of these simulators reflecting the overall view of Pioneer Investments and linking different risk factors across divisions of the firm.

**CASM**

CASM is based on a hierarchical econometric model with EU *GDP growth* and *inflation* considered as the basic driving forces from which further economic factors, such as interest rate term structure and asset prices, are derived, as shown in Exhibit 2. The economic and financial risk factors are modelled with parameters and correlations fitted to quarterly data in a three level 'cascade' fashion:

Level 1: Macro factors: GDP cycle, GDP trend, oil cycle, real short rates

Level 2: Cyclical financial variables and term structure

Level 3: Equities, credit spreads, FX.

The pension scheme asset classes used in the examples of Section 6 are a selection proxied by nominal and inflation-linked bond indices of various durations, a money market fund index and an equity index. For this set of asset classes the key quantities that determine their returns are interest rates and term structure, inflation and equity performance.
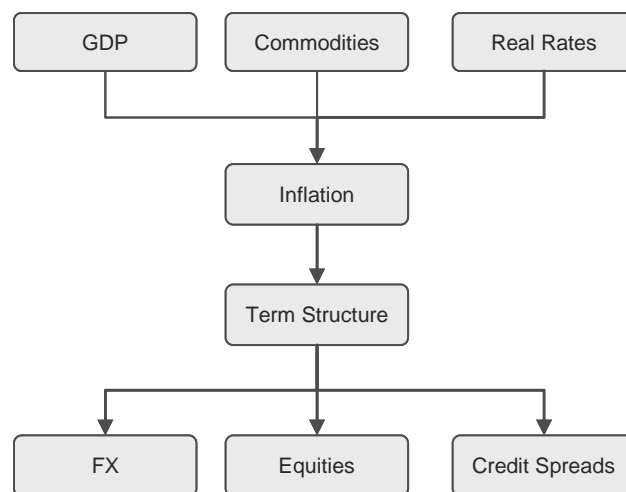


**EXHIBIT 2  Cascade Asset Simulation Model (CASM)**

The asset classes in Exhibit 3 were simulated for potential investment using CASM. The complex nature of the nonlinear interaction between asset returns and economic variables in CASM result in dynamic interdependencies between the variables. For each asset class a proxy index of similar instruments (see Exhibit 3) was used to assess the quality of the simulated scenarios.

| Asset class | Code | Index |
|---|---|---|
| Money Market | EUR300M | 3 month EURIBOR rate |
| 1-3 year duration Euro bonds | EUM1TR | Bloomberg 1-3 year Euro bond market tracker |
| 3-5 year Euro bonds | EUM2TR | Bloomberg 3-5 year Euro bond market tracker |
| 5-7 year Euro bonds | EUM3TR | Bloomberg 5-7 year Euro bond market tracker |
| 7-10 year Euro bonds | EUM4TR | Bloomberg 7-10 year Euro bond market tracker |
| 10+ year Euro bonds | EUM5TR | Bloomberg 10+ year Euro bond market tracker |
| 1-3 year inflation indexed bonds | BEIG1T | Barclays Euro Govt Inflation linked bond tracker for all maturities |
| 5 year inflation indexed bonds | BEIG2T | Barclays Euro Govt Over 5 Year |
| 5-7 year inflation indexed bonds | BEIG3T | Barclays Euro Govt 1-10 Year |
| 10+ year inflation indexed bonds | BEIG4T | Barclays Euro Govt Over 10 Year |
| 15+ year inflation indexed bonds | BEIG5T | Barclays Euro Govt Over 15 Year |
| European equity | DJST | Dow Jones EuroStoxx 50 stock index |

**EXHIBIT 3  Asset Classes and Related Indices**

To illustrate the complex nature of the output from CASM, forward simulations of the corresponding asset class returns have been performed quarterly from Q1 2006 to Q4 2014.  Representative scenarios are shown in Exhibit 4 in the form of fan charts.  These charts illustrate the distribution of the scenarios across time as a fan plot, with the intensity of the shading declining through quantiles either side of the median (so that, for example, the 20th and 80th percentile values across all scenarios are coloured the same). The median scenario, inter-quartile range and envelope (a faint dotted line at the edge of the fans showing the most extreme value in any scenario in the given period) are also marked for clarity.
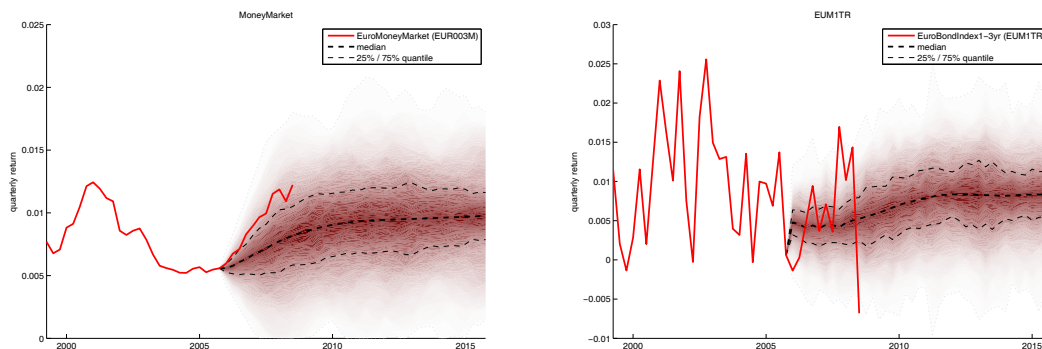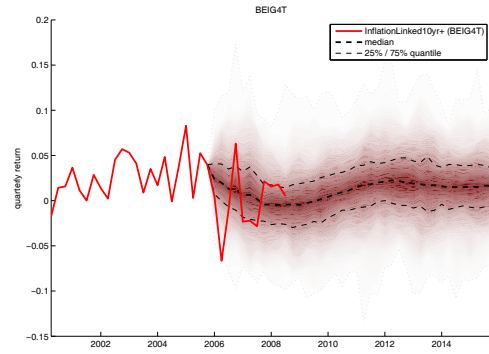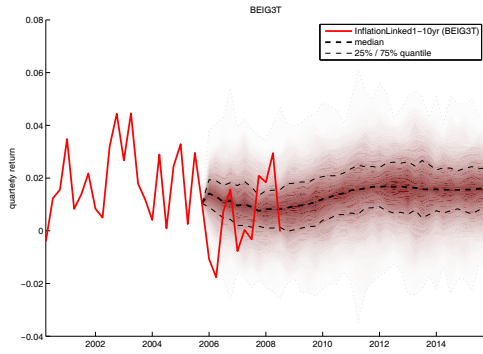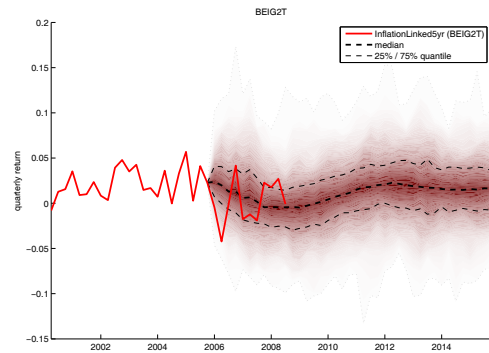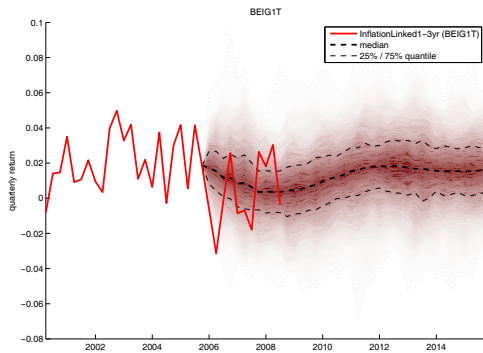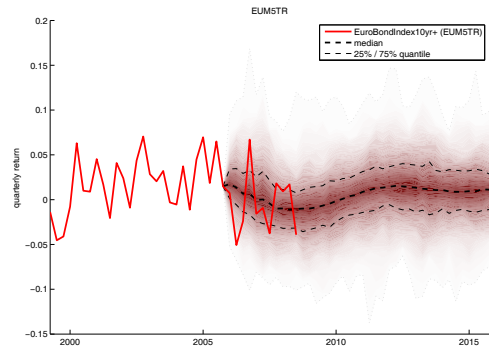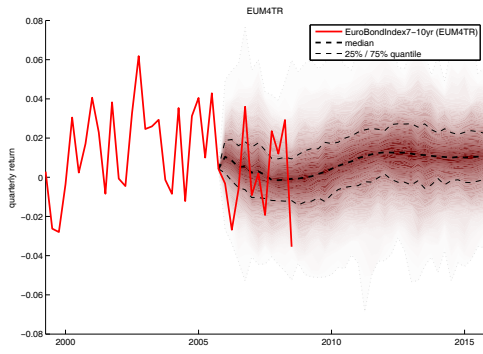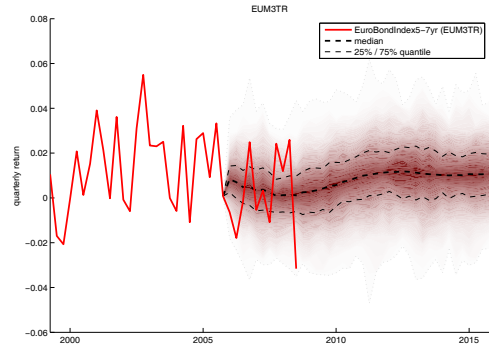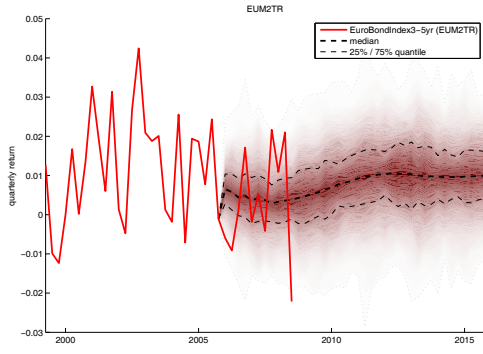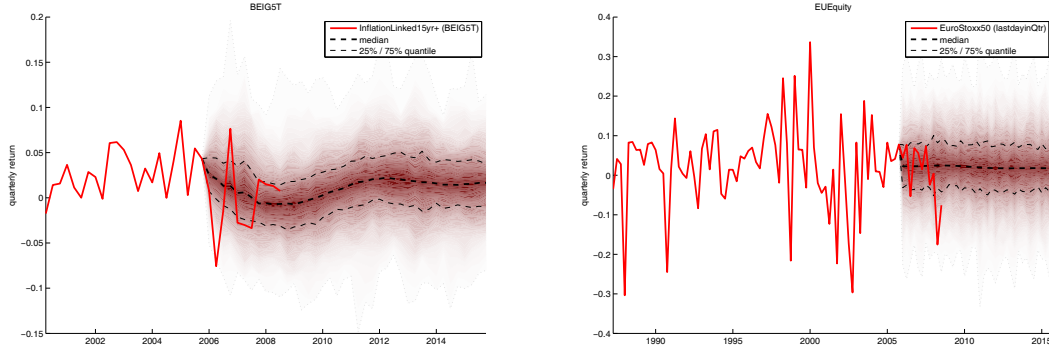
**EXHIBIT 4 Fan Charts of Simulated Quarterly Asset Returns for Investment Assets**

These diagrams also allow us to make some assessment of the plausibility of our asset simulations with respect to historical data up to August 2008. In almost all cases the envelope of the simulated scenarios includes realized returns over the simulation period seen to that date (the exceptions are in short-duration nominal bonds where recent large movements have gone beyond the scenario envelope). The median of the simulated scenarios also fits with the observed returns during this period. In all cases the volatility of the scenarios is on the same scale as the historical return volatility.

## MODELLING LIABILITIES

Pension fund liabilities arise both from the number of members in the scheme and from the cost of purchasing annuities now and in the future to fund members' retirement pensions. The sizes of the annuities that must be purchased depend on the final salary of the scheme members at retirement and hence on interest and inflation rates. The simplified model of the total defined benefit liabilities we use for illustration in this paper is based on Cairns (2004). The techniques required to consider mortality risk may however be incorporated in the DSP framework (see e.g. Medova *et al.*, 2008).

### Scheme Structure Model

The number of members of different ages in a scheme ( $M^t$ ) is modelled starting with an *initial age profile* for the scheme

$$M^0 = (M^0_{x_0}, M^0_{x_1}, ..., M^0_{x_n}),$$ (1)

where $M^t_x$ is the number of members in the scheme at time *t* of age *x*.

This evolves deterministically over time with members moving from one age band to the next each year

$$M^{t+1}_{x_i} = M^t_{x_{i-1}} \qquad\qquad i = 1,...,n.$$ (2)

All members are assumed to retire at *age* $x_n$. Our simplified model only allows members to join the scheme at age $x_0$ and does not account for death in service or early retirement. However, it would be straightforward to add these features to our model and to allow

members to join at any age.  The process by which members join the scheme ($\mathbf{M}_{x_0}^t$) can be specified by a fund's own model or modelled using an appropriate stochastic process[1].

**Salary Model**

A model of the *salaries* of scheme members is important for two reasons: *contributions* from scheme members are made as a fixed proportion of their salary and, being a *final salary* pension scheme, liabilities depend on scheme members' final salaries.  The first of these means that it is necessary to have a complete salary model, rather than just a model of salary at retirement.

We use a simple model for salary evolution.  Real salaries increase with age each year, with the proportional increase fixed for each particular age.  Salaries also increase with inflation, so that their real value is not eroded over time.

We start with an *initial profile of salaries*

$$S^0 = (S_{x_0}^0, S_{x_1}^0, ..., S_{x_{n-1}}^0).$$ 

(3)

We use this to define the *(real) age related salary increase* from age $x_i$ to $x_{i+1}$ that will be used throughout the simulation as $\left( S_{x_{i+1}}^0 / S_{x_i}^0 \right)$.

*Nominal salary evolution* is therefore

$$\mathbf{S}_{x_i}^{t+1} = \left( \frac{S_{x_i}^0}{S_{x_{i-1}}^0} \right) \mathbf{c}(t, t+1) \mathbf{S}_{x_{i-1}}^t,$$ 

(4)

where $\mathbf{c}(t, t+1)$ is the CPI inflation factor from time $t$ to $t + 1$.  Note that since inflation is a simulated stochastic quantity the resulting salaries in (4) are scenario dependent.

The expected *final salary* of a scheme member of age $x_i$ can be written as

$$\mathbf{F}_{x_i}^t = \mathbb{E}_t \left( \mathbf{c}(t, t+n-i) \right) \mathbf{S}_{x_i}^t \left( \frac{S_{x_n}^0}{S_{x_i}^0} \right).$$ 

(5)

Since, the inflation process from $t$ to $t + n - i$ is not known at $t$ we must take its conditional expectation (over forward scenarios) in (5) in order to estimate the final salaries of scheme members at time $t$, which is necessary for calculating liabilities.

As with the scheme structure model, it would be completely straightforward to replace this salary model with a more nuanced model of salary evolution corresponding to a fund's actuarial forecasts.

**Contributions**

Contributions are made to the scheme from two sources: employees and the employer to whom the scheme belongs.  *Employee contributions* are modelled as a *fixed* proportion of

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[1] We used bold face throughout the paper to denote random scenario dependent entities.

salary[2]. Employer contributions are a *variable* proportion of the total salary roll, typically somewhere up to 20%, whose exact level is to be determined at the optimization stage.

**Annuity Model**

Generally, pension benefits to retirees are funded through the purchase of *life annuities*. These are life-linked products, paying a constant (or inflation linked) payment at some regular interval until the death of the annuity holder. Many schemes have retirement payments that are linked to *consumer price inflation* (CPI) over the lifetime of the annuity.

We use a simplified model of mortality, assuming that all scheme members live for a fixed number of years from retirement. This allows us to price the annuities that fund their pensions as the value of all future payments for this deterministic period (Milevsky, 2006). Though such deterministic mortality seems like a major simplification, in the absence of any currently traded liquid asset capable of hedging such mortality risk this simplification will not have a qualitative effect on scheme asset allocations; allocations will concentrate on hedging inflation and duration exposure of liabilities. It is a consequence of Jensen's inequality (Milevsky, 2006, p.116) that the price of an annuity calculated with stochastic mortality is strictly less than the price of an annuity with a fixed length equal to expected life remaining. Our estimates of liabilities based on fixed length annuities will therefore be *pessimistic* when compared to those made with a stochastic mortality model.

To price the life annuities used to fund retiree benefits therefore we only need price an annuity with constant *tenor* (i.e. payment length), where this tenor is the expectation of employee survival after retirement. The time *t* price of a *non-inflation-linked unit annuity-due* (i.e. one paying a single unit of currency each year with immediate initial payment) is

$$\ddot{\mathbf{a}}_{\overline{n}|}(t) = \sum_{i=0}^{n-1} \mathbf{v}(t, t+i),$$ (6)

where $n$ is the tenor of the annuity in years, and

$\mathbf{v}(t, t+i)$ is the time *t* value of a unit cashflow occurring at time $t + i$.

The rates $\mathbf{v}(t, \tau)$ depend on the simulated term structure of interest rates given by

$$\mathbf{v}(t, \tau) = \left(1 + \mathbf{r}(t, \tau)\right)^{-(\tau - t)},$$ (7)

where $\mathbf{r}(t, \tau)$ is the interest rate on bonds maturing at $\tau > t$ taken from the yield curve simulated at *t*. These are scenario dependent because the term structure of interest rates is simulated on each scenario, thus making the annuity prices in (6) scenario dependent. If the annuity is *inflation-linked* then the formula for a *real* unit annuity becomes

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[2] With a mean of 4.9% in the UK in 2007 according to the Office for National Statistics, ONS (2009a).

$$\ddot{\mathbf{a}}_{\overline{n}|}^{real}(t) = \sum_{i=0}^{n-1} \mathbf{v}_{real}(t, t+i), \tag{8}$$

where $\mathbf{v}_{real}(t, t+i)$ is the *real* time $t$ value of a unit cashflow occurring at time $t + i$.

The rate $\mathbf{v}_{real}$ is defined as

$$\mathbf{v}_{real}(t, T) = \left(1 + \mathbf{r}_{real}(t, T)\right)^{-(T-t)}, \tag{9}$$

where $\mathbf{r}_{real}(t, T)$ is the real (as opposed to nominal) interest rate at maturity $T$ taken from the simulated *real* yield curve at time $t$.

## Total Liabilities

In the UK FRS17 demands that the total liability of a pension scheme be calculated as the present value of all future expected payments, taking into account only payments from service given to date but calculating the benefits on the basis of the inflation-adjusted expected final salary.

According to this rule (and on the basis of our simplified scheme and salary structures) the *member liability* to the scheme at time $t$ of a member of *age $x_i$*, who has provided *i years of service* and has another *(n – i) years until retirement*, is

$$\underbrace{\frac{(x_i - x_0)}{60} \mathbf{F}_{x_i}^t}_{\substack{\text{expected liability based} \\ \text{on years of service}}} \cdot \underbrace{\mathbf{v}(t, t+n-i)\ddot{\mathbf{a}}_{\overline{n}|}(t+n-i)}_{\text{current value of future annuity}}. \tag{10}$$

The *total scheme liability* at time $t$ is the sum of liabilities for all scheme members

$$\mathbf{L}_t = \sum_{i=0}^{n-1} M_{x_i}^t \frac{(x_i - x_0)}{60} \mathbf{F}_{x_i}^t \cdot \mathbf{v}(t, t+n-i)\ddot{\mathbf{a}}_{\overline{n}|}(t+n-i). \tag{11}$$

The fund liability is scenario-dependent and here is shown for nominal annuities but these could be substituted for by real annuities using (8) if required. Using a deterministic model of fund membership and salary structure means that the stochastic component of fund liabilities is driven by inflation and interest rates; adding a stochastic component to membership or salary structure would add a further random element to fund liability. Depending on the model used, this random component might be difficult to hedge using the assets currently available in the market.

## OPTIMIZATION PROBLEM

Annuity prices, and therefore fund liabilities, are heavily dependent on interest rates and, when the annuities are inflation linked, on inflation rates. Since these basic processes underlie our asset return models, the fund's liabilities are linked to its returns. The evolution of the fund's portfolio must be constructed in such a way as to optimize its exposure to various instruments in order to match its liabilities. This is the basis of the *asset liability management* (ALM) problem in the absence of mortality risk.

To solve this problem optimally we must trade off considerations of three conflicting objectives in each time period: maximizing fund wealth, minimizing shortfall relative to a target wealth and minimizing employer contributions. We construct a target wealth for the fund for each period in terms of a *target funding ratio* which is a desired ratio of current assets to liabilities for the period. The target funding ratio in each period $TR_t$ is set to vary linearly between the initial funding ratio $TR_1$ and the final target funding ratio $TR_T$ at the *amortization horizon T*, i.e.

$$TR_t = \frac{T-t}{T-1}TR_1 + \frac{t-1}{T-1}TR_T \qquad\qquad t = 2,\ldots,T{-}1. \qquad (12)$$

The *target wealth* of the fund is then simply this target funding ratio multiplied by the total liability of the fund at that period.

$$\mathbf{TG}_t = TR_t\mathbf{L}_t \qquad\qquad t = 1,\ldots,T. \qquad (13)$$

Note that the fund liabilities $\mathbf{L}_t$ are stochastic, so that unlike the target funding ratio the target wealth is scenario dependent.

**Fund Shortfall and Surplus**

The *shortfall* of the fund $\mathbf{w}_t^-$ is defined as the amount by which the fund *wealth* $\mathbf{w}_t$ is below the target wealth $\mathbf{TG}_t$, or zero if there is a surplus. The fund *surplus* $\mathbf{w}_t^+$ is defined the amount by which the fund wealth is above the target wealth, or zero if there is a shortfall. Clearly these possibilities are mutually exclusive on each scenario at any given point of time, i.e. the fund is either below or above its current target. The relations between the variables are therefore

$$\mathbf{w}_t^+ - \mathbf{w}_t^- = \mathbf{w}_t - \mathbf{TG}_t$$

$$\mathbf{w}_t^+ \geq 0 \qquad\qquad\qquad (14)$$

$$\mathbf{w}_t^- \geq 0 \qquad\qquad t = 1,\ldots,T.$$

**Objective**

Pension funds would ideally like to reach their target funding ratio whilst minimizing both their exposure to risk and the level of additional contributions made by the employer to whom the scheme belongs. The objective function to be maximized reflects these conflicting objectives as

$$\mathbb{E}\left( \sum_{t=1}^{T} \alpha\mathbf{w}_t^+ - \beta\mathbf{w}_t^- - \gamma(1-d)^{t-1}\mathbf{c}_t\mathbf{S}_t \right), \qquad (15)$$

where $\mathbf{c}_t$ is the proportion of the total salary roll contributed by the employer at time $t$,

$d$ is the discount rate per period used by the employer,

$\mathbf{S}_t$ is the total salary roll at time $t$,

$T$ is the number of periods over which the dynamic optimization takes place,

$\alpha$, $\beta$ and $\gamma$ are weights on the objective terms, which set the balance of importance between the competing objectives.

The first term in the objective aims to maximize total expected surplus over periods when the fund exceeds its target, the second to minimize total expected shortfall over periods when the fund falls short of its target and the third to minimize total expected employer contributions over all periods.

All these quantities are in units of currency. Shortfall and surplus would be expected to be comparable in magnitude and might be in the region of 10% of the fund wealth. We can roughly determine the size of the contributions in terms of fund wealth by assuming full funding and then equating fund wealth with fund liability defined in terms of salary. For full funding the liability is approximately equal to the value of, say, a 15 year annuity of two thirds of the final salary for all retired scheme members, which for a fund in the steady state is something around two thirds of the total salary roll of employed scheme members. This gives a fund wealth in the region of 10 times total salary roll and, since contributions are in the region of 0-20% of this (ONS, 2009a), this means that the contribution term will be in the region of 1% of the fund wealth. So, the objective term values are in magnitude ratio of roughly 10:10:1 for surplus, shortfall and contributions, respectively.

If we assume that varying decisions has a proportionally similar effect on the objective terms, this can give us some indication as to the appropriate relative magnitudes of the objective term weights. This is a significant assumption but it can at least serve as a heuristic to help us in the task of finding appropriate parameter values within the parameter space of the model. Since fund shortfall is much more important than surplus (which is not beneficial *per se*) $\beta$ should be much greater than $\alpha$. The contribution term is of similar importance to the shortfall, but can be expected to be around ten times smaller than the shortfall term, so $\gamma$ should be roughly ten times the size of $\beta$

$$\alpha \ll \beta,$$
$$\gamma \approx 10\beta. \tag{16}$$

We also discount the employer contributions by some amount to reflect the fact that future payments to the scheme are less costly (and so more desirable) for the employer than current ones. The discount rate $d$ used by the employer depends on the company's cost of capital, so it would be expected to be in the region of 10% per annum.

## Cashflow Constraints

The cashflow model represents the basic movement of value in the fund and ensures proper accounting of cashflows into, out of and within the fund. It can be represented by Exhibit 5, where circles represent stores of value and arrows represent paths along which value can be transferred.
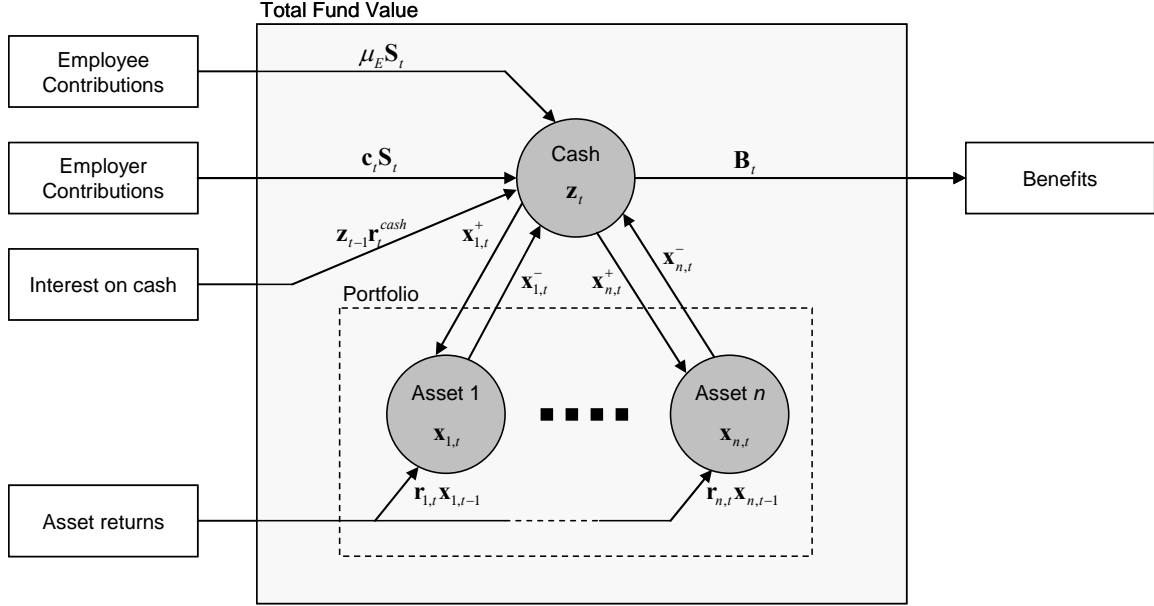


**EXHIBIT 5  Cashflow Diagram for Defined Benefit Pension Fund**

This leads to the *cashflow constraint*

$$\mathbf{z}_t = \left(1 + \mathbf{r}_t^{cash}\right)\mathbf{z}_{t-1} + \mu_E \mathbf{S}_t + \mathbf{c}_t \mathbf{S}_t + \sum_{a \in A} \mathbf{x}_{a,t}^- - \sum_{a \in A} \mathbf{x}_{a,t}^+ - \mathbf{B}_t \qquad a \in A,\, t = 2,\ldots,T, \quad (17)$$

where  *A* is the *set of assets* from which the portfolio can be composed,

$\mathbf{B}_t$ is the payment of employee benefits (annuity purchases) at time *t*,

$\mathbf{r}_t^{cash}$ is the interest rate on bank cash at time *t*,

$\mathbf{x}_{a,t}^-$ is the *sale* by value of asset *a* at time *t*,

$\mathbf{x}_{a,t}^+$ is the *purchase* by value of asset *a* at time *t*,

$\mathbf{z}_t$ is the holding of bank cash at time *t*,

$\mu_E$ is the (fixed) proportion of salary contributed by employees,

and to the *asset balance constraints*, which account for the asset holdings and their accrual of value due to returns:

$$\mathbf{x}_{a,t} = \mathbf{x}_{a,t-1}(1 + \mathbf{r}_{a,t}) + \mathbf{x}_{a,t}^+ - \mathbf{x}_{a,t}^- \qquad\qquad a \in A,\; t = 2,\ldots,T, \quad (18)$$

where $\mathbf{x}_{a,t}$ is the fund's *holding* of asset $a$ at time $t$,

$\mathbf{r}_{a,t}$ is the return on asset $a$ from time $t$–1 to $t$.

These two constraints do not apply in the first period (since $\mathbf{z}_0$ and the $\mathbf{x}_{a,0}$ are not defined) and are replaced in that period by the initial cashflow constraint

$$z_1 = W_0 + \mu_E S_1 + c_1 S_1 + \sum_{a \in A} x^-_{a,1} - \sum_{a \in A} x^+_{a,1} - B_1, \tag{19}$$

where $W_0$ is the initial wealth of the fund, and the initial asset balance constraints

$$x_{a,1} = x^+_{a,1} - x^-_{a,1}, \qquad\qquad\qquad a \in A \tag{20}$$

which simply states that the initial asset holding is equal to the amount of asset bought less the amount of asset sold at time 1.

For simplicity we abstract from fund management fees, transactions costs and turnover constraints but these may easily be included (see e.g. Dempster *et al*. 2006, 2007b).

**Total Wealth**

The *total wealth* of the fund is defined as the sum of all asset holdings, i.e.

$$\mathbf{w}_t = \sum_{a \in A} \mathbf{x}_{t,a} . \qquad\qquad\qquad t = 1,\dots,T. \tag{18}$$

**Employer Contribution Rate**

The level of *employer contributions* is constrained to lie between an upper and lower limit ($c_{min}$ and $c_{max}$ respectively) specified by the user, i.e.

$$c_{min} \le \mathbf{c}_t \le c_{max} \qquad\qquad\qquad t = 1,\dots,T. \tag{19}$$

**Short Selling Constraint**

In this model we restrict the fund to *long-only* positions, meaning that all asset holdings must be non-negative

$$\mathbf{x}_{a,t} \ge 0 \qquad\qquad\qquad a \in A, \ t = 2,\dots,T. \tag{20}$$

**Cash Constraint**

Cash is constrained to be nonnegative, so that the fund is not allowed to borrow from the bank at the cash rate

$$\mathbf{z}_t \ge 0 \qquad\qquad\qquad t = 1,\dots,T. \tag{21}$$

**Solution**

In order to make a large scale dynamic stochastic programming problem such as ours computationally tractable we must limit the number of scenarios and branch points at which portfolio rebalances are undertaken. However, the CASM simulator produces asset returns at *quarterly* time steps and this is used as the basic frequency of accounting

and decisions in the model.  In order to limit the number of scenarios whilst still using quarterly simulation data and allowing quarterly portfolio and contribution adjustment we adopt a tree that only branches at a limited subset of stages, as shown in Exhibit 6.
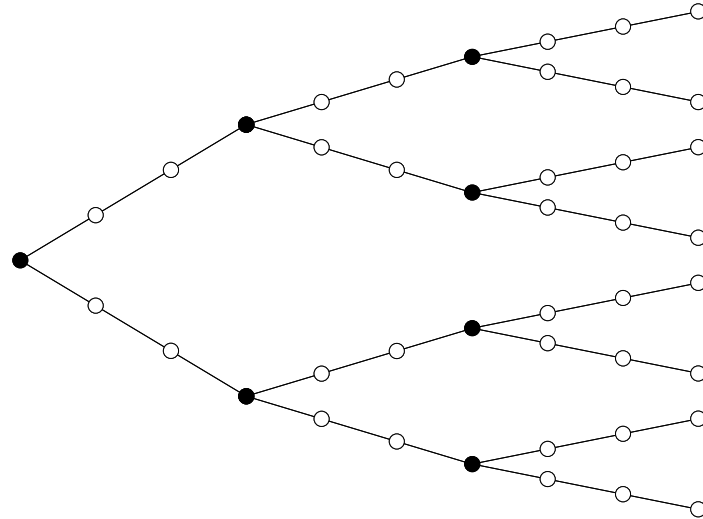


**EXHIBIT 6  2.2.2 Tree with Ten Stages and Three Branching Stages (shown in black)**

The difficulty with this two time scale scenario tree structure without further enhancement is that at periods between major rebalance points, the exact evolution of the scenario up to the next major rebalance point is visible to the optimizer.  In order to prevent the optimizer exploiting this information we insist that it makes these portfolio adjustment and contribution assignment decisions for the non-branching periods at the preceding *branching stage* (or *major rebalance point*), where there is still uncertainty as to which scenario will be realized.

This introduces the further problem that, at the preceding branch point, the exact value of an asset at forthcoming stages is not known.  Thus the exact value of purchases or sales of the asset are unknown, which could result in a surfeit or shortage of *bank cash* on a particular scenario.  In fact it is possible to exclude one of these mutually exclusive possibilities by restricting the sign of the cash variable and so we choose to only allow positive cash holdings, enforced by the nonnegativity constraint (21) above.  Overcoming this effect is the reason for introducing non-investable bank cash, which acts as a slack variable able to absorb excess value between major rebalance points.  Bank cash holdings tend to grow slowly between major rebalance points and return to zero at the major rebalance points.  Bank cash is not intended to be an investment asset and so we make the cash interest rate uncompetitive with the returns on other assets (we have used a zero cash interest rate in our examples).

We formulate the model and solve the example problems using tools from the *STOCHASTICS* [TM] suite for dynamic stochastic programming problem formulation, analysis and visualization (CSA, 2008).

We present two sample problems. In the first we have a severely under funded scheme that aims to become fully funded within ten years. In the second we have an almost fully funded scheme that aspires to improve its funding ratio over fifteen years. We present two solutions for each scheme, one with variable employer contributions and a second with employer contributions fixed to a constant proportion of the total salary roll.

**Under-funded Scheme**

We consider a hypothetical under-funded pension scheme with a joining age of $x_0 = 25$ and retirement age of $x_n = 65$. The initial age profile is

$$M^0 = (20, 20, ..., 20),\qquad\qquad(22)$$

and the joining rate is taken to be constant, $M_{x_0}^t = 20$ throughout the problem. The initial salary profile is

$$S^0 = (20, 21, ..., 59),\qquad\qquad(23)$$

where the salaries are given in thousands of euros. The annuities provided for retiring scheme members are taken to have a *constant* tenor of 15 years with inflation linked payments.

The initial liability in Q2 2007 is calculated to be €150,235,000 and the initial fund wealth is €125,000,000 from which a payment of €10,439,000 is immediately made to pay for annuities for retirees at time 0. Thus the initial wealth is €114,561,000 giving the scheme an initial funding ratio of 76.37%.

In the first solution, employer contributions are constrained to lie between 0% and 20% of the total payroll. This range covers the realistic range of pension contributions made by UK employers; in 2007, 40.7% of UK schemes had employer contributions between 8% and 15%, whilst 58.7% had contributions above 15% (ONS, 2009a). The mean was 15.6%.

In the second solution, contributions are fixed to a constant 12%. This would place the scheme in the 24th percentile of UK defined benefit schemes in terms of employer contributions in 2007 (ONS, 2009a).

Employee contributions are fixed at 5% of salary, which places the scheme around the 21st percentile of UK defined benefit schemes in terms of 2007 employee contribution rate according to ONS 2009a, though just above the mean of 4.9%. As with employer contributions, the distribution of employee contributions is negatively skewed.

**Model Parameters**

The asset return and liability simulation is quarterly with a 10 year horizon from Q2 2007, quarterly portfolio rebalancing and major rebalancing at the branching times. We use a (50.8.4.4) regular branching tree with branching times of 0, 2½, 5 and 7½ years and the investable asset classes are those described in Section 3. The aim is to achieve a fully funded scheme within 10 years, giving a target funding ratio of 100%.

The optimization uses the parameters $\alpha := 10^{-6}$, $\beta := 0.1$ and $\gamma := 1$, which fit closely with the values in (16) determined heuristically in Section 5. By using such a small value of $\alpha$ relative to $\beta$ we are saying that the negative impact of €1 of shortfall is equivalent in magnitude to the positive impact of €100,000 of surplus. In other words, we are effectively requiring the optimizer never to build up surplus above the target on some scenarios at the expense of increasing shortfall on other scenarios. We use a discount rate for employer contributions of 6.5% per annum.

**Results**

In Exhibit 7 we chart the average fund wealth across all scenarios along with its target and liability. The drop in wealth at every fourth step is due to the annual purchase of annuities for retiring members. Initially the expected wealth is far less than the liability but by the amortization horizon the expected wealth has reached the target in both cases.
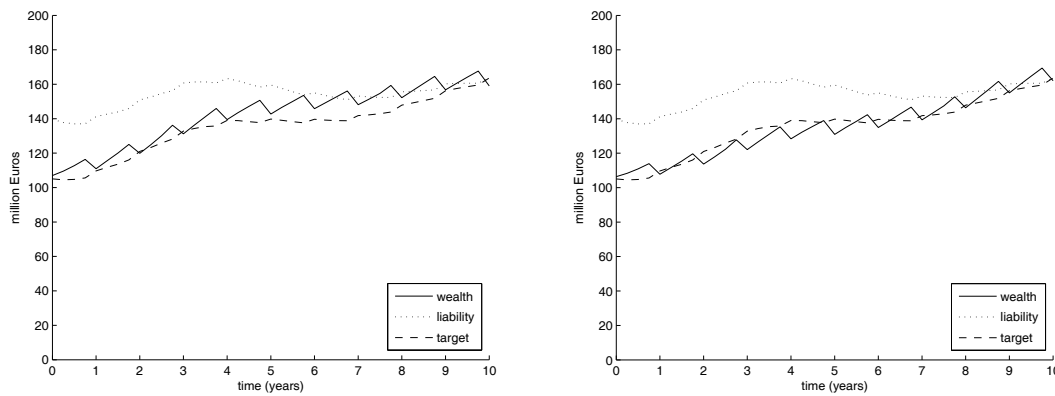


**EXHIBIT 7  Expected Wealth, Liability and Funding Target with Variable (left) and Constant (right) Employer Contributions**

In Exhibit 8 we chart the average level of employer contributions. For the problem with variable contributions initial employer contribution rates are at their maximum permitted level. As the fund starts to surpass its funding target the contributions drop steadily from 2½ years into the problem. The pattern of contributions seen in the variable contribution optimizations makes intuitive sense, as the optimizer chooses to take employer contributions early when they can contribute to fund growth.
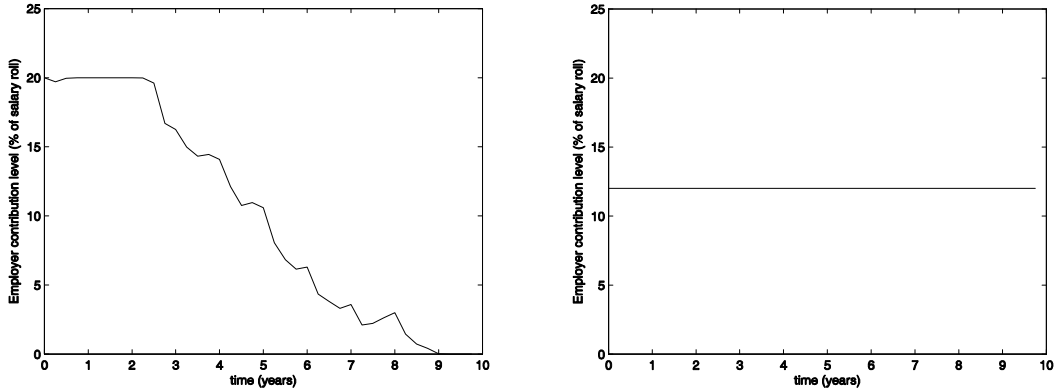
**EXHIBIT 8 Expected Employer Contribution Rate with Variable (left) and Constant (right) Employer Contributions**

In Exhibit 9 we show the (expected) *dynamic asset allocations* averaged across scenarios. In both instances, the initial asset allocations are largely made up of long maturity inflation linked bonds. This is because all scenarios start with a small shortfall to the target which the optimizer must reduce, but is penalized heavily for increasing. As well-performing scenarios rise above the target they can afford to increase their exposure to equity, which can provide them with enhanced returns. The bulk of the assets are still held in inflation-linked bonds, which makes sense in light of the fund's inflation-linked liabilities. Notice that the final allocation contains a high proportion of long maturity inflation-indexed bonds which reflects the long maturity of many of the promised payments and an attempt to hedge exposure to future inflation. The allocation for both problems is quite similar, although the constant contribution problem contains slightly more equity in the first 2½ years to compensate for lower contributions during this period.
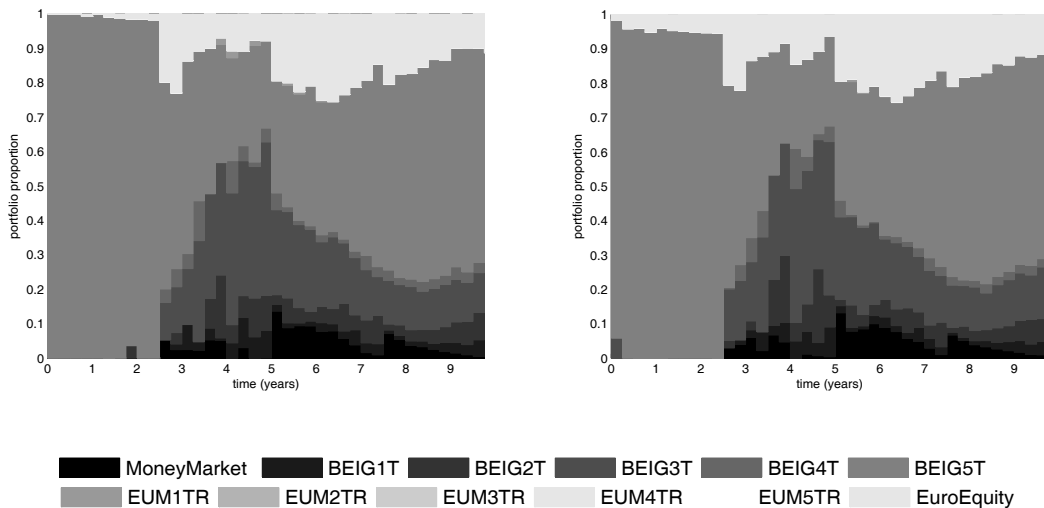


**EXHIBIT 9 Expected Dynamic Asset Allocation with Variable (top) and Constant (bottom) Employer Contributions (equity topmost)**

21

Exhibit 10 shows the evolution of the fund's shortfall (against the target) across all scenarios. As with the fan plots of the asset simulations in Section 3, these plots show the distribution of the fund shortfall throughout the optimization, with the median, inter-quartile range and scenario envelope marked.

The two charts are fairly similar, with variable contributions leading to lower early shortfalls. With constant contributions the range of the extreme scenarios is increased somewhat as the fund is unable to use variable contributions to ameliorate funding difficulties or reduce contributions when funding levels are high.

The range of fund shortfall across the scenarios can also be influenced using the $\alpha$ and $\beta$ parameters; increasing the emphasis on minimizing shortfall (increasing $\beta$ relative to $\alpha$) will decrease the range of fund wealth, whereas increasing the emphasis on wealth will increase the range of fund wealth, as more risk will be taken in order to increase expected return.
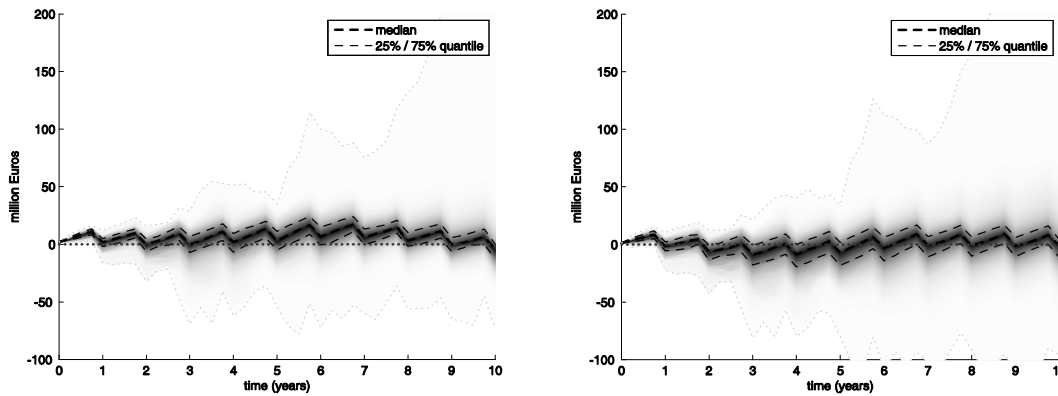


**EXHIBIT 10  Fan Plots of Fund Shortfall Across Scenarios with Variable (left) and Constant (right) Employer Contributions**

The variable and constant contribution cases of the under-funded problem took 149s and 137s to solve, respectively, using a dual Xeon 3GHz computer.

**Well-funded Scheme**

We now consider the case of a fund with a healthier funding ratio which seeks to achieve a high funding ratio over a longer 15 year horizon from Q2 2007. Again we set $x_0 = 25$ and $x_n = 65$. The initial age profile is

$$M^0 = (25, 25, ..., 25) \tag{24}$$

with a *constant* joining rate $M_{x_0}^t = 25$ at all times.

The initial salary profile is

$$S^0 = (25, 26.25, 27.5, ..., 73.75) \tag{25}$$

in thousands of euros. The annuities purchased to fund retirees' pensions are inflation linked with a constant tenor of 15 years.

In this example, the initial liability is €234,743,000 and after an initial lump sum annuity payment of €16,311,000 from the fund value of €235,000,000 the initial wealth is €218,689,000, giving a funding ratio of 93.16%.

In the first solution, employer contributions to the scheme are constrained to lie between 0% and 20% of the total payroll. In the second they are fixed to be a constant 7%, which would place the scheme around the 4th percentile of UK pension schemes for employer contributions in 2007 (ONS, 2009a). Employee contributions are fixed at 5% of salary.

### Model Parameters

We use a (50.8.4.2.2) regular branching tree with branching times of 0, 3, 6, 9 and 12 years. The simulation is quarterly with a 15 year horizon, quarterly portfolio rebalancing and major rebalancing at the branching times. The aim is to achieve a well funded scheme within 15 years, with a target funding ratio of 120%. As in the under-funded case, the optimization uses the parameter values $\alpha := 10^{-6}$, $\beta := 0.1$ and $\gamma := 1$. Again, we use an employer discount rate of 6.5% per annum.

### Results

Exhibit 11 shows the liability and wealth for this scheme. The fund wealth starts just below the initial liability and for both constant and variable contributions increases to a final level which is above 120% of the fund's liability at the 15 year amortization horizon. In both regimes the expected fund wealth considerably exceeds the target wealth. As will be seen in the surplus plots below, this reflects the fact the fund is above the funding target in nearly all scenarios.
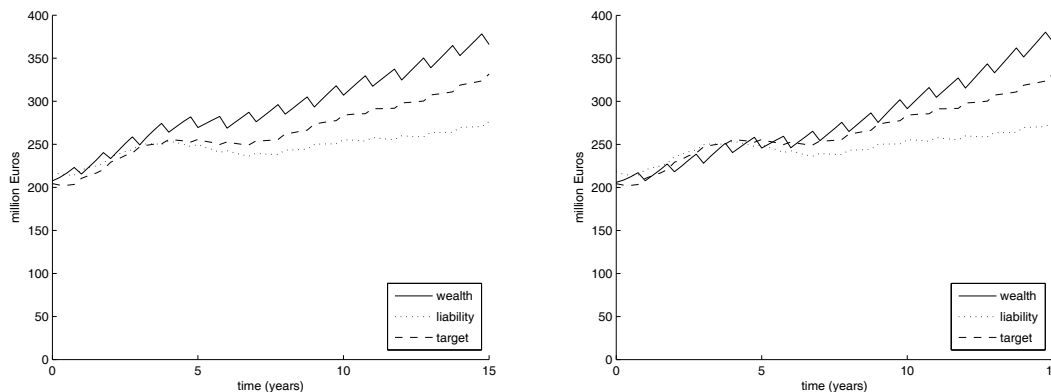


**EXHIBIT 11 Expected Wealth, Liability and Funding Target with Variable (left) and Constant (right) Employer Contributions**

Exhibit 12 charts the average employer contribution rates. Here we see a similar pattern to the under funded case, with the variable contribution fund using early contributions to reach the target funding ratio and then reducing them once a high level of funding has been achieved. Total employer contributions made under the variable contribution

23

regime are €40.9m (present value €34.8m), lower than the €61.0m (present value €37.4m) made in the fixed-contribution rate scheme, yet the expected fund wealth is just €3.1m (present value €0.94m) higher at the problem horizon in the constant contribution scheme. In reality the value of early contributions to a pension scheme is determined by the rate of discounting used by the employer. If the company uses a high discount rate (indicating, for example, a high cost of capital) early contributions will appear less attractive.
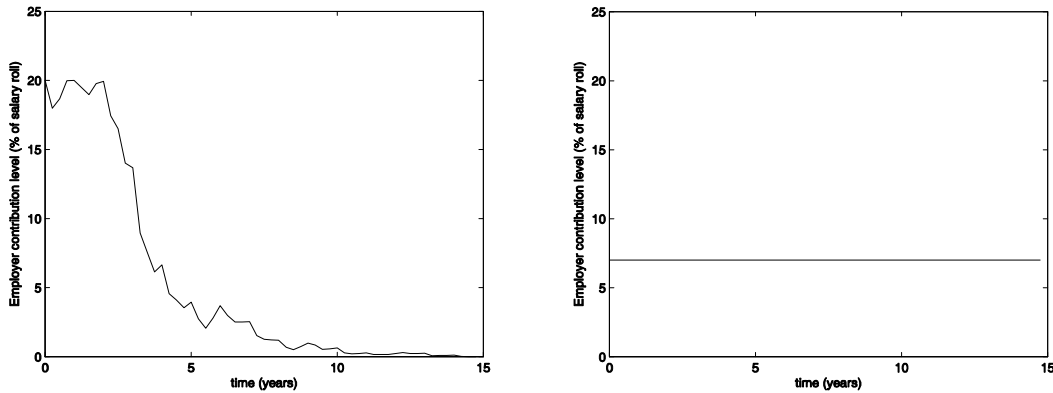


**EXHIBIT 12  Expected Employer Contribution Rates with Variable (left) and Constant (right) Employer Contribution Rates**

Exhibit 13 presents the dynamic asset allocation averaged across scenarios. The asset allocations here are similar to those seen in the under-funded case, with the fund relying on inflation-linked bonds to make up the bulk of its assets, in line with its liabilities. As funding levels increase the fund can afford to allocate more of its wealth to equity. The allocations for the two contribution schemes are similar, with the constant-contribution fund taking on more risk through a higher equity holding in the first three years.
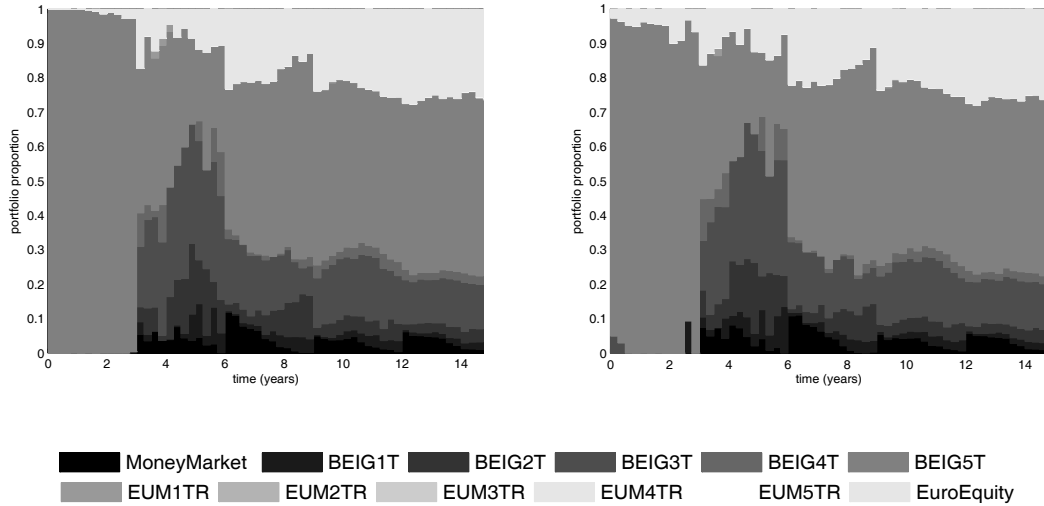
**EXHIBIT 13  Expected Dynamic Asset Allocation with Variable (left) and Constant (right) Employer Contributions (equity topmost)**

Finally, the fan plots of the fund shortfall in Exhibit 14 show that under both regimes the fund is expected to stay above its funding target in almost all scenarios. As with the under-funded case the constant contribution regime initially falls below the target in many scenarios as variable contributions cannot be used to make up the shortfall.
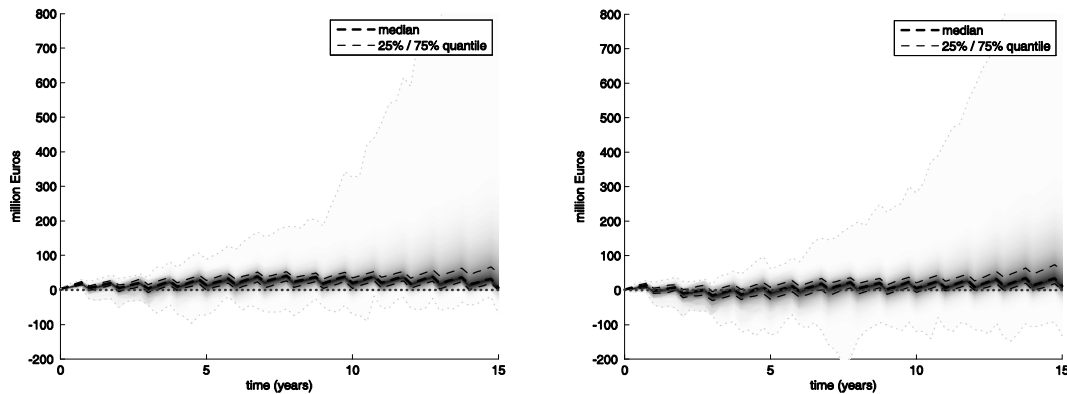


**EXHIBIT 14  Fan Plots of Fund Shortfall Across Scenarios with Variable (left) and Constant (right) Employer Contributions**

The variable and constant contribution cases of the well-funded problem took 362s and 333s to solve, respectively, using a dual Xeon 3GHz computer.

**CONCLUSION**

In this paper we have described a strategic level dynamic stochastic optimization model which can be used to advise under-funded defined benefit pension schemes on best practice for returning the fund to solvency and long term stability. We have presented an

overview of the dynamic stochastic programming techniques involved and briefly described the nature of Pioneer Investment's proprietary CASM simulator from which the asset class returns and pension scheme liabilities were generated. The stochastic optimization model was set out precisely and its solution by linear programming discussed. To illustrate the approach, two examples of defined benefit schemes using simple conservative fund liability models were presented and discussed. The optimal dynamic asset allocations of these examples reflect the motivation of second generation liability driven investment schemes discussed in the introduction. Although the final salary scheme models used in our examples are simple, more complex models can be incorporated into the system described with little extra effort. Most actuarial assessments used in practice (see e.g. Clark *et al*., 2006 for examples) can be modelled for this purpose.

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