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Designing Minimum Guaranteed Return Funds

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ABSTRACT

In recent years there has been a significant growth of investment products aimed at attracting investors who are worried about the downside potential of the financial markets. This paper introduces a dynamic stochastic optimization model for the design of such products. An optimal dynamic portfolio allocation strategy combined with risk management allows us to provide the best possible portfolio returns that fit clients' risk aversion. The pricing of the minimum guarantee as well as the valuation of a portfolio of bonds are based on a three-factor term structure model. The implementation of our investment strategy is illustrated on real market data and back-tested through a period of the last five years.

Keywords: Dynamic Stochastic Programming; Asset & Liability Management; Guaranteed Returns; Backtests

1. Introduction

In recent years there has been a significant growth of investment products aimed at attracting investors who are worried about the downside potential of the financial markets. There are several different guarantees available in the market. The one most commonly used is the *nominal guarantee* in which a fixed percentage of the initial wealth is guaranteed at a specified date in the future. There also exist funds with a 'real' or *flexible guarantee* linked to an inflation index or some other capital market index. Sometimes the guarantee of a minimum rate of return is even set in relation to the performance of other funds.

Life insurance companies often include guarantees in their products. These guarantees provide options to their policyholders which in some cases can be valuable. In the past these options have sometimes been viewed by insurers as having negligible value, as they were far out-of-the-money, and were not taken into account in pricing products. The pricing of option-embedded policies for early products with guarantees was addressed in the papers by Brennan and Schwartz (1976) and Boyle and Schwartz (1977). They analysed *unit-linked maturity guarantee* policies, in which the interest accrued is linked directly and without lags to the return on some reference portfolio – the unit.

Significant research has also been done into life insurance products where interest is credited to the policy periodically according to some mechanism which smoothes past returns on the life insurance company's assets – a *participating policy*. Grosen and Jørgensen (2000) decompose a typical participating policy into a risk-free bond element, a bonus option and a surrender option. However, a continuous-time frictionless economy with a perfect financial market is assumed in these papers. In Consiglio, Cocco and Zenios (2000), a scenario optimization asset and liability management model for multi-period participating policies with guarantees is developed. Their model is evaluated in a single-period framework and there is no immediate straightforward extension of their model to a multi-stage framework. As this paper concentrates on long-term insurance products, investment decisions should be evaluated with regard to more appropriate temporal issues than static risk-reward trade-offs, as was advocated in the papers by Dempster *et al.* (2003) and Mulvey *et al.* (2003). Consiglio, Cocco and Zenios (2001) confirm that traditional Markowitz mean-variance optimization is inefficient in solving guarantee policies. Another stream of research into insurance products are *guaranteed annuity options*, where an insurer guarantees to convert a policyholder's accumulated funds to a life annuity at a fixed rate when the policy matures. The pricing and risk management of these products is described in Boyle and Hardy (2003) and Wilkie, Waters and Yang (2003).

All the papers cited concentrate on the pricing of the option liability created by introducing insurance products with guarantees. In this paper however, rather than concentrating on the pricing of these products, we focus on optimal strategic asset allocation for the insurance company once the guaranteed return products have been issued. The asset and liability sides of the problem are priced in a consistent manner by allowing a multistage stochastic programme to be applied to allow for temporal adjustments to the portfolio mix. Insurance products have become increasingly more innovative in order to face competitive pressures and over recent years the focus has shifted from static models to stochastic models (Vanderhoof and Altman (1998), Babbel and Merrill (1999) and Embrechts (2000)). Other examples of the use of dynamic portfolio optimization models for asset and liability management for insurance companies are given by the Yasuda-Kasai model in Cariño and Ziemba (1998), the Towers Perrin model by Mulvey and Thorlacius (1998) and the CALM model of Consigli and Dempster (1998). These models have been successfully used in a practical setting but their application does not cover policies with guarantees. One way in which the guarantee can be achieved is by investing in zero-coupon Treasury bonds with maturity equal to the time horizon of the product in question. However using this option will forego all upside potential provided by an equity component of the fund's portfolio over the contract horizon. Even though the aim is to protect the investor from the downside, a reasonable expectation of higher than guaranteed returns must remain.

This paper reports on results of research with Pioneer Investments and concentrates on the design of the fund with dynamic portfolio and liability management. In this paper we will consider long-term nominal minimum guaranteed return plans with a fixed time horizon. As such these are *closed end funds* -- after the initial contribution there is no possibility of making any contributions during the lifetime of the product. Our main focus will be on optimal portfolio allocation subject to risk attitude and monitoring. This requires long-term forecasting of all investment class returns and dealing with a stochastic liability in the form of the current market value of the guarantee. *Dynamic stochastic programming* is the technique of choice to solve this kind of problem. Such a model will automatically hedge current portfolio allocations against the future uncertainties in asset returns and liabilities over a long horizon (Dempster *et al.*, 2003). A practical method must have the flexibility to take into account multiple time periods, portfolio constraints such as the prohibition of short selling, and varying degrees of risk aversion in the portfolio allocation. In addition, it should be based on a realistic representation of the dynamics of the relevant asset prices. All these factors have been carefully addressed here and are explained further in the sequel.

The paper is organized as follows. In Section 2 we explain the structure of the fund, define stochastic guarantees and give the strategy for portfolio re-balancing. In Section 3 we describe the stochastic optimization framework, which includes the problem set up, model constraints and possible objective functions. Section 4 then briefly presents a three-factor term structure model detailed elsewhere (Dempster *et al.*, 2004) for pricing the bond components of the portfolio and the liability side of the fund. Section 5 presents several historical backtests to show how the framework would have performed had it been implemented in practice over the period 1999 - 2004, paying particular attention to the effects of using different objective functions and varying tree structures. Section 6 concludes.

2. Management of a Closed End Guaranteed Return Fund

In Japan, Nissan Mutual Life failed on a $2.56 billion liability arising from a 4.7% guaranteed investment in 1997. In Europe, the EU authorities have now responded to the threat of insolvency from return guarantees. More specifically, Article 18 of the Third EU Life Insurance Directive, which became effective as of 10 November 1992, requires that interest rate guarantees do not exceed 60% of the rate of return on government debt (of unspecified maturity). In response to market pressures and regulatory conditions, insurers currently offer more conservative guaranteed returns. However, policyholders are compensated by participating in the fund's profits, receiving a bonus whenever the return of the fund's portfolio exceeds the guarantee.

In this paper we will concentrate on a *closed end guaranteed return fund* in which after the initial cash outlay no contributions will be allowed. The time horizon of the fund involved will be five years, and even though capital guarantees and 1% guarantees are very common nowadays we will investigate the more taxing 2% guarantee. The focus lies in combining strategic asset allocation together with the risk management of the fund. So for simulation purposes, we calibrated a three-factor term structure model with a closed form solution for the yields (see Medova *et al.*, 2004), to price individual bonds and the liability. This guarantee liability, later referred to as the *barrier*, consists of the final guarantee multiplied by the price of a zero-coupon bond paying one at expiration and with maturity equal to the remaining life of the fund. This is the minimum value for which we can be absolutely certain to attain the guarantee. Due to this formulation both asset values and liabilities are priced in a consistent way across all future scenarios.

We have ignored service and management fees for simplicity although the liability model could easily be extended to include these and other fees. Similarly, no transaction costs have been included as the fund trades internally. However proportional transaction costs can be included into the model without significant alterations (Dempster *et al.*, 2003).

Solving a chance-constrained stochastic programme in which the probability of the portfolio's wealth falling below the barrier is restricted to a small percentage might render the problem non-convex (see Prékopa, 1980). We therefore adopted an approach in which the risk-return trade-off is incorporated into the objective function. Defining *shortfall* as the amount by which the portfolio's wealth falls below the barrier, the risk of the policy is quantified in two ways. In the first approach we consider the average shortfall over time for each scenario and then take the expectation over all scenarios (the *expected average shortfall* approach). In the second case we look at the maximum shortfall over time for each scenario and then as before take the expectation over all scenarios (the *expected maximum shortfall* approach). A scaling factor which can be interpreted as a measure of risk aversion links the portfolio wealth and the shortfall/risk factor for the guarantee in the objective function.

To test the potential of the model, we applied a five-year backtest to the period 1999-2004, a period in which the Eurostoxx 50 index lost 24%. We will allow rebalancing on an annual basis. However as the risk should be monitored on a more frequent basis, we developed a model in which risk management is applied on a monthly basis -- the data frequency. The exact specification is given in Section 3.3.

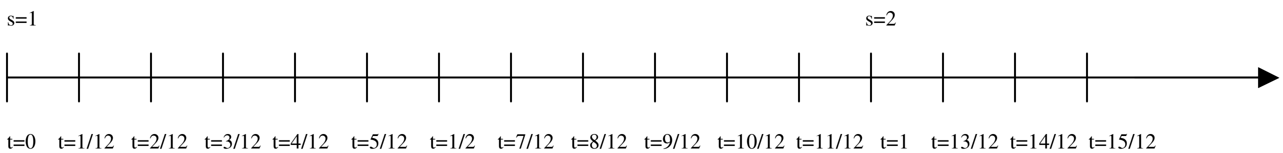
3. Stochastic Optimization Framework

In this section we describe a general framework for solving minimum guaranteed return funds using stochastic optimization. We consider both risk management and *strategic* asset allocation concerned with the allocation across broad asset classes. Given this set of assets, a fixed planning horizon and a set of decision times, the objective is to find a trading strategy that maximizes the risk adjusted wealth accumulation process subject to the constraints.

3.1 Set up

We look at alternative formulations to optimally allocate assets for a minimum guaranteed return fund involving respectively expected average and maximum shortfalls. The two models will be applied to eight different assets: coupon-bearing Treasury bonds with maturity equal to 1, 2, 3, 4, 5, 10 and 30 years and an equity index. All assets are denominated in Euros.

Stages:



Time

Figure 1: Time and Stage Setting

We consider a discrete time and space setting. The time interval considered is given by $\left\{0, \frac{1}{12}, \frac{2}{12}, ..., T\right\}$, where the times indexed by $t = 0, 1, ..., T-1$ correspond to the *decision times* at which the fund will trade to rebalance its portfolio and T is the planning horizon at which no further decision is made and the guarantee will be paid out. In this paper we use a five-year horizon $(T = 5)$ for illustration.

Uncertainty is represented by a *scenario tree* Ω, in which each data path through the tree corresponds to a *scenario* ω in Ω and each node in the tree corresponds to a time along one or more scenarios. An example scenario tree schema is given in Figure 2. The probability $p(\omega)$ of scenario ω in Ω is the reciprocal of the total number of scenarios as these data paths are generated by Monte Carlo simulation, i.e. each scenario is equally likely.

To represent the scenario tree structure, we use a *treestring* which is a string of integers specifying for each stage s the number of branches for each node in that stage. This specification gives rise to *balanced scenario trees*, in which each subtree in the same period has the same number of branches. The balanced scenario tree of Figure 2 can be described by the treestring 3.3, giving a total of $3 \cdot 3 = 9$ scenarios.

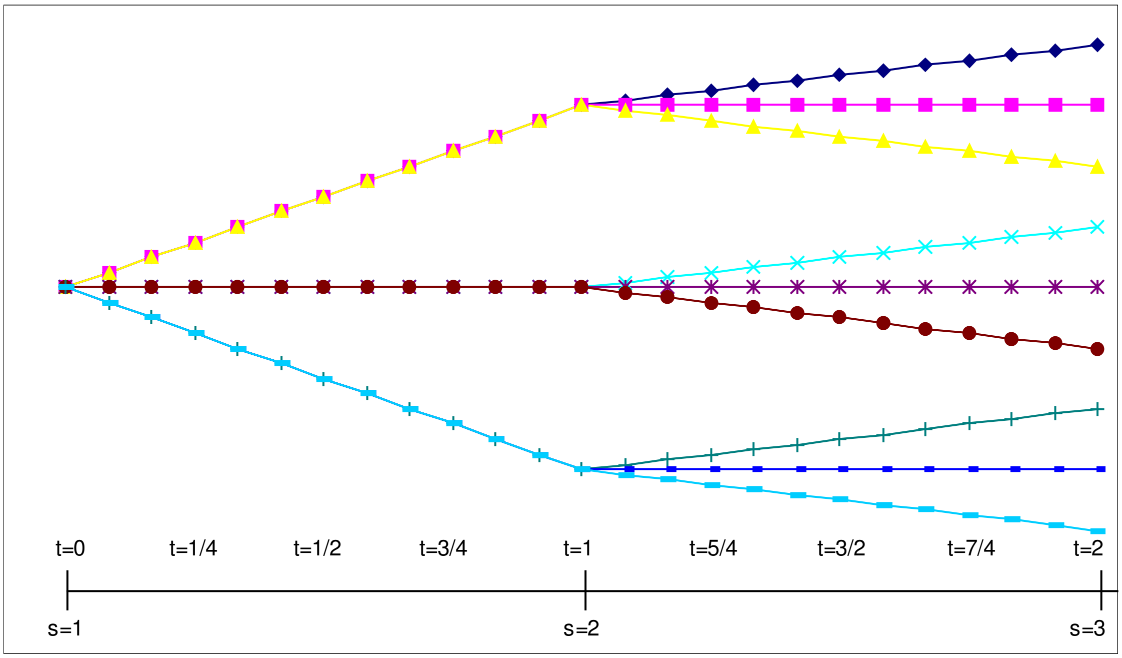


Figure 2: Graphical Representation of Scenarios

Table 1 defines the variables and parameters of our stochastic programming problem. Throughout the paper we will use boldface to denote random entities.

Time Sets

$T^{\text{total}} = \left\{0, \frac{1}{12}, \ldots, T\right\}$	*set of all times* considered in the stochastic programme
$T^{d} = \{0, 1, \ldots, T-1\}$	*set of decision times*
$T^{i} = T^{\text{total}} \setminus T^{d}$	*set of intermediate times*
$T^{c} = \left\{\frac{1}{2}, \frac{3}{2}, \ldots, T - \frac{1}{2}\right\}$	*times when a coupon is paid out in between decision times*

Instruments

$S_t(\omega)$	*Dow Jones Eurostoxx 50 index level* at time t in scenario ω
$B_t^T(\omega)$	*EU Treasury bond with maturity T* at time t in scenario ω
$\delta_t^{B^T}(\omega)$	*coupon rate* of EU Treasury bond with maturity T at time t in scenario ω
F^{B^T}	*face value* of EU Treasury bond with maturity T at time t
$Z_t(\omega)$	*EU zero-coupon Treasury bond price* at time t in scenario ω

Risk Management Barrier

$y_{t,T}(\omega)$	*EU zero-coupon Treasury yield* with maturity T at time t in scenario ω
G	*annual guaranteed return*
$L_t(\omega)$	*barrier* at time t in scenario ω

Portfolio Evolution

A	*set of all assets*
$P_{t,a}^{buy}(\omega) / P_{t,a}^{sell}(\omega)$	*buy/sell price* of asset $a \in A$ at time t in scenario ω
$x_{t,a}(\omega)$	*quantity held* of asset $a \in A$ between time t and $t + 1/12$ in scenario ω
$x_{t,a}^{+}(\omega) / x_{t,a}^{-}(\omega)$	*quantity bought/sold* of assets $a \in A$ at time t in scenario ω
$W_t(\omega)$	*portfolio wealth* at time $t \in T^{\text{total}}$ in scenario ω
$h_t(\omega) := \max(0, L_t(\omega) - W_t(\omega))$	*shortfall* at time t in scenario ω

Table 1: Variables and Parameteres of the Model

3.2 Model constraints

The constraints considered for the minimum guaranteed return problem are:

- *cash balance constraints*. These constraints ensure that the net cash flow at each decision time and at each scenario is equal to zero

$$\sum_{a \in A} P_{0,a}^{\text{buy}}(\omega)\, x_{0,a}^{+}(\omega) = W_0(\omega) \qquad \omega \in \Omega \qquad (1)$$

$$\sum_{a\in A\backslash\{S\}} \frac{1}{2}\delta_{t-1}^{a}(\omega) F^{a} x_{t,a}^{-}(\omega) + \sum_{a\in A} P_{t,a}^{\text{sell}}(\omega) x_{t,a}^{-}(\omega) = \sum_{a\in A} P_{t,a}^{\text{buy}}(\omega) x_{t,a}^{+}(\omega)$$
$$\omega\in\Omega \quad t\in T^{d}\backslash\{0\}. \tag{2}$$

In (2) the left hand side represents the cash freed up to be reinvested at time $t\in T^{d}\backslash\{0\}$ and consists of two distinct components. The first term represents the coupons received on the coupon-bearing Treasury bonds held between times $t-1$ and t, the second term represents the cash obtained from selling part of the portfolio. This must equal the value of the new assets bought (in the absence of transaction costs), which is given by the right hand side of (2).

- *short sale constraints*. In our model we assume we will not be able to short any stocks or bonds

$$x_{t,a}(\omega)\geq 0 \qquad a\in A \quad \omega\in\Omega \quad t\in T^{\text{total}}\backslash\{T\} \tag{3}$$

$$x_{t,a}^{+}(\omega)\geq 0 \qquad a\in A \quad \omega\in\Omega \quad t\in T^{\text{total}}\backslash\{T\} \tag{4}$$

$$x_{t,a}^{-}(\omega)\geq 0 \qquad a\in A \quad \omega\in\Omega \quad t\in T^{\text{total}}\backslash\{0\}. \tag{5}$$

- *wealth constraint.* This constraint determines the portfolio wealth at each point in time

$$W_{t}(\omega) = \sum_{a\in A} P_{t,a}^{\text{buy}}(\omega) x_{t,a}(\omega) \qquad \omega\in\Omega \quad t\in T^{\text{total}}\backslash\{T\} \tag{6}$$

$$W_{T}(\omega) = \sum_{a\in A} P_{T,a}^{\text{sell}}(\omega) x_{T-\frac{1}{12},a}(\omega) + \sum_{a\in A\backslash\{S\}} \frac{1}{2}\delta_{T-1}^{a}(\omega) F^{a} x_{T-\frac{1}{12},a}(\omega) \qquad \omega\in\Omega. \tag{7}$$

As $x_{t,a}(\omega)$ represents the quantity held in asset $a \in A$ *after* rebalancing, we must use the *buy* price. If transaction costs were included two wealth variables would be defined; one before and one after rebalancing once the transaction costs have been incurred.

- *accounting balance constraints.* These constraints give the quantity invested in each asset at each time and for each scenario

$$x_{0,a}(\omega) = x_{0,a}^{+}(\omega) \qquad a \in A \quad \omega \in \Omega \tag{8}$$

$$x_{t,a}(\omega) = x_{t-\frac{1}{12},a}(\omega) + x_{t,a}^{+}(\omega) - x_{t,a}^{-}(\omega) \qquad a \in A \quad \omega \in \Omega \quad t \in T^{\text{total}} \setminus \{0\}. \tag{9}$$

The total quantity invested in asset $a \in A$ between time *t* and $t + \frac{1}{12}$ is equal to the total quantity invested in asset $a \in A$ between time $t - \frac{1}{12}$ and *t* plus the quantity of asset $a \in A$ bought at time *t* minus the quantity of asset $a \in A$ sold at time *t*.

- *information constraints.* These constraints ensure that the portfolio allocation can not be changed from one decision time to the next

$$x_{t,a}^{+}(\omega) = x_{t,a}^{-}(\omega) = 0 \qquad a \in A \quad \omega \in \Omega \quad t \in T^{i} \setminus T^{c}. \tag{10}$$

- *coupon re-investment constraints.* We assume that the coupon paid each six months will be reinvested in the same coupon-bearing Treasury bond

$$x_{t,a}^{+}(\omega) = \frac{\frac{1}{2}\delta_t^a(\omega) F^a x_{t-\frac{1}{12},a}(\omega)}{P_{t,a}^{\text{buy}}(\omega)} \qquad x_{t,a}^{-}(\omega) = 0$$

$$x_{t,S}^{+}(\omega) = x_{t,S}^{-}(\omega) = 0 \tag{11}$$

$$a \in A \setminus \{S\} \quad \omega \in \Omega \quad t \in T^c.$$

Note that $x_{t,a}^{+}(\omega)$ is not a decision variable here as it is fixed once the portfolio rebalancing decisions have been made at the previous decision time so that the information constraints are not violated.

- *annual roll-over constraint*. This constraint ensures that at each decision time all the coupon-bearing Treasury bond holdings are sold

$$x_{t,a}^{-}(\omega) = x_{t-\frac{1}{12},a}(\omega) \qquad a \in A \setminus \{S\} \quad \omega \in \Omega \quad t \in T^d \setminus \{0\}. \tag{12}$$

- *barrier constraints.* These constraints determine the shortfall of the portfolio at each time and scenario as defined in Table 1

$$h_t(\omega) + W_t(\omega) \geq L_t(\omega) \qquad \omega \in \Omega \quad t \in T^{\text{total}} \tag{13}$$

$$h_t(\omega) \geq 0 \qquad \omega \in \Omega \quad t \in T^{\text{total}}. \tag{14}$$

As the objective of the stochastic programme will put a penalty on any shortfall, optimizing will ensure that $h_t(\omega)$ will be zero if possible and as small as possible otherwise.

To obtain the maximum shortfall for each scenario, we need to add one of the following two constraints

$$H(\omega) \geq h_t(\omega) \qquad \omega \in \Omega \quad t \in T^{\mathrm{d}} \cup \{T\} \tag{15}$$

$$H(\omega) \geq h_t(\omega) \qquad \omega \in \Omega \quad t \in T^{\mathrm{total}}. \tag{16}$$

Constraint (15) must be added if the maximum shortfall is taken into account on a yearly basis, while (16) considers maximum shortfall on a monthly basis.

3.3 Objective functions: expected average and expected maximum shortfall

Starting with an initial wealth W_0 and a guarantee of $G\%$ annually, we have the guarantee liability at the planning horizon given by

$$W_0(1+G)^T. \tag{17}$$

The price at time *t* in scenario ω of the zero-coupon bond (which pays 1 at time *T*) is

$$Z_t(\omega) = e^{-y_{t,T-t}(\omega)(T-t)}. \tag{18}$$

Therefore, investing the amount given by (19) in these zero-coupon bonds will exactly give the guarantee at time *T* since irrespective of price fluctuations $Z_T(\omega) = 1$ and hence $L_T(\omega) = W_0(1+G)^T$ for all $\omega \in \Omega$. Thus the *barrier* for the minimum guaranteed return fund

$$L_t(\omega) = W_0(1+G)^T Z_t(\omega) = \frac{W_0(1+G)^T}{\left(1+y_{t,T-t}(\omega)\right)^{T-t}} \tag{19}$$

serves as a lower boundary.

For a minimum guaranteed return fund the objective of the fund manager is twofold; firstly to manage the investment strategies of the fund and secondly to take into account the guarantees given to all investors. Investment strategies must ensure that the guarantee for all participants of the fund is met with a high probability. Thus we might add a constraint limiting the probability of falling below the barrier in a VaR-type minimum guarantee constraint, i.e.

$$P\left(\max_{t\in T^{\text{total}}} h_t(\boldsymbol{\omega})>0\right)\leq\alpha \tag{20}$$

for α small. However, such scenario-based probabilistic constraints are extremely difficult to implement and tune to historical data and may give rise to non-convexities. We therefore use two convex alternatives in which the risk of falling below the barrier is traded off against return in the form of the expected wealth at each decision point.

Firstly, we will consider the *expected average shortfall* (EAS) model, in which the objective function is given by:

$$\begin{aligned}&\max_{\substack{\left\{x_{t,a}(\omega),x^+_{t,a}(\omega),x^-_{t,a}(\omega):\right\}\\ \left\{a\in A,\omega\in\Omega,t\in T^d\cup\{T\}\right\}}}\left\{\sum_{\omega\in\Omega}\sum_{t\in T^d\cup\{T\}}p(\omega)\left((1-\beta)W_t(\omega)-\beta\frac{h_t(\omega)}{\left|T^d\cup\{T\}\right|}\right)\right\}=\\&\max_{\substack{\left\{x_{t,a}(\omega),x^+_{t,a}(\omega),x^-_{t,a}(\omega):\right\}\\ \left\{a\in A,\omega\in\Omega,t\in T^d\cup\{T\}\right\}}}\left\{(1-\beta)\left(\sum_{\omega\in\Omega}p(\omega)\sum_{t\in T^d\cup\{T\}}W_t(\omega)\right)-\beta\left(\sum_{\omega\in\Omega}p(\omega)\sum_{t\in T^d\cup\{T\}}\frac{h_t(\omega)}{\left|T^d\cup\{T\}\right|}\right)\right\}\end{aligned} \tag{21}$$

In this case we maximize the expected sum of wealth over time while penalizing each time the wealth falls below the barrier. For each scenario $\omega \in \Omega$, we can calculate the average shortfall and then take expectations over all scenarios.

In this case only shortfalls at decision times are taken into account and any serious loss in portfolio wealth in between decision times is ignored. However the position of the portfolio's wealth relative to the fund's barrier is significant on a continuous basis from the fund manager's perspective and serious or repeated drops below this barrier might force the purchase of expensive insurance. To capture this feature specific to minimum guaranteed return funds, we also consider an objective function in which the shortfall of the portfolio is considered on a monthly basis.

For the *expected average shortfall with monthly checking* (EAS MC) model, the objective function is given by

$$\max_{\substack{\left\{x_{t,a}(\omega),x_{t,a}^{+}(\omega),x_{t,a}^{-}(\omega):\right.\\ \left.a\in A,\omega\in\Omega,t\in T^{d}\cup\{T\}\right\}}} \left\{(1-\beta)\left(\sum_{\omega\in\Omega} p(\omega)\sum_{t\in T^{d}\cup\{T\}} W_t(\omega)\right)-\beta\left(\sum_{\omega\in\Omega} p(\omega)\sum_{t\in T^{\text{total}}} \frac{h_t(\omega)}{\left|T^{\text{total}}\right|}\right)\right\} \quad (22)$$

Note that although we still only rebalance once a year but shortfall is now being measured in the objective on a monthly basis the annual decisions must also take into account the possible effects they will have on monthly shortfall.

The value of $0 \le \beta \le 1$ can be chosen freely and sets the level of risk aversion. The higher the value of β, the higher the importance given to shortfall and the less to the expected sum of wealth and hence the more risk-averse will be the optimal portfolio allocation. The two extreme cases are represented by $\beta = 0$, corresponding to the 'unconstrained' situation, which is indifferent to the probability of falling below the barrier, and $\beta = 1$, corresponding to the situation in which only the shortfall is penalized and the expected sum of wealth is ignored.

In general short-horizon funds are likely to attract more risk-averse participants than long-horizon funds, whose participants can afford to tolerate more risk in the short run. This natural division between short and long-horizon funds is automatically incorporated in the problem set up, as the barrier will initially be lower for long-term funds than for short-term funds. This feature of the model is demonstrated in Figure 3.

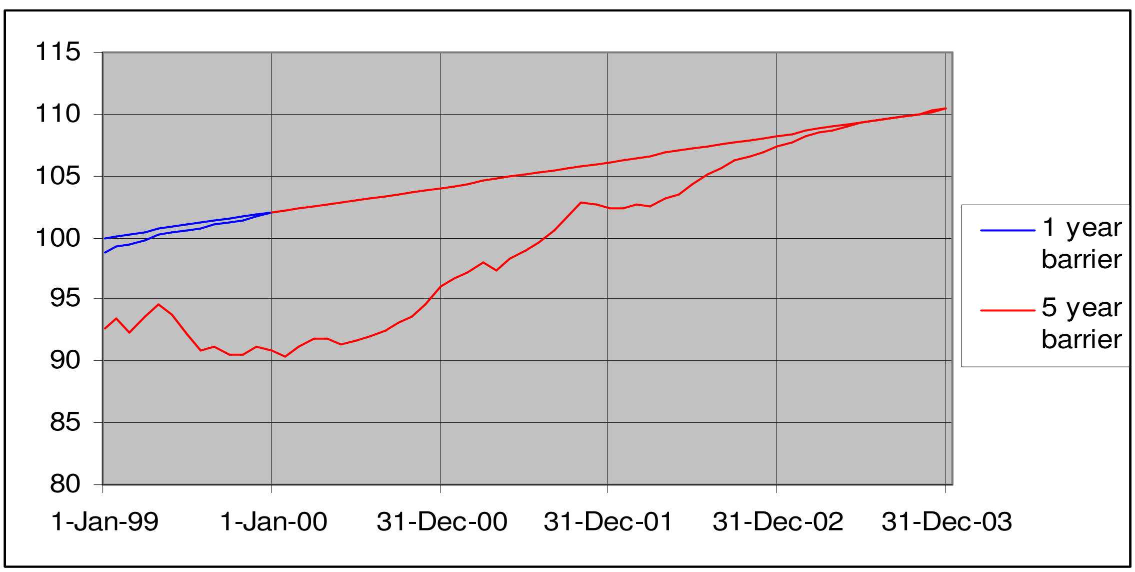


Figure 3: Barrier for one-year and five-year 2% guaranteed fund

The second model we will consider is the *expected maximum shortfall* (EMS) given by

$$\max_{\substack{\left\{x_{t,a}(\omega),x_{t,a}^{+}(\omega),x_{t,a}^{-}(\omega):\right.\\ \left.a\in A,\omega\in\Omega,t\in T^{d}\cup\{T\}\right\}}} \left\{(1-\beta)\left(\sum_{\omega\in\Omega} p(\omega)\sum_{t\in T^{d}\cup\{T\}} W_{t}(\omega)\right)-\beta\left(\sum_{\omega\in\Omega} p(\omega)H(\omega)\right)\right\} \qquad (23)$$

using constraint (15) to define $H(\omega)$.

For the *expected maximum shortfall with monthly checking* (EMS MC) model, the objective function remains the same but $H(\omega)$ is now defined by (16). In this model we penalize the expected maximum shortfall over time along scenarios which ensures that for each scenario $\omega \in \Omega$, $H(\omega)$ is as small as possible. Combining this with the constraints (15) or (16) ensures that $H(\omega)$ is equal to the maximum shortfall for scenario ω.

The EMS model focusses on limiting the maximum shortfall and therefore does not penalize portfolio wealth falling just slightly below the barrier several times. The EAS model on the other hand, incurs a penalty every time the portfolio's wealth falls below the barrier, but does not differentiate between a substantial shortfall at one point in time and a series of small shortfalls over time. So one model limits wealth from falling below the barrier *substantially* and the other limits the *number* of times it does so.

4. Asset Return Models

For minimum guaranteed return funds we must deal with a long-term liability and Treasury bonds of varying maturities and therefore must capture the dynamics of the whole term structure. For this we will use a Gaussian *economic factor model* (EFM) whose evolution, under the real world measure is determined by the stochastic differential equations

$$d\mathbf{R}_t = k\left(X_t + Y_t - R_t\right) + \gamma_R \sigma_R dt + \sigma_R d\mathbf{W}_t^R \tag{24}$$

$$d\mathbf{X}_t = \left(\mu_X - \lambda_X X_t + \gamma_X \sigma_X\right) dt + \sigma_X d\mathbf{W}_t^X \tag{25}$$

$$d\mathbf{Y}_t = \left(\mu_Y - \lambda_Y Y_t + \gamma_Y \sigma_Y\right) dt + \sigma_Y d\mathbf{W}_t^Y, \tag{26}$$

where the Wiener process increment $d\mathbf{W}$ terms are correlated. The three unobservable Gaussian factors ***R***, ***X*** and ***Y*** represent respectively a *short rate*, a *long rate* and the *slope* between an *instantaneous short rate* and the long rate. The short rate factor is mean reverting at rate *k* and the drifts of all three factors contain a *market price of risk* γ in volatility units. The 14 parameters of the factor dynamics are estimated by a generalized process form of the *EM algorithm* (Dempster *et al.*, 1977) in which Kalman filtering of factor evolution is alternated with maximum likelihood parameter estimation until convergence. For a complete description of the model, derivation of the yields, calibration of the model parameters and its simulation possibilities, see Medova *et al.* (2004) and Villaverde (2003). A typical yield curve simulation is shown in Figure 4.

The stock index price process $\mathbf{S}$ is assumed to follow a geometric Brownian motion, i.e.

$$\frac{d\mathbf{S}_t}{\mathbf{S}_t} = \mu_S dt + \sigma_S d\mathbf{W}_t^S \tag{27}$$

where $d\mathbf{W}_t^S$ is correlated with the $d\mathbf{W}_t$ terms driving the three term structure factors.

As sufficient historical data on Euro coupon-bearing bonds is difficult to obtain, we use the zero-coupon yield curve to construct the relevant bonds. Coupons on newly-issued bonds are closely related to the corresponding spot rate at the time, so we will use the current zero yield with maturity *T* as a proxy for the coupon rate of a coupon-bearing bond with maturity *T*, i.e. the coupon rate $\delta_2^{B^{10}}(\omega)$ on a newly issued 10-year Treasury bond at time $t = 2$ will be set equal to the projected 10-year spot rate $y_{2,10}(\omega)$ at time $t = 2$.

Generally

$$\delta_t^{B^T}(\omega) := y_{t,T}(\omega) \qquad t \in T^d \quad \omega \in \Omega \tag{28}$$

$$\delta_t^{B^T}(\omega) := \delta_{\lfloor t \rfloor}^{(T)}(\omega) \qquad t \in T^i \quad \omega \in \Omega, \tag{29}$$

where $\lfloor . \rfloor$ denotes the integral part. This ensures that as the yield curve falls, coupons on newly-issued bonds will go down correspondingly and each coupon cash flow will be discounted at the appropriate zero yield.

Bonds are assumed to pay coupons semi-annually and to be rolled over on an annual basis so that a coupon will be received after six months and again after a year just before selling the bond. This forces us to distinguish between the price at which we will sell the bond at decision times and the price for which we buy the new bond.

Let P_{t,B^T}^{sell} denote the *selling price* of the bond B^T of maturity *T* at time *t*, assuming two coupons have now been paid out and the time to maturity is equal to $T-1$, and let P_{t,B^T}^{buy} denote the *buying price* of a newly issued coupon-bearing Treasury bond with maturity *T*.

The 'buy' bond price at time *t* is given by

$$B_t^T(\omega) = F^{B^T} e^{-(T+\lfloor t \rfloor - t) y_{t,T+\lfloor t \rfloor - t}(\omega)} + \sum_{s=\frac{\lfloor 2t \rfloor}{2}+\frac{1}{2}, \frac{\lfloor 2t \rfloor}{2}+1, \ldots, \lfloor t \rfloor + T} \frac{\delta_t^{B^T}(\omega)}{2} F^{B^T} e^{-(s-t) y_{t,(s-t)}}$$

$$\omega \in \Omega \quad t \in T^{\text{total}}, \tag{30}$$

where the principal of the bond is discounted in the first term and the stream of coupon payments in the second. At decision times the 'sell' bond price is given by

$$B_t^T(\omega) = F^{B^T} e^{-(T-1)y_{t,T-1}(\omega)} + \sum_{s=\frac{1}{2},1,\ldots,T-1} \frac{\delta_{t-1}^{B^T}(\omega)}{2} F^{B^T} e^{-(s-t)y_{t,(s-t)}}$$

$$\omega \in \Omega \quad t \in \{T^d \setminus \{0\}\} \cup \{T\}. \tag{31}$$

Note that the coupon rate $\delta_{t-1}^{B^T}(\omega)$ is in (31) as this is the rate of the bond sold at time *t* the coupon rate is reset for the newly issued Treasury bonds. We also assume that the coupon that is being paid out after six months will be reinvested in the same coupon-bearing Treasury bond. This gives the following adjustment to the amount held in bond B^T at time *t*

$$x_{t,B^T}(\omega) = x_{t-\frac{1}{12},B^T}(\omega) + \frac{\frac{1}{2}\delta_t^{B^T}(\omega) F^{B^T} x_{t-\frac{1}{12},B^T}(\omega)}{P_{t,B^T}^{\text{buy}}} \qquad t \in T^c \quad \omega \in \Omega. \tag{32}$$

5. Historical Backtest

We will look at an *historical backtest* in which statistical models are fitted to market data up to an initial model decision time *t* and scenario trees are generated from *t* to some chosen horizon $t+T$. The optimal *first stage/root node decisions* are then implemented at time *t* and to generate the historical portfolio returns with these decisions at time $t+1$. Afterwards the whole procedure is rolled forward for *T* trading times.

Our backtest will involve a *telescoping horizon* as depicted in Figure 4.

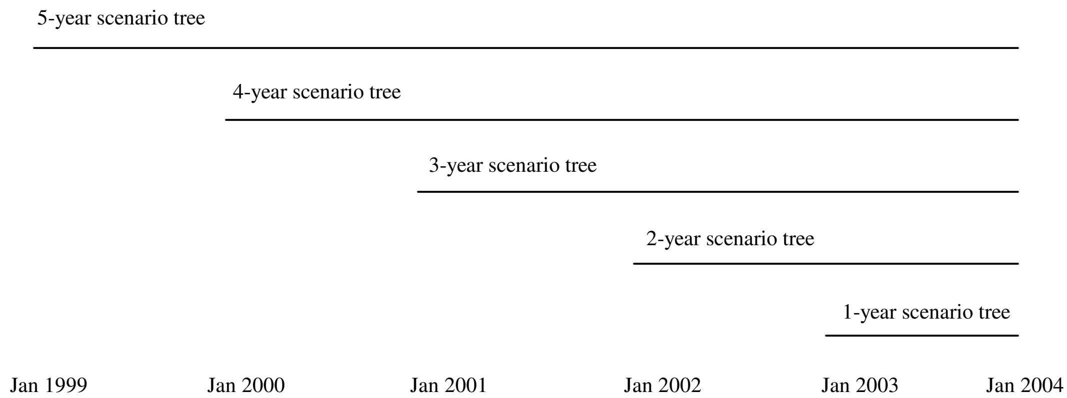


Figure 4: Telescoping Horizon Backtest Schema

At each decision time *t*, the parameters of the stochastic processes driving the stock return and the three factors of the term structure model are re-estimated and re-calibrated using historical data up to and including time *t*, and the initial values of the simulated scenarios are given by the actual historical values of the variables at these times. Re-estimating and re-calibrating the simulator's parameters at each successive initial decision time *t* captures information in the history of the variables up to that point.

Although the optimal second and later stage decisions of a given problem may be of "what-if" interest, managers' and decision makers' focus is on the implementable first-stage decisions which are hedged against the simulated future uncertainties. The reasons for implementing stochastic optimization in this way are twofold. Firstly, after one year has passed the actual values of the variables realized may not coincide with any of the values of the variables in the simulated scenarios. In this case the optimal investment policy would be undefined, as the model only has optimal decisions for the nodes on the simulated scenarios. Secondly, as one more year has passed new information has become available to re-estimate and re-calibrate the simulator's parameters. Relying on the original optimal investment strategies will ignore this information.

For the backtest, we will use three different tree structures with approximately the same number of data scenarios, but with an increasing *initial branching factor*, which defines the number of branches from the root node at stage one. We will start of by solving the five-year problem using a 6.6.6.6.6 tree, which gives a total of $6^5 = 7776$ scenarios. The second option is to use 32.4.4.4.4 = 8192 scenarios and the third is the extreme case of 512.2.2.2.2 = 8192 scenarios, thereby increasing the initial branching factor from 6 to 32 and finally to 512. For the subsequent stages of the backtest we will adjust the branching factor in such a way that the total number of scenarios stays as close to the original number of scenarios as possible and such that the same ratio is maintained. This gives us the following tree structures, described in Table 2.

Jan 1999	6.6.6.6.6 = 7776	32.4.4.4.4 = 8192	512.2.2.2.2 = 8192
Jan 2000	9.9.9.9 = 6561	48.6.6.6 = 10368	512.2.2.2 = 4096
Jan 2001	20.20.20 = 8000	80.10.10 = 8000	768.3.3 = 6912
Jan 2002	88.88 = 7744	256.32 = 8192	1024.4 = 8192
Jan 2003	7776	8192	8192

Table 2: Tree Structure for Different Backtests

5.1 Results

Figures 5 to 10 plot the barrier, the portfolio performance of the annual and monthly checking methods, together with the models expectation one period ahead. Starting with an initial wealth W_0 equal to 100 and an annual guaranteed return G of 2% the barrier can be calculated using (19) for each month between January 1999 and January 2004 using the appropriate historical zero-yields. The barrier is independent of the choice of objective function or tree structure and therefore identical in all six figures. For January 1999, three five-year monthly scenario trees were generated for each of the annual tree structures described in Table 2. For each scenario tree, the corresponding stochastic programme was solved for each of the four different objective functions presented in Section 3.3 and the first-stage decisions implemented. This gives us the fund's portfolio strategy between January 1999 and January 2000, the next decision time. Starting from the initial wealth of 100, historical returns for the various assets determine the wealth for the portfolio over the year 1999. For January 2000, having observed the price variation of the various assets in 1999, the stochastic processes driving the asset prices were re-calibrated using all the data available up to January 2000 and three four-year scenario trees generated (as the time horizon of the fund has now gone down by one year). Using the portfolio's wealth at January 2000 as an input variable, four stochastic programmes were again solved, one for each objective functions. Again the first-stage decisions were implemented and the same procedure is repeated for January 2001, January 2002 and finally January 2003 for three-, two- and one-year scenario trees respectively. Included in the figures are the one-year ahead in-sample expectations of the portfolio's wealth. Implementing the first-stage decisions, the portfolio's wealth is calculated for each scenario in the simulated tree one year later, after which an expectation over the scenarios is taken.

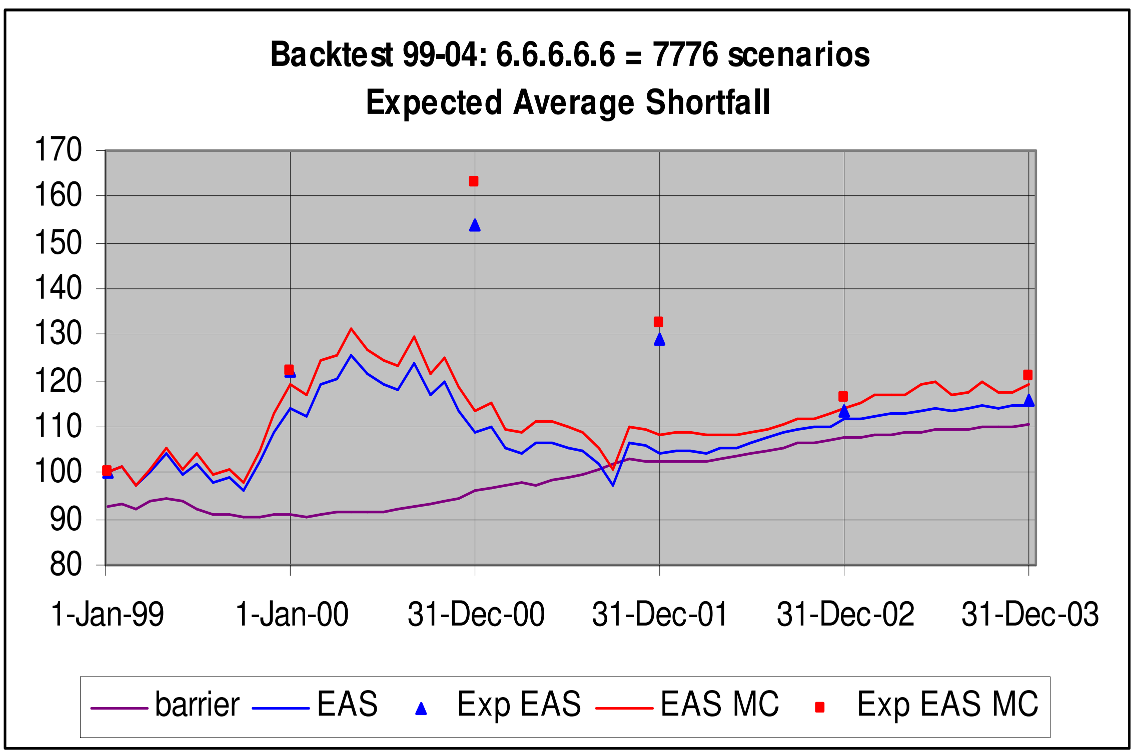


Figure 5: Expected Average Shortfall using 6.6.6.6.6 tree

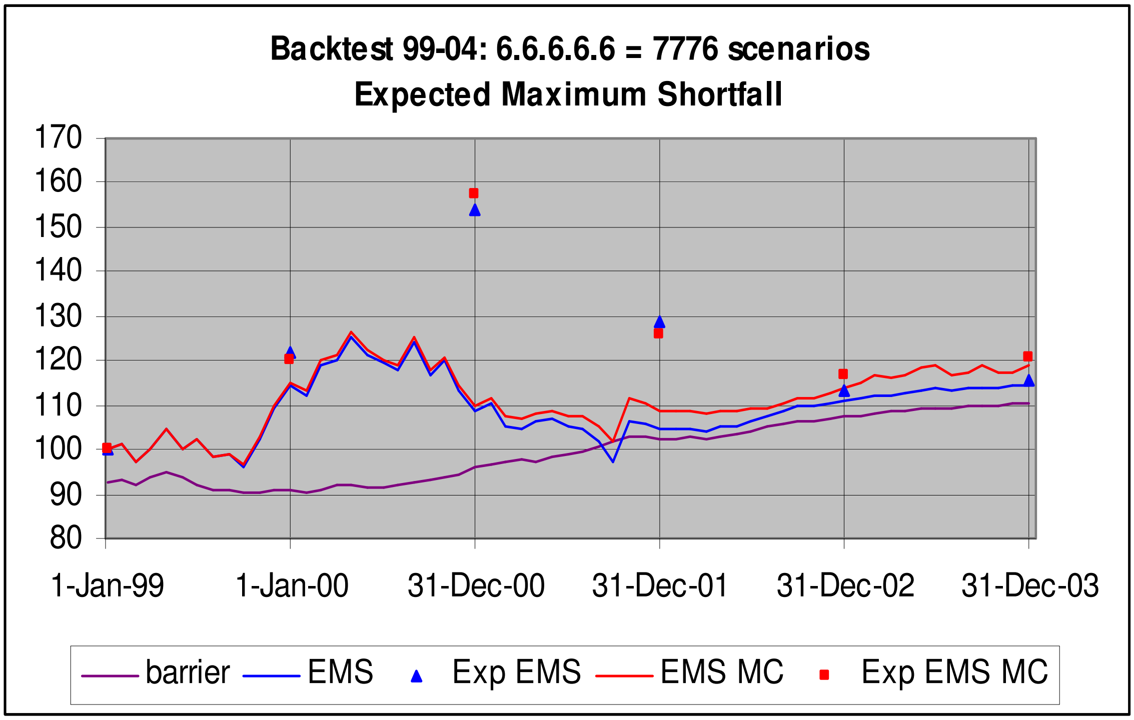


Figure 6: Expected Maximum Shortfall using 6.6.6.6.6 tree

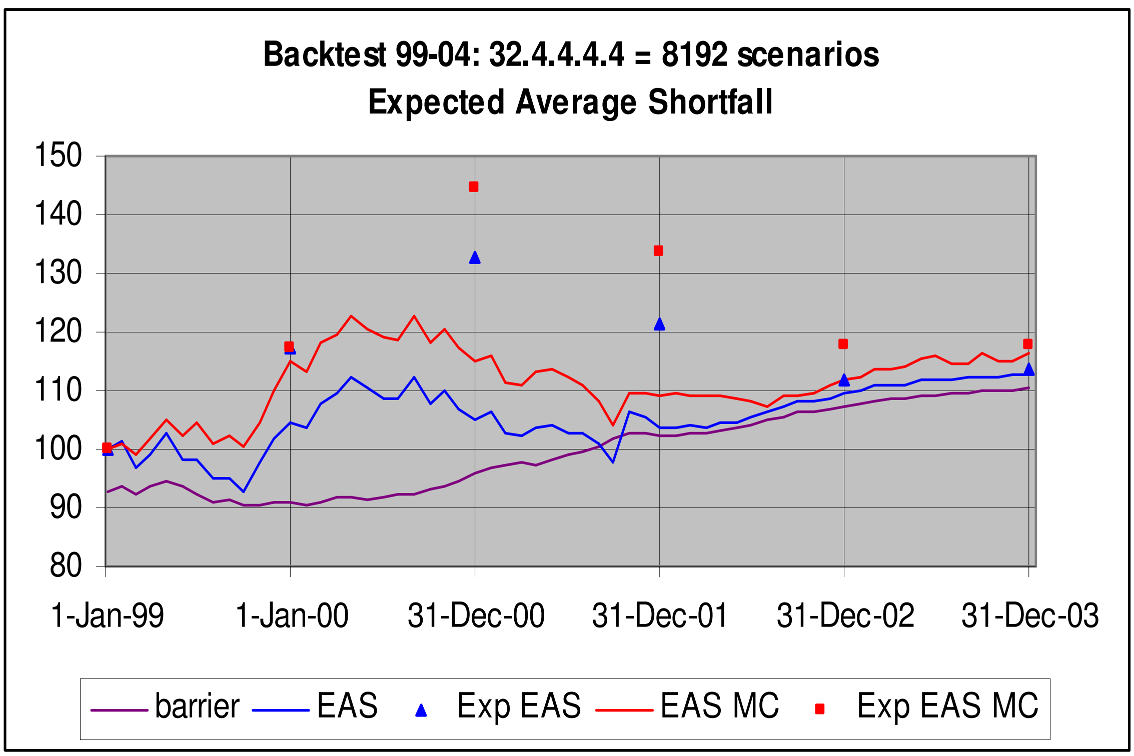


Figure 7: Expected Average Shortfall using 32.4.4.4.4 tree

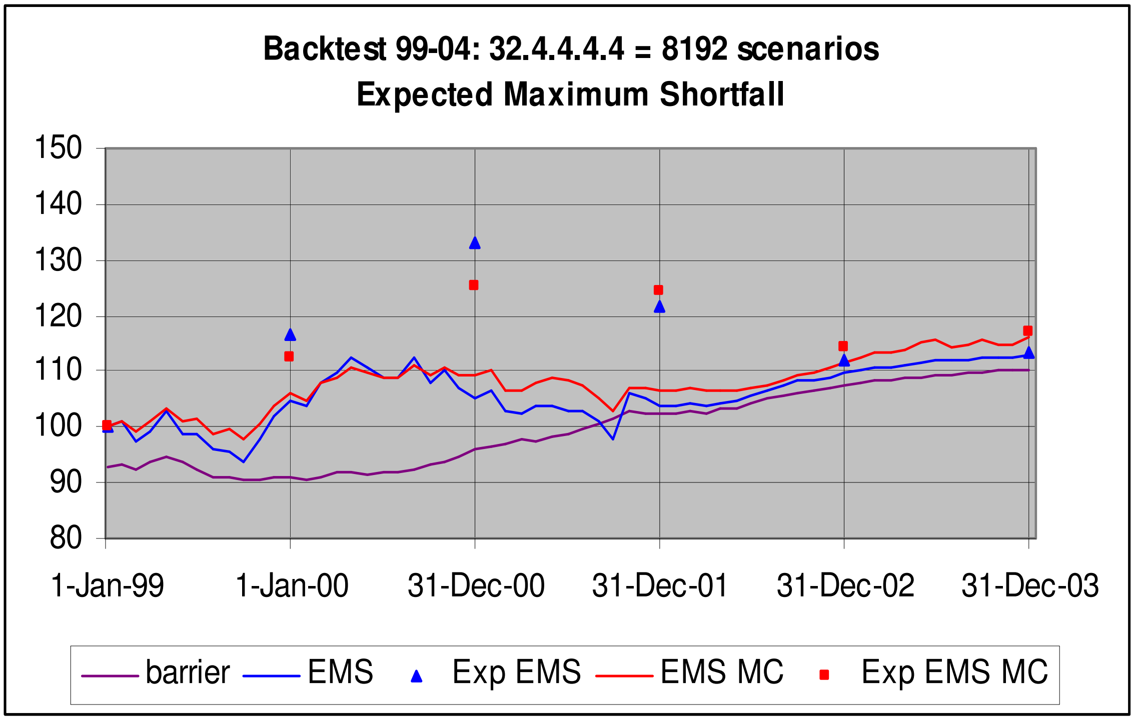


Figure 8: Expected Maximum Shortfall using 32.4.4.4.4 tree

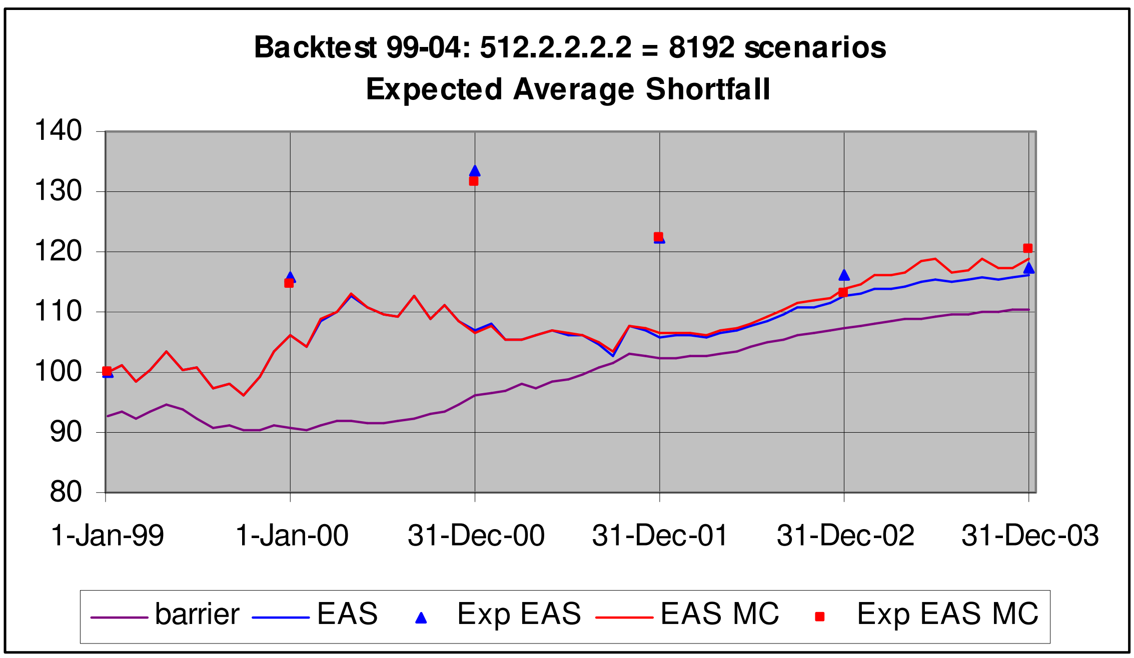


Figure 9: Expected Average Shortfall using 512.2.2.2.2 tree

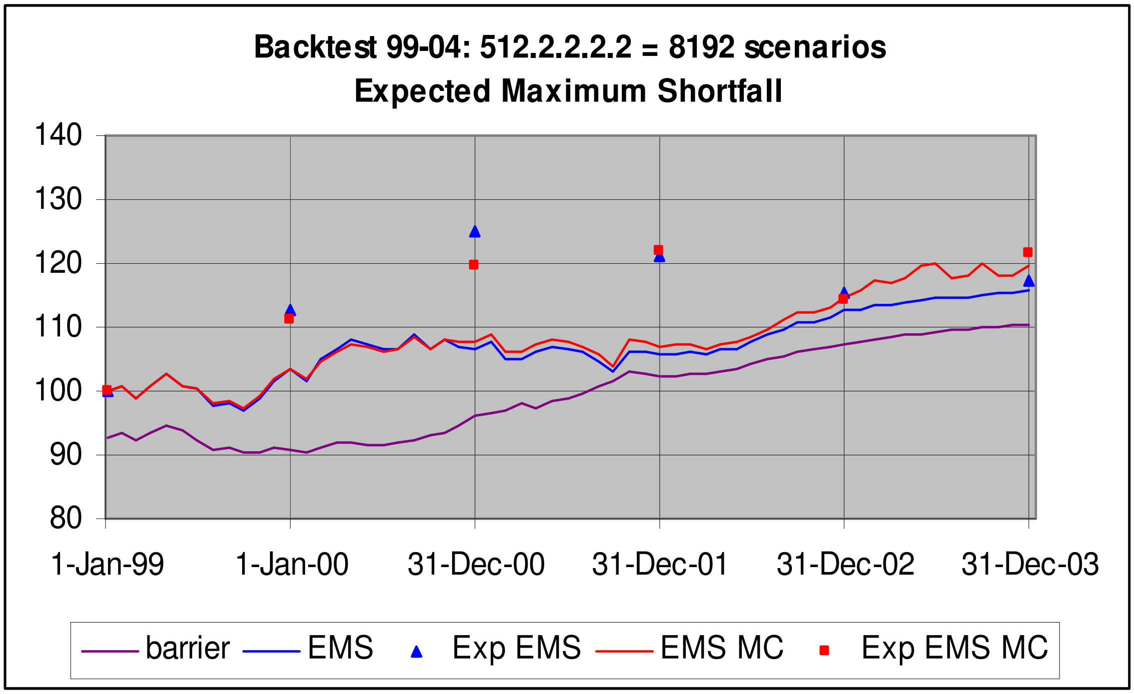


Figure 10: Expected Maximum Shortfall using 512.2.2.2.2 tree

First observe that the risk management monitoring incorporated into the model appears to work well. In all cases the only time portfolio wealth dips below the barrier, if at all, is on September 11 2001. The initial in-sample overestimation of the model is likely mainly due to the short time series for parameter estimation which leads to hugely inflated stock return expectations. However as time progresses and more data points to re-calibrate the model are obtained, the model expectation and real-life realization very closely approximate each other. For reference we have included the performance of the Eurostoxx 50 in Figure 11 to give an indication of how the stock market performed over the backtesting period. Even though this was a difficult period for the portfolio to perform well in, it leads to an excellent demonstration that the risk management incorporated into the model operates as required. It is in periods of economic downturn that one wants the model to survive.

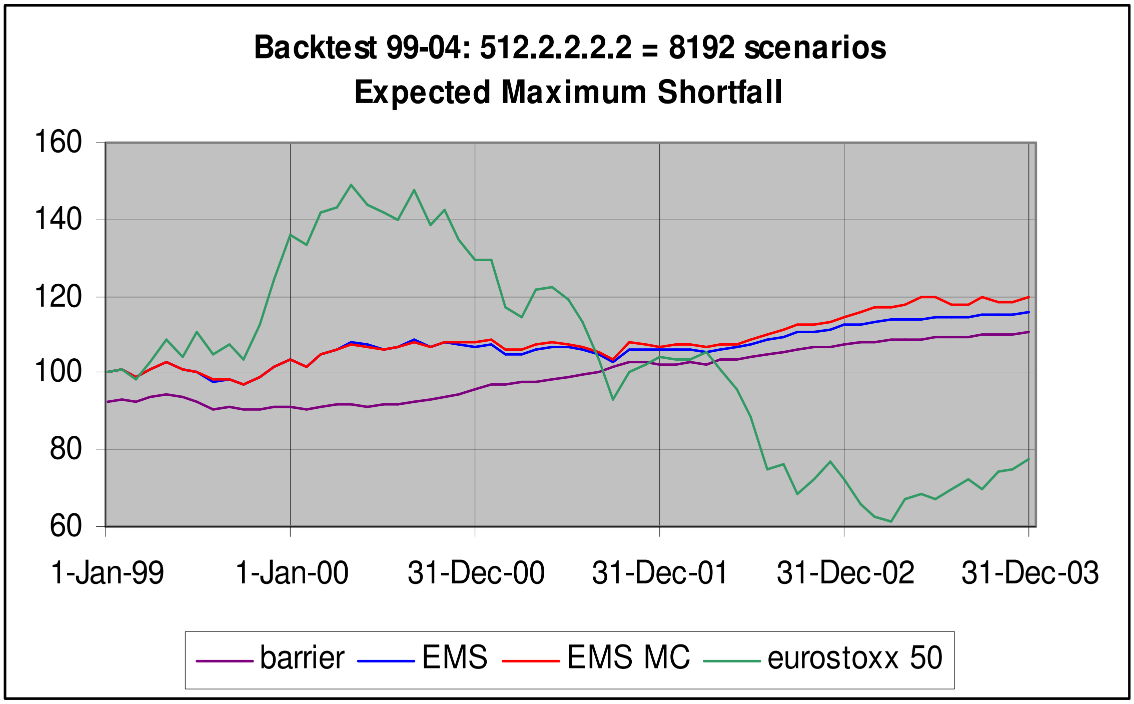


Figure 11: Comparison of the Fund's Portfolio to the Eurostoxx 50

Tables 3 and 4 give the backtest portfolio allocations for the 32.4.4.4.4 tree using the maximum shortfall objectives. In both cases we can identify a tendency for the portfolio to move to the safer, shorter-term assets as time progresses. This feature is built into the model as shown in Figure 3. Furthermore, for the decisions to be made on January 2002 and 2003, the portfolio wealth is significantly closer to the barrier for the EMS model than it is for the EMS MC model. This increased risk for the fund is taken into account by the model and results in an investment in safer short-term bonds. Whereas the EMS MC model invests mainly in bonds with a maturity in the range of three to five years, the EMS model stays in the one to three year range. As a result the portfolio's wealth manages to stay above the barrier for both models.

	1y	2y	3y	4y	5y	10y	30y	Stock
Jan 99	0	0	0	0	0	**0.23**	**0.45**	**0.32**
Jan 00	0	0	0	0	0	0	**0.37**	**0.63**
Jan 01	**0.04**	0	0	0	0	**0.39**	**0.53**	**0.40**
Jan 02	**0.08**	**0.16**	**0.74**	0	0	0	0	**0.01**
Jan 03	**0.92**	0	0	0	0	**0.07**	0	**0.01**

Table 3: Portfolio Allocation Expected Maximum Shortfall using 32.4.4.4.4 tree

	1y	2y	3y	4y	5y	10y	30y	Stock
Jan 99	0	0	0	0	**0.49**	**0.27**	0	**0.24**
Jan 00	0	0	0	0	**0.25**	**0.38**	0	**0.36**
Jan 01	0	0	0	0	**0.49**	**0.15**	0	**0.36**
Jan 02	0	0	0	**0.47**	**0.44**	0	0	**0.10**
Jan 03	0	0	**0.78**	**0.22**	0	0	0	**0.01**

Table 4: Portfolio Allocation Expected Maximum Shortfall with Monthly Checking using 32.4.4.4.4 tree

From Figures 5 to 10 we can see that in all cases the method with monthly checking outperforms the equivalent method with just annual checks. However as the initial branching factor is increased, the models increasingly improve their performance. For the 512.2.2.2.2 = 8196 scenario tree, all four objective functions give portfolio allocations which keep the portfolio wealth above the barrier at all times, but models with the monthly checking still outperform the others. The more important difference however seems to lie in the deviation of the expected in-sample portfolio wealth from the actual historical realization of the portfolio value. Table 5 displays this average annual deviation and shows clearly a reduction in this deviation for all four models as the initial branching factor is increased. The model which uses the expected maximum shortfall with monthly checking as its objective function significantly outperforms the other models.

	EAS	EAS MC	EMS	EMS MC
6.6.6.6.6	14.78 %	14.40 %	14.73 %	13.59 %
32.4.4.4.4	11.58 %	11.33 %	11.46 %	8.17 %
512.2.2.2.2	10.70 %	9.60 %	8.93 %	6.93 %

Table 5: Average Annual Deviation

Overall these backtests have shown that the stochastic optimization framework described in this paper carefully considers the liability risks created by the guarantee. The EMS MC model especially produced well-diversified portfolios that did not change drastically from one year to the next and resulted in a portfolio which, even in a period of economic downturn and uncertainty, remained above the barrier.

6. Conclusions

This paper considers the construction of investment products which give a minimum guaranteed return. We have concentrated on the design of the liability side of the product, paying particular attention to the pricing of bonds using a three-factor Gaussian term structure model which gives reliable results for both long-term and short-term yields. We constructed several objective functions for the dynamic stochastic optimization fund management model using expected average shortfall and expected maximum shortfall in order to combine risk management with strategic asset allocation. Finally we introduced the concept of monthly checking, which we showed improved the results considerably.

In future research we intend to relax the assumption of a closed end fund, allowing for contributions throughout the lifetime of the product. A second extension is to look at inflation-linked barriers, as there has been an increased demand recently for such products. We will also be investigating how to extend this model and use it as a building block for an open multi-link pension fund, in which we will deal with several unit links of varying risk aversion and apply risk management to each individual client's portfolio.

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