

Extreme Value Theory

Extreme values and the measurement of operational risk

Below, in the first of a two-part discussion, Elena Medova introduces extreme value theory and looks at how the approach fits in with recent initiatives to improve operational risk management in financial institutions. Next month, she offers a technical description of a new EVT approach that makes use of Bayesian simulation techniques to measure firm-wide op risk.

Since the publication of Gnedenko's limit theorems for maxima in 1941, and Gumbel's *Statistics of Extremes* in 1958, the engineering community has applied a body of theory concerning the calculation of extreme values to a great variety of practical problems.

Extreme value theory (EVT) has found applications in structural, aerospace, ocean and hydraulic engineering as well as in studies of pollution, meteorology and highway traffic.

Actuaries also now use EVT extensively to model casualty insurance claims. So perhaps it's not surprising that researchers have begun to explore whether EVT can be used to measure operational risk in financial institutions.

The key attraction of EVT is that it offers a set of ready-made approaches to the most difficult problem in op risk analysis: how can risks that are both extreme, and extremely rare, be modelled appropriately?

But, as we discuss, applying EVT to financial institution op

risk raises some difficult issues. Some of these arise from the nature of the data that is available to analysts. Others relate to the purpose of any op risk analysis, the definition of an "extreme" event, and the meaning of the term "operational risk".

Key EVT literature

The literature on EVT is now quite extensive. Galambos et al (1994) offers a useful exposition of EVT theory and practice, while Castillo (1998) describes its applications in engineering.

Perhaps the most complete exposition of EVT is given in a series of working papers and a monograph by Embrechts et al. (1997)¹. The monograph also examines the application of extreme values to insurance and risk management.

Other key texts include the

significant theoretical and experimental results in Smith (1987, 1997) and McNeil and Saladin (1997); McNeil's extreme value software library written in S-plus; and Danielson and de Vries (1997).

Key components of EVT

The principal results of extreme value theory concern the limiting distribution of sample extrema (maxima or minima).

Suppose that $X = (X_1, \dots, X_n)$ is a sequence of independent identically distributed observations with distribution function F , not necessarily known, and let the sample maximum be denoted by $M_n = \max \{X_1, \dots, X_n\}$.

Under certain assumptions - subexponential distributions - the tail of the maximum determines the tail of the sum as $n \rightarrow \infty$.

More generally, the *generalized extreme value distribution* (GEV) given by $H_\xi(x)$ describes the limit distribution of suitably normalized maxima. The random variable X may be replaced by $(X-\mu)/\sigma$ to obtain a standard GEV with a distribution function that is specified as shown below, where μ , σ and ξ are the *location*, *scale* and *shape parameters*, respectively.

Three standard distributions correspond to different values of ξ . They are the:

Gumbel distribution Λ , $\xi=0$;

Fréchet distribution Φ_α , $\xi=\alpha^{-1} > 0$;

$$H_{\xi, \mu, \sigma}(x) = \begin{cases} \exp \left[- \left(1 + \xi \frac{x - \mu}{\sigma} \right)^{-1/\xi} \right] & \text{if } \xi \neq 0, 1 + \xi \frac{x - \mu}{\sigma} > 0 \\ \exp \left[- \exp \left(- \frac{x - \mu}{\sigma} \right) \right] & \text{if } \xi = 0, \end{cases}$$

1. In August this year Risk Books will publish a collection of key papers and new research on EVT edited and introduced by Paul Embrechts, with title of "Extremes and Integrated Risk Management".

Weibull distribution $\Psi\alpha, \xi = -\alpha^{-1} < 0$.

The purpose of tail estimation procedures is to estimate the values of X outside the range of existing data. To do this, researchers have employed both *extreme epochs* (events), and *exceedances* of a specified level. The standard approach assumes that the tail of the population follows the selected family of distributions.

Pickands (1975) showed (with some additional assumptions) that the *generalized Pareto distribution* (GPD) – the limit distribution of *excesses* $Y := \max\{X-u, 0\}$ over sufficiently high thresholds u – offers a good approximation of the tail of F for some fixed ξ and β which depend upon u . Similar results have been obtained for stationary sequences of observations whose dependence extends only to a finite number of previous values, see Leadbetter *et al.*, (1983).

Thus the distribution of Y may be thought of as the conditional distribution of X given

$$X > u.$$

The GPD with *shape parameter* ξ and *scale parameter* β is specified as

$$G_{\xi, \beta}(y) = \begin{cases} 1 - \left(1 + \xi \frac{y}{\beta}\right)^{-1/\xi} & \text{if } \xi \neq 0 \\ 1 - \exp\left(-\frac{y}{\beta}\right) & \text{if } \xi = 0 \end{cases}$$

$$\text{where } y \in \begin{cases} [0, \infty] & \text{if } \xi \geq 0 \\ [0, -\frac{\beta}{\xi}] & \text{if } \xi < 0 \end{cases}$$

and the sign of the shape parameter ξ determines its tail behaviour and thus the tail behaviour of the original distribution.

For $\xi > 0$, the tail of the distribution function F of X decays like a *power function* $x^{-1/\xi}$. In this case, F

belongs to a family of *heavy-tailed* distributions that includes, among others, the Pareto, log-gamma, Cauchy and t-distributions.

For $\xi = 0$, the tail of F decreases *exponentially*, and belongs to a class of *medium-tailed* distributions that include the normal, exponential, gamma and log-normal distributions.

Finally, for $\xi < 0$, the underlying distribution F is characterised by a *finite right endpoint*, which class of *short-tailed* distributions includes the uniform and beta distributions.

It can be shown that the mean excess function (expectation) of the GDP is given by the expression

$$e(u) = E(X - u \mid X > u) = \frac{\beta + \xi u}{1 - \xi},$$

where

$$\beta = \sigma + \xi(u - \mu)$$

and $\max_n Y_n$ follows a GEV distribution with parameters ξ, μ, σ .

The POT model

The *peaks over threshold* (POT) model can be used to estimate the excess distribution with respect to a threshold level u , and to estimate the tail shape of the original distribution. It should be noted that the threshold setting of the POT model is data dependent. The model defines a two-dimensional (Y_n, N_u) space-time point process on $X_n \geq u, n=1, \dots, N_u$.

Here Y_n and N_u are independent random variables such that $Y_n \sim \text{GPD}(\xi, \beta)$, and the *number of excesses* N_u follows a Poisson process with *intensity* λ , representing the average number of exceedances over the time interval used for the sampling process and given by

$$\lambda = \left(1 + \xi \frac{(x-\mu)}{\sigma}\right)^{-1/\xi} \text{ for } x \geq u.$$

The threshold u is usually chosen using mean excess plots, and other statistical devices developed by Smith (1987, 1997), that consider the trade off between bias in estimating the excess distribution function parameters and their variance.

The POT model for operational risk

The estimation of the parameters of the POT model is usually based on the *maximum likelihood method*, which requires a relatively large number of observations above the threshold (e.g., more than 100).

But in an operational risk situation, it might be more realistic to think in terms of 20 or 30 excesses. This suggests that another estimation technique will be necessary for operational risk, and we propose the use of Bayesian simulation techniques (Medova, 1999).

A presentation of this, rather technical, estimation procedure will form the subject of an article in August's *Operational Risk*.

Another complex issue is the question of consistency in deciding upon threshold values. In an ideal world, the threshold obtained from the POT model should correspond to the integrated market and credit risk VaR quantile. This is currently one topic of our programme of research into integrated risk management.

Some problems

To justify the modelling of operational risk using EVT, many obstacles must be overcome.

But not all the obstacles are technical in nature. Many are

caused by the fact that operational risk continues to be ill-defined for the purpose of calculating risk capital.

For example, one might ask how any approach to operational risk using extreme value theory relate to definitions of “normality” and the problem of internal bank controls and external supervision?

More topically, how does EVT relate to the Basel Committee on Banking Supervision’s present proposals for controlling operational risk?

The Committee has attempted to clarify the complex issues of risk management by adopting a “three-pillared” approach. The first pillar concerns capital allocation, the second supervision and controls, and the third transparency and consistency of risk management procedures. What is the relation of EVT to these three pillars – most problematically, the second and third pillars?

Another problem is that, while risk capital is generally understood as a way of protecting a bank against “unexpected” losses – expected losses are covered by business-level reserves – it is not clear to what degree it is used to cover the most extreme risks.

Some practitioners and regulators have made it clear that they do not intend to include the risk of the most extreme losses in their calculations of either economic risk capital or regulatory risk capital.² So in what way is extreme value theory useful in measuring operational risk?

Lastly, how can an analyst deal with market and credit risk

management without double-counting? Some framework that identifies the roles of credit, market, and other risks must be constructed.

Below we suggest some thoughts on these issues that help to show how they relate to the nature of extreme value theory.

Some solutions

Let us assume that a bank’s market and credit risk management is informed by quantitative models that compute the value at risk (VaR) for market risk and credit risk and that allocate economic capital to these risks.

Is such a capital allocation sufficient for unexpected losses due to human errors, natural disasters, fraudulent activities and other external factors including acts of God? Clearly not, for two reasons.

Firstly, the models do not take into account operational risks (extreme or not). Secondly, they make various assumptions about “normality”, and so exclude extreme and rare events. Such events include natural disasters as well as major social or political events.

How can we think clearly about operational and extreme events? In our research, we termed the related risk factors *primal* (catastrophic). Processing all incoming information and taking decisions at different levels of the bank may lead to further losses reflected in increased business costs. Some such *secondary* causes are human or technological errors, lack of control to prevent unauthorised or inappropriate transactions

being made, fraud and faulty reporting.

Many of these secondary causes are used in one or other definition of operational risk. Some of them, such as the failure of a bank’s internal computer system, may themselves be regarded as primal and catastrophic.

The first step in operational risk management should be a careful analysis of all available data to identify the statistical patterns of losses related to identifiable primal and secondary risk factors.

Ideally, this analysis would form part of the financial surveillance system for the bank. In the future, perhaps such an analysis might also form part of the duties of bank supervisors. In other words, at a conceptual level, it relates to the second of the Basel Committee’s three pillars.

Here, the important point for analysts is that this surveillance is concerned with the identification of the “normality” of business processes. The identification of suitable of market and credit risk models also forms a natural part of this operational risk assessment.

Such an analysis should allow an analyst to classify a bank’s losses into two categories: (1) *significant* in value but *rare*, corresponding to extreme loss events distributions; (2) *low value* but *frequently* occurring, corresponding to ‘normal’ loss event distributions.

Next, we might take the view that control procedures will be developed for the reduction of the low value/frequent losses, and for

2. For example, see practitioner Tony Peccia’s comments on capital allocation at CIBC, one of Canada’s leading banks, in April’s *Operational Risk*, page 12, and comments by Jeremy Quick of the UK’s Financial Services Authority, in February’s *Operational Risk*, page 10.

their illumination and disclosure (the third pillar of the Basel approach).

These control procedures, and any continuing expected level of loss, should be accounted for in the operational budget. This allows us to assume that only losses of large magnitude need be considered for operational risk economic capital provision.

Again, an analysis of the profit and loss data, and the verification or rejection of the assumption of normality, related to the universe of primary and secondary risks, are all the part of the (usually internal) risk supervisory process.

From value at risk to extreme event analysis

Value at risk (VaR) has been adopted as the central measure of market risk by many organisations that trade in the world’s financial markets.

Under normal market conditions VaR provides a measure of the market risk due to adverse market movements. Any deviation from normality will tend to underestimate the value at risk.

Similarly, under normal conditions for credit risk, which correspond to credit ratings higher than BBB, credit models provide measures for credit risk.

But there are theoretical alternatives to VaR that also offer a coherent risk measure (Artzner et al., 1997, 1999). One approach is to define a measure for the expected shortfall or tail conditional expectation with respect to the unknown maximal loss distribution.

We adopt a similar conditional measure for operational risk. But we assume that a threshold has been derived from the marginal statistical distribution of losses as a part of the operational risk supervisory process – as discussed above, and in next month’s technical paper.

This gives a slight twist to the usual definition of operational risk. For the purpose of calculating capital provision, operational risk is everything which is not credit and market risk under normal conditions – including catastrophic market and credit losses where appropriate.

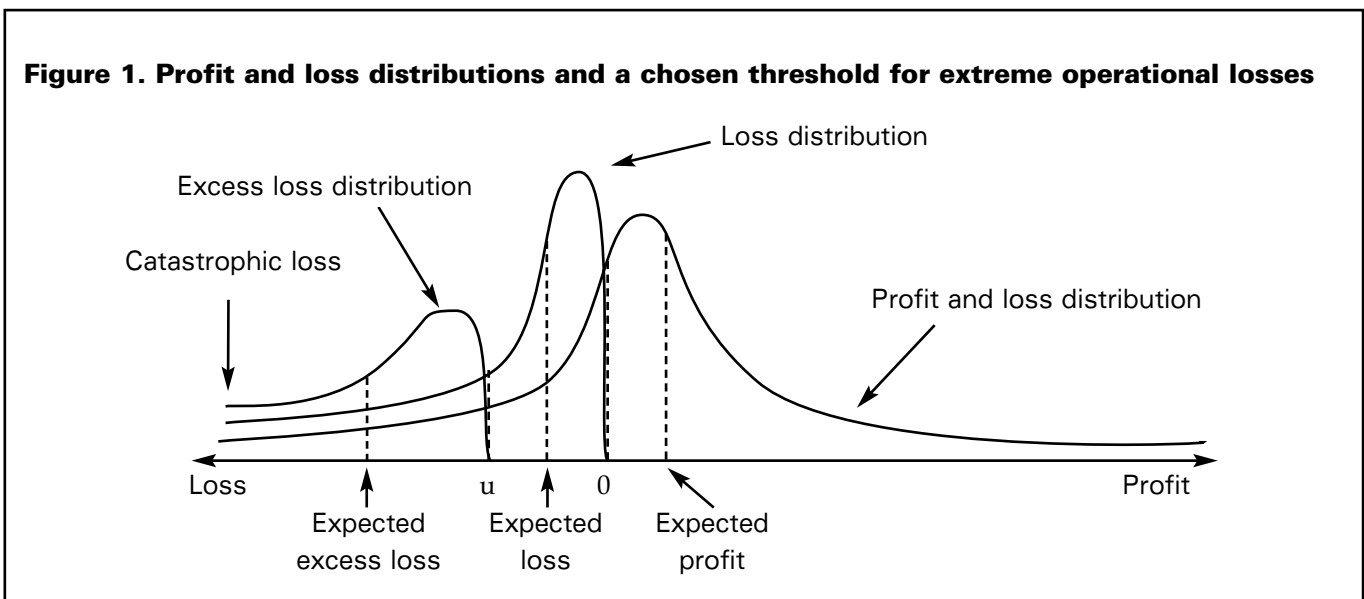
In effect, operational risk is redefined as a tail of the profit and loss distribution of the appropriate level – business unit, or enterprise-wide – of the bank (see Figure).

In the presence of extremes, further analysis will be required for the identification of a threshold, and for the evaluation of a capital requirement for unexpected operational losses.

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References

- Artzner P., F. Delbaen, J.M. Eber & D.Heath (1997). Thinking coherently. *Risk* 10, 68-71.
- Artzner P., F. Delbaen, J.M. Eber & D.Heath (1999).



- Coherent measures of risk.
Mathematical Finance **9**, 203-228.
- Castillo E. (1988). *Extreme Value Theory in Engineering*. Academic Press, Inc.
 - Danielson J. and C. G. de Vries (1997). Tail index and quantile estimation with very high frequency data. *Journal of Empirical Finance* **4**, 241-257.
 - Embrechts P., C. Kluppelberg & T. Mikosch (1997). *Modelling Extremal Events*. Springer, Berlin.
 - Galambos J., Lechner J. & Simiu E., eds. (1994). *Extreme Value Theory and Applications*. Proceedings of the Conference on Extreme Value Theory and Applications, Gaithersburg, Maryland 1993.
 - Gnedenko B.V. (1941). Limit theorems for the maximal term of a variational series. *Comptes Rendus de l'Academie des Sciences del'URSS* **32**, 7-9.
 - Gumbel E.J. (1958). *Statistics of Extremes*. Columbia Univ. Press., New York.
 - Leadbetter M.R., Lindgren G. and Rootzen H. (1983). *Extremes and Related Properties of Random Sequences and Processes*. Springer Verlag, Berlin.
 - McNeil A.J. & T. Saladin (1997). The peaks over thresholds method for estimating high quantiles of loss distributions. In *Proceedings of XXVII International ASTIN Colloquium*, 23-43.
 - Medova E.A.(1999). Operational risk measures and Bayesian simulation methods for capital allocation. Working Paper 27/99, Judge Institute of Management Studies, University of Cambridge.
 - Quick J. (1999). Other risks. The FSA conference: *Reforming the Basel Capital Accord*, 2.12.1999, London.
 - Smith R.L. (1987). Estimating Tails of Probability Distributions. *The Annals of Statistics* **15**, 1174-1207.

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