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LP VALUATION OF EXOTIC AMERICAN OPTIONS EXPLOITING STRUCTURE

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We investigate the numerical solution of American financial option pricing problems, using a novel formulation of the valuation problem as a linear programme (LP) introduced in [14, 21]. By exploiting the structure of the constraint matrices derived from standard Black-Scholes "vanilla" problems we obtain a fast and accurate revised simplex method which performs at most a linear number of pivots in the temporal discretization. When empirically compared with projected successive overrelaxation (PSOR) or a commercial LP solver the new method is faster for all the vanilla problems tested. Utilising this method we value discretely-sampled Asian and lookback American options and show that path-dependent PDE problems can be solved in 'desktop' solution times. We conclude that LP solution techniques – which are robust to parameter changes [15] – can be tuned to provide fast efficient valuation methods for finite-difference approximations to many vanilla and exotic option valuation problems.

1 Introduction

In this paper we describe the implementation of a new *linear programming* (LP) technique to solve financial option valuation problems. The method exploits the specific structure of finite difference approximations to the standard Black-Scholes *partial differential equation* (PDE) to value American vanilla options and discretely-sampled exotic options with early-exercise features. We show that path-dependent American option valuation problems in two dimensions can be solved to penny accuracy in desktop solution times – even for the most general implementation of our methodology. This is a feature not readily attainable by standard PDE complementarity problem solution techniques such as *projected successive over-relaxation* (PSOR) – or indeed by alternatives such as tree, convolution or Monte-Carlo techniques.

In Section 2 we describe the LP formulation of the American put valuation problem. In the classical Black-Scholes framework we note that the American put is the solution of an optimal stopping problem and, in terms of a suitable domain partition, the solution to a free-boundary value problem. Standard results from the variational inequality (VI) literature [2] show that the value is the *unique* solution of a *linear order complementarity problem* (OCP) [24, 14].

The OCP may be solved iteratively using PSOR and in Section 5 we present PSOR results for comparison with our new method. For coercive type-Z parabolic partial differential operators such as the Black-Scholes operator OCP may be formulated as a *least element* (LE) problem and hence as an abstract linear programme [14]. Upon suitable domain truncation and the use of finite difference approximations to the Black-Scholes operator we obtain an ordinary linear programme which can be solved numerically with standard techniques such as simplex or interior point algorithms.

However, Section 3 exploits the *tridiagonal structure* of constraint matrices formed from the above procedure to derive a fast revised simplex method for solution of standard valuation problems in LP form. This time-stepping method utilises the monotone properties of the option exercise boundary to update the basis factorization from the previous time stage in a 'parametric' fashion. For one state variable this method is fast and requires little storage. For higher dimensional problems we describe new basis factorization updating techniques which reduce memory requirements to result in a fast solution algorithm applicable to both constant and non-constant coefficient Black-Scholes operators.

In Section 4 we outline a framework for pricing exotic options using the LP techniques described above. Exotic options have a value which involves a path-dependent variable and if this continuously-sampled variable is incorporated into the PDE a *degenerate* equation of the *advection-diffusion* type is created. This has several consequences for numerical solution. The added dimension leads to exponential growth in solution times for PSOR methods and the degeneracy can create numerical oscillations in solutions. However, for the more practical discretely-sampled versions of exotic options this degeneracy disappears. We can formulate their values as solutions to the Black-Scholes PDE with jump conditions across sampling dates [32]. We show that the LP techniques used to obtain speed-ups in the solution of one-dimensional problems can be modified to solve two-dimensional lookback and Asian options with either European or American exercise features in 'desktop' solution times of several seconds on current machines.

Numerical results are presented in Section 5. Firstly, we test the new structure-exploiting linear programming approach for the American vanilla put option and compare it with the *simplex algorithm* from the IBM-OSL library and PSOR. As noted in [14] the basic LP method has performance linear in each of the variable discretizations separately. The two LP methods have similar performance, but the new tridiagonal approach significantly out-performs the standard method. For this tridiagonal approach we give results for three different basis factorization algorithms, and we find that the constant-coefficient method and the time-dependent coefficient update method both out-perform OSL and PSOR at all the tested levels of discretisation, whilst even recalculation of the basis factorization at each time-step is faster than PSOR at higher levels of spatial discretisation. We show that the speed-up obtained with the tridiagonal method allows the pricing of discretely-sampled exotic options in reasonable solution times, whereas PSOR exhibits exponential solution-time behaviour as spatial discretisations increase. The *accuracy* of the new method is compared with published sources [3, 7, 35] which solve the degenerate PDE directly to value Asian options. Our results are accurate relative to the few benchmark closed-form solutions available, even considering the possible misspecification of sampling schemes not described in detail in the quoted literature. Conclusions are drawn in Section 6 and directions for further work indicated.

2 LP Valuation of American Options

In this section we review the formulation of the American put option valuation problem as described in [14, 15, 21]. We first formulate the problem as a classical optimal stopping problem and then, considering the domain properties of this problem, as a free-boundary problem. Removing any explicit reference to the free boundary we obtain the value as the unique solution of an order complementarity problem by considering its equivalent formulation as a variational inequality and utilising standard results for coercive operators. Finally, we see that the value is the solution of an abstract linear programme which can be solved with standard LP techniques upon suitable domain truncation and discretisation.

2.1 Theory

Throughout this paper we work with the standard Black-Scholes [4] economy, where we have two financial instruments – a 'risky' asset with *price* $\mathbf{S}$ modelled by a *geometric Brownian motion* (GBM) and a savings account whose balance is continuously compounded at a constant *risk-free rate* $r \geq 0$. The Black-Scholes assumptions apply – namely no dividends, continuous trading, perfect information, no transaction costs, perfectly divisible assets, no short-sales restrictions and no arbitrage opportunities.

Under these assumptions, and with some technical restrictions, we can define an *equivalent martingale* probability *measure* (EMM) (see Harrison-Kreps-Pliska [19, 20]) under which the discounted stock price process $e^{-rt}\mathbf{S}(t)$ is a *martingale.* Using Itô's Lemma, we obtain the *stochastic differential equation* (SDE) for the stock price process

$$\frac{d\mathbf{S}}{\mathbf{S}} = r dt + \sigma d\tilde{\mathbf{W}} \qquad t \in [0,T] \qquad S(0) > 0 \tag{1}$$

as GBM, where $\sigma > 0$ is the constant *volatility* of the stock price and $\tilde{\mathbf{W}}$ is a Wiener process under the EMM.

An *option* is a risky asset whose value is determined entirely by other underlying risky assets and hence is a *derivative security.* A *European* (*vanilla*) *call* or *put option* confers the right (but not the obligation) to the holder to buy or sell respectively one unit of the asset for a price K, the *strike price*, only at a *maturity* date T. The *American* equivalent on the other hand may be exercised at any *exercise time* $\tau \in [0,T]$.

Under our assumptions an American call stock option will be optimally held to maturity. We therefore concentrate in this section on obtaining a formulation of the American put problem which is suitable for numerical solution, as closed-form solutions are not in general available for early-exercise option valuation problems. We define the value function $v\colon \mathbb{R}^+ \times [0,T] \to \mathbb{R}$, giving an option's fair *value* $v(x,t)$ to the holder at stock price $x > 0$ and time $t \in [0,T]$. This value is partially determined by the *payoff* function $\psi : \mathbb{R}^+ \to \mathbb{R}$, which for the American put is defined to be $\psi(\mathbf{S}(\boldsymbol{\tau})) := (K - \mathbf{S}(\boldsymbol{\tau}))^+$ and is received by the holder upon exercise at a general stopping time $\boldsymbol{\tau} \in [0,T]$.

The value function of an American put option can be formulated as the solution of a classical *optimal stopping problem* – choose the stopping time which maximises the conditional expectation of the discounted payoff. This stopping time $\boldsymbol{\rho}(t)$ may be shown to be the first time the value falls to the payoff at exercise, *viz.*

$$\boldsymbol{\rho}(t) := \inf \{ s \in [t,T] : \; v(\mathbf{S}(s), s) = \psi(\mathbf{S}(s)) \} . \tag{2}$$

The domain of the value function can thus be partitioned into a *continuation region* $\mathcal{C}$, on which the option has value greater than the payoff for early exercise, and a *stopping region* $\mathcal{S}$, where the value equals the payoff since exercise occurs at the first time that the value falls to the payoff. Hence

$$\mathcal{C} := \left\{(x,t) \in \mathbb{R}^+ \times [0,T) : \quad v(x,t) > \psi(x)\right\} \tag{3}$$

and

$$\mathcal{S} := \left\{(x,t) \in \mathbb{R}^+ \times [0,T) : \quad v(x,t) = \psi(x)\right\}. \tag{4}$$

On the continuation region, the value function satisfies the Black-Scholes PDE

$$\mathcal{L}_{BS}v + \frac{\partial v}{\partial t} = 0 \tag{5}$$

for $(x,t) \in \mathbb{R}^+ \times [0,T]$, where $\mathcal{L}_{BS} := \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2} + rx\frac{\partial}{\partial x} - r$, since the discounted stopped price process of the option is a martingale, whilst as soon as the process crosses into $\mathcal{S}$, $v = \psi$ and

$$\mathcal{L}_{BS}v + \frac{\partial v}{\partial t} \leq 0 \tag{6}$$

to preclude arbitrage. Hence if we require the opposite inequality to (6) we have

$$\left(\mathcal{L}_{BS}v + \frac{\partial v}{\partial t}\right) \wedge (v - \psi) = 0 \tag{7}$$

on the whole domain $\mathbb{R}^+ \times [0,T)$, where $\wedge$ signifies pointwise minimum of two functions. We now have a *free-boundary* formulation where $v(x,t) = \psi(x,t)$ for (x,t) on the optimal stopping or *exercise boundary.* To implicitly define the exercise boundary we could impose the *smooth fit condition* $\frac{\partial v}{\partial x} = -1$ along it [28]. However, we can remove any reference to the optimal stopping boundary by formulating the problem in terms of (7) as a linear order complementarity problem (OCP), using the log-transformed stock price variable $\xi := \ln x$, with respect to which the Black-Scholes operator is given by $\mathcal{L}v + \frac{\partial v}{\partial t}$, where $\mathcal{L}$ is the constant coefficient elliptic operator

$$\mathcal{L} := \frac{1}{2}\sigma^2\frac{\partial^2}{\partial \xi^2} + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial}{\partial \xi} - r, \tag{8}$$

and v is now the option value as a function of ξ. The various inequalities carry through the domain transformation and the new payoff function is given by $\psi(\xi) := \left(K - e^{\xi}\right)$. As shown in [14] the American put value function is the unique solution to

$$(\text{OCP}) \begin{cases} v(\cdot,T) = \psi \\ v \geq \psi \\ \mathcal{L}v + \frac{\partial v}{\partial t} \geq 0 \\ \left(\mathcal{L}v + \frac{\partial v}{\partial t}\right) \wedge (v - \psi) = 0 \quad \text{a.e. in } \mathbb{R} \times [0,T]. \end{cases} \tag{9}$$

For (OCP) to be well-posed, we must restrict it to a *vector lattice* Hilbert space, which is a Hilbert space H with inner product $\langle\cdot,\cdot\rangle$ and partial order defined by a positive cone P such that for any points x and y the maximum $x \vee y$ and the minimum $x \wedge y$ exist in the given order. See Borwein and Dempster [5] and Cryer and Dempster [12] for further discussion. Dempster and Hutton [14] use another equivalent formulation of the value function problem as a *variational inequality* (VI) to show the uniqueness of the solution to (OCP) if the differential operator is *coercive*, i.e. $\exists \alpha \in \mathbb{R}^+$ s.t. $\langle u, \mathcal{L}u \rangle \geq \alpha \|u\|^2 \quad \forall u \in H$. For a full treatment of the precise setting of the variational inequality see [14] which generalises that of Jaillet *et al* [24].

The final stage of the formulation shows that the value function, as the unique solution to (OCP), can be expressed as the unique solution of an abstract *linear programme* given by

$$\text{(LP)} \qquad \inf_v \langle v, c \rangle \text{ s.t. } v \in \mathcal{F} \text{ for any } c > 0 \text{ a.e. on } \mathbb{R} \times [0, T], \tag{10}$$

where

$$\mathcal{F} := \left\{ v : v(\cdot, T) = \psi,\ v \geq \psi,\ \mathcal{L}v + \frac{\partial v}{\partial t} \geq 0 \right\} \tag{11}$$

and the linear operator $\mathcal{L}$ on the Hilbert space H is of *type-Z*, i.e. $\langle v, y \rangle = 0 \Rightarrow \langle v, \mathcal{L}y \rangle \leq 0 \ \forall v, y \in H$. See [14] for proofs of this equivalence and establishing the coercive and type-Z properties for the operator $\mathcal{L}$.

From this abstract LP formulation it is a small step to the reduction of the problem from infinite to finite dimensions through space and time discretisations and solution of the resultant ordinary LP to find a numerical approximation to the value function. To this end, we restrict the value function on $\mathbb{R} \times [0, T]$ to a finite region $[L, U] \times [0, T]$ with explicit conditions on the boundaries of the domain. Then, defining a localised inner product with integration over the reduced domain, we have a *localised* LP with new constraint set

$$\mathcal{F} := \left\{ v : v(L, \cdot) = \psi(L), v(U, \cdot) = \psi(U), v(\cdot, T) = \psi,\ v \geq \psi,\ \mathcal{L}v + \frac{\partial v}{\partial t} \geq 0 \right\} \tag{12}$$

which in the limit, as $L \to -\infty$ and $U \to \infty$, converges to the solution of the abstract problem.

2.2 Numerical Methods

We approximate the value function by a function which is piecewise constant on rectangular intervals between points in a regular lattice of dimension $I \times M$. Denote the value at a general point $(L + i\Delta\xi, T - m\Delta t)$ by $v_i^m := v\,(L + i\Delta\xi, T - m\Delta t)$ where $m \in \{0, 1, \ldots, M\} =: \mathcal{M}$ and $i \in \{0, 1, \ldots, I\} =: \mathcal{I}$. Approximating the partial derivatives by standard *Crank-Nicolson finite differences* [32] we obtain a discrete form of (OCP) which, upon collapsing the space index, can be rewritten in matrix form. The complementarity condition (line 3 of (9)) is given in matrix form by

$$\left(Bv^{m-1} + Av^m - \phi\right) \wedge (v^m - \psi) = 0 \qquad m \in M \backslash \{0\}, \tag{13}$$

where A and B are $I - 1$ square tridiagonal matrices with constant nonzero entries denoted by $\{a, b, c\}$ and $\{d, e, f\}$ respectively, and

$$v^m := \begin{pmatrix} v_1^m \\ \vdots \\ v_{I-1}^m \end{pmatrix} \quad \psi := \begin{pmatrix} \psi_1^m \\ \vdots \\ \psi_{I-1}^m \end{pmatrix} \quad \phi := \begin{pmatrix} -(a+d)\psi_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{14}$$

It is easy to see that the matrix A represents the discrete approximation of the continuous linear type-Z operator $\mathcal{L}$, so it is necessary to find conditions for the matrix A to be type-Z. By definition [5] a matrix is *type-Z* if it has non-negative off-diagonal elements, which in the case of A occurs when $|r - \sigma^2/2| \leq \sigma^2/\Delta\xi$ and can be satisfied by adjusting the number of space steps I in the discretisation. From this condition it can also be shown that A is coercive [21, 24]. Hence we can formulate the discretized OCP (DOCP) by considering the finite (time step) sequence of order complementarity problems

$$\begin{aligned} & v^m \geq \psi \\ & Bv^{m-1} + Av^m - \phi \geq 0 \\ & \left(Bv^{m-1} + Av^m - \phi\right) \wedge (v^m - \psi) = 0 \end{aligned} \tag{15}$$

with equivalent sequence of ordinary LPs

$$\begin{aligned} & \min c' v^m \\ & \text{s.t.} \;\; v^m \geq \phi \\ & Av^m \geq \phi - Bv^{m-1} \qquad m = 1, \dots, M. \end{aligned} \tag{16}$$

It is possible to solve (DOCP) in an iterative manner without using the equivalent formulation as an LP, and we will present results in Section 5 for both this method and the main LP solution algorithms. The *projected successive over-relaxation* (PSOR) method due to Cryer [11] is an algorithm for solving order-complementarity problems subject to obstacles. The method splits the matrix A into upper-triangular, lower-triangular and diagonal matrices, then uses an iterative scheme to solve (DOCP) subject to a user-specified tolerance. A suitable value of the relaxation parameter ω was chosen by experimentation in our work. Other methods exist to solve the complementarity problem directly by *pivoting methods.* See [26, 9] for a review and [24] for their application to the American put problem.

The linear programme (LP) formulation can similarly be solved either directly or iteratively. The simplex algorithm [13] is a direct solution method which steps between vertices of the polytope in an objective function descent direction until the optimal solution is found. By the fundamental theorem of linear programming an optimal solution must be at a vertex of the polytope. The *interior point* method, first applied to linear programmes in [25], starts from a point in the interior of the polytope described by the inequalities in (LP) and steps by Newton descent of a nonlinear error function representing the current discrepancy in solving the first order optimality conditions towards an optimal solution.

Commercial linear programme solvers such as CPLEX [10] and IBM's Optimization Subroutine Library (OSL)[23] are available. In Section 5 we quote numerical results for PSOR and OSL-simplex, but the interested reader can find further comparisons of solution methods in [14, 21]. In the next section we describe an efficient simplex algorithm that exploits the structure of the tridiagonal constraint matrix in (16).

3 Tridiagonal Revised Simplex Method

In this section we describe a simplified revised simplex method for solution of the LP formulation of the vanilla American put option valuation problem. We take advantage of the specific structure of the constraint matrix, formed from standard Crank-Nicolson finite difference approximations, to produce a fast accurate direct solution method. For more details on the terminology in this section see a standard LP text such as [27].

To rewrite (16) in *standard form* we define a new variable u^m which is the value of the option in excess of the payoff function, $u^m := v^m - \psi$. Substituting gives

$$\begin{array}{rll} \min & c'u^m & \\ \text{s.t.} & u^m & \geq 0 \\ & Au^m & \geq b, \end{array} \tag{17}$$

where the right-hand side vector b is given by $b := \phi - B\left(u^{m-1} + \psi\right) - A\psi$.

Setting $n := I-1$, we convert (17) to an underdetermined $n \times 2n$ system of linear equations by adding non-negative *slack* variables $s := (s_1, s_2, \ldots, s_n)$, giving

$$\min \left(c' \; 0'\right)\begin{pmatrix} u^m \\ s \end{pmatrix} \quad \text{s.t.} \;\; (A \quad -I)\begin{pmatrix} u^m \\ s \end{pmatrix} = b, \quad u^m \geq 0, \;\; s \geq 0. \tag{18}$$

The constraints of (18) describe a polytope in $\mathbb{R}^{2n}$, with the (unique) optimal solution of (18) at a vertex of this polytope. We can identify a vertex by setting n of the (slack and real) variables (*non-basics*) to zero and solving the modified system $D\bar{u} = b$ for the remaining n *basic* variables, where D is the $n \times n$ *basis* matrix constructed from the columns of the constraint matrix corresponding to the basic variables and $\bar{u}$ is the corresponding vector of basic slack and real variables.

We first choose an initial basis, which simply amounts to excluding *nb real* (i.e. not slack) non-basic variables from the basis so that it comprises $\bar{u}_{nb} = \left(s_1 \ldots s_{nb} \; u^m_{nb+1} \ldots u^m_n\right)'$. Note that we are assuming the connectedness of the index sets of the real basic variables and the slack basic variables as subsets of $\mathbb{N}$. This is implied by the connectedness of the stopping and continuation regions in $[L, U] \times [0, T]$ (see Figure 1). We also assume that the optimal basis contains u^m_n, which can be guaranteed by appropriate indexing given connectedness. With this basis specified, we next find the solution of the linear system

$$D\bar{u}_{nb} = b,$$

where

$$D := \begin{pmatrix} -1 & & & & & & \\ & \ddots & & & & & \\ & & -1 & a & & & \\ & & & b & \ddots & & \\ & & & c & \ddots & a & \\ & & & & \ddots & b & a \\ & & & & & c & b \end{pmatrix}. \tag{19}$$

This solution may be found as

$$\bar{u}_{nb} = \begin{pmatrix} -I_{nb-1} & \\ & \tilde{D}^{-1}_{n-nb+1} \end{pmatrix} b, \tag{20}$$

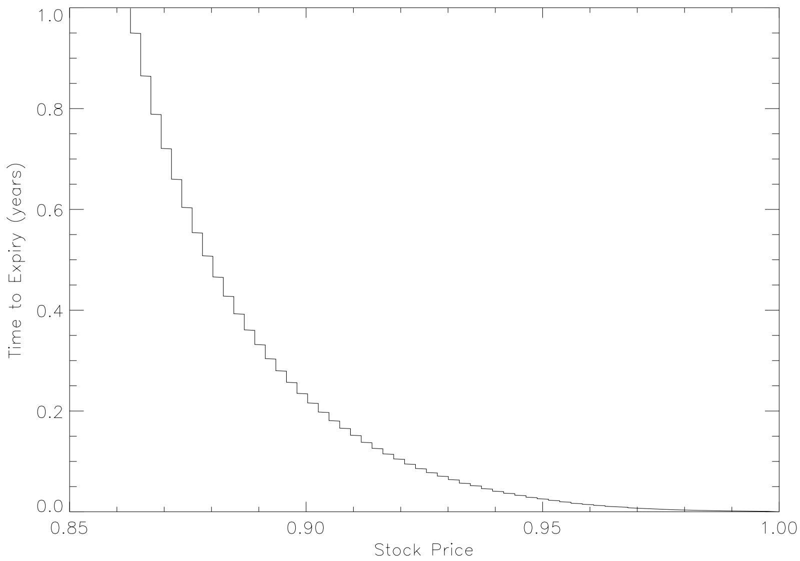

Figure 1: The calculated optimal exercise boundary for the American put option $r := 10\%$, $\sigma := 20\%$ with discretization $M := 1000$ and $I := 1200$ and stock price range $[0.37, 7.39]$.

where I_{nb-1} is the $nb - 1$ identity matrix and $\tilde{D}^{-1}_{n-nb+1}$ is the tridiagonal $n - nb + 1$ square matrix given by

$$\tilde{D}_{n-nb+1} := \begin{pmatrix} -1 & a & & & \\ & b & \ddots & & \\ & c & \ddots & a & \\ & & \ddots & b & a \\ & & & c & b \end{pmatrix}. \tag{21}$$

Since the objective function coefficients $c_1, \dots, c_n$ are arbitrary positive numbers, we can choose these so that a basis with $n - nb$ real variables always produces a solution (20) with a smaller optimal value than a basis with $n - nb - 1$ real variables. Therefore, from a given feasible basis one may find an optimal basis by repeatedly adding the next real variable u^m_{nb} into the current basis until the corresponding solution (20) becomes infeasible – then the new optimal basis is that with one fewer real basic variable than the first infeasible basis. We may start the iterative process for u^m from the previous time-step's solution vector u^{m-1} – in the case of the American put we know that this will be feasible since the exercise boundary has a convex graph in $[L, U] \times [0, T]$ (see Figure 1). With options for which this graph has a positive slope, one can reverse the iterative procedure by removing real variables from the basis until feasibility is achieved.

3.1 Computing the decomposition

The procedure outlined above requires repeated solution of the tridiagonal system (20) for a sequence of basic real variables. This may be done efficiently by factorization of the tridiagonal matrix $\tilde{D}$ defined by (21) by noting that the factorization need only be computed once for $nb = 1$, i.e. $\tilde{D}_n := A$, as follows. In general writing $\tilde{D} = LU$, the lower and upper triangular bidiagonal matrices L and U respectively are computed via the forward recursion:

$$\begin{aligned}
U_{1\ 1} &= \tilde{D}_{1\ 1} \\
U_{1\ 2} &= \tilde{D}_{1\ 2} \\
\vdots\ &\quad \vdots \\
L_{i\ i-1} &= \frac{\tilde{D}_{i\ i-1}}{U_{i-1\ i-1}} \\
L_{i\ i} &= 1 \\
U_{i\ i} &= \tilde{D}_{i\ i} - L_{i\ i-1}U_{i-1\ i} \\
U_{i\ i+1} &= \tilde{D}_{i\ i+1} \qquad i = 2, \ldots, n.
\end{aligned} \tag{22}$$

It follows that the factorization for $\tilde{D}_{n-nb+1}$ is simply formed from a submatrix of columns $1, \ldots, n-1$ from $\tilde{D}_n$. Therefore we need only compute the factorization for the matrix $\tilde{D}_n := A$ of size n and read-off the required factorizations of smaller matrices $\tilde{D}_{n-nb+1}$ as we progressively increase the number of real variables $n - nb$ in the basis. This algorithm can be implemented efficiently in any computing system using only 3 storage arrays containing the nontrivial coefficients of the LU decomposition along the 3 diagonals.

3.2 Non-constant coefficients and UL update

The procedure outlined above is suitable for any standard constant parameter Black-Scholes type formulation, but in this section we outline a procedure which yields significant computational savings for valuation problems with volatility and drift parameters which are functions of time. It also incorporates a technique for the solution of problems with non-constant constraint matrix coefficients such as those involving the untransformed Black-Scholes PDE, which has coefficients given by functions of the underlying asset price, or for exotic option pricing problems, where the coefficients vary with the third variable representing the path-dependency. In Section 5 we present results for this updating procedure which show that even for a general constraint matrix the procedure greatly out-performs standard commercial LP solvers.

The greatest efficiency saving in the standard LU factorization above follows from the observation that for the constant coefficient constraint matrix the factorization need only be performed once at the outset of the algorithm. This would not be the case using the above technique with time-dependent parameters, for these would require a full factorization of the intial basis at each time-step. With the LU formulation it is not so easy a task to update the factorization with the introduction of new real basic variables due to the recursive 'above-diagonal' nature of the computation of the diagonal of the U matrix. We therefore 'reverse' the factorization to allow for computationally efficient updating. Define

$$\bar{D}_{nb,n} := \begin{pmatrix} -1 & a_{nb} & & & & \\ & b_{nb+1} & \ddots & & & \\ & c_{nb+2} & \ddots & \ddots & & \\ & & \ddots & \ddots & a_{n-2} & \\ & & & c_{n-1} & b_{n-1} & a_{n-1} \\ & & & & c_n & b_n \end{pmatrix} \tag{23}$$

as the $n - nb + 1$ square submatrix of the basis matrix corresponding to (20) when nb non-basic real variables have been excluded. The subscripts in (23) represent entries of a general $n-nb+1$ square tridiagonal matrix with entries which vary with their indices, for example, to be dependent on the asset value. We factorize (23) by writing $\bar{D}_{nb,n} = U_{nb,n} L_{nb,n}$ where $U_{nb,n}$ and $L_{nb,n}$ are upper and lower triangular bidiagonal $n - nb + 1$ square matrices respectively, with $U_{i\ i} := 1 \quad i = nb, \ldots, n$. With this 'reversed' factorization we remove the need to recursively calculate all the factor matrices on introduction of a new real variable - instead we perform a simple update.

Setting c_{nb} and b_{nb} equal to zero for notational simplicity, the factorization proceeds as follows. At each iteration we start from a basis with $n - nb$ real variables and factorize by backwards recursion as

$$\begin{aligned} U_{n\ n} &= 1 \\ L_{n\ n-1} &= c_n \\ L_{n\ n} &= b_n \\ &\vdots \quad \vdots \\ L_{i\ i-1} &= c_i \\ U_{i\ i+1} &= \frac{a_i}{L_{i+1\ i+1}} \\ L_{i\ i} &= b_i - U_{i\ i+1} L_{i+1\ i} & \qquad i = n-1, \ldots, nb+1 \\ U_{i\ i} &= 1 \\ &\vdots \quad \vdots \\ U_{nb\ nb+1} &= \frac{a_{nb}}{L_{nb+1\ nb+1}} \\ L_{nb\ nb} &= -1. \end{aligned} \tag{24}$$

When another real variable enters the basis, we perform a simple update by increasing by one the dimension of the square matrices, calculating the new columns of L and U corresponding to the new variable, and re-calculating certain elements in the previous columns, *viz.*

$$
\begin{aligned}
L_{nb+1\ nb} &= c_{nb+1} \\
L_{nb\ nb-1} &= 0 \\
L_{nb\ nb} &= b_{nb} - U_{nb\ nb+1} L_{nb+1\ nb} \\
\vdots \quad & \quad \vdots \\
U_{nb-1\ nb} &= \frac{a_{nb-1}}{L_{nb\ nb}} \\
L_{nb-1\ nb-1} &= -1.
\end{aligned} \tag{25}
$$

The number nb of real non-basic variables is then decremented and the procedure continues as above. The full UL factorization has the same computational complexity as the LU decomposition described in §3.1 for a full factorization, but only three floating point operations are required at each update using (25).

3.3 Computational Complexity

To gain an understanding of the exact computational savings of the above methods, we first analyze the complexity of the one-factor American put option valuation problem after transformation to the constant-coefficient Black-Scholes operator.

At each time-step the maximum number of real variables which can enter the basis is given by $\lfloor \frac{\ln K - L}{\Delta x} \rfloor$ where K is the strike price L and Δx are respectively the lower bound and space step size of the discretization of the space domain, and $\lfloor \cdot \rfloor$ denotes integer part. Thus we have $\mathcal{O}(I)$ possible new basis variables, i.e. iterations, at each time step, where I is the number of points in the spatial discretization. In fact, after the first few time steps – where the exercise boundary has greatest curvature away from $\ln K$ (see Figure 1) – at most one new basic variable enters at each time step and far from maturity calculations for several time steps may utilize the *same* basis. Each iteration requires $\mathcal{O}(n)$ operations to solve, where $n \leq I$, giving $\mathcal{O}(I)$ operations at each time step. Hence the space complexity of the algorithm is linear and the total operation count is $\mathcal{O}(TI)$, where T is the number of time-steps.

For the updating technique the calculations result in a similar complexity, but extra solution time is needed for the dynamic allocation of the UL factorization at each iteration. For the full recalculation method it is necessary to include the UL factorization calculation at each iteration, resulting in an extra $\mathcal{O}(I)$ operations at each time point but still $\mathcal{O}(I)$ complexity – a significant saving over the $\mathcal{O}(I^3)$ operations required for a full $I \times I$ matrix LU factorization and equation solution.

In Section 5 we report results for the constant coefficient method and for the non-constant coefficient updating technique. We also present the results for a complete calculation of the full LU factorization at each iteration to highlight the overheads of using general commercial solvers.

4 Discretely Sampled Exotic Options

An *exotic option* is any derivative security which has a path-dependent component in its payoff at exercise. Vanilla options on the other hand have payoffs which are at most functions of the stock price at exercise and time. In this section we formulate exotic option valuation problems as linear programs, particularly for *discretely-sampled American lookback* and *Asian options* whose values are dependent on the underlying stock price, time and an additional 'independent' variable which encapsulates the required path information.

In Section 5 we compare our numerical results with two procedures [3, 35] which use the full *augmented* PDE. This PDE is derived by augmenting the state-space with a new independent variable representing the path-dependent quantity to create a degenerate two-dimensional PDE [3] with no diffusion (i.e. second derivative term) in the new variable. It can be shown (see [32]) that when a path-dependent quantity is sampled discretely on a finite number of occasions the option value satisfies a fixed parameter one-dimensional Black-Scholes equation with jump conditions across sampling dates. As a result the degeneracy can be removed and we can express discretely-sampled exotics in the LP form of Section 2. The problem must still be solved in two space dimensions, but the extra variable enters only as a parameter in the valuation problem. Our effort is directed towards the evaluation of these discretely-sampled traded options, rather than their less realistic continuously-sampled approximations.

4.1 Discretely-sampled American lookback options

Lookback options are derivative securities whose payoff at exercise depends on the maximum or minimum realised asset price over the option's lifetime. Several different versions of lookback options are traded in the market including lookback strike and lookback rate puts and calls. Such options can have either European or American exercise features.

A *lookback strike option* has a payoff similar to the corresponding (put or call) vanilla option, but with the strike price replaced by the maximum or minimum realised asset price. For example, an American lookback strike put might have payoff $\psi(S, M) := \max(M - S, 0)$ at exercise, where M is the *maximum asset price* over the life of the option until exercise, and S is the asset price at exercise as usual. We will concentrate on strike options rather than on *rate* options [33]; the latter have payoffs in which the relevant extremal value of the asset price replaces the stock price in the corresponding vanilla payoff[1]. The valuation of discretely-sampled American lookbacks is usually achieved using tree methods [1], while some closed-form solutions are available for the European continuously-sampled case [8, 17, 18, 34].

We outline the formulation of the American lookback put in a discretely-sampled setting using a dynamic programming algorithm for the option valuation based on the unifying framework of [32]. Denote by $V(S, M, t)$ the *value function* of the option with $V : \mathbb{R}^+ \times \mathbb{R}^+ \times [0, T] \to \mathbb{R}$, where S denotes the asset price and M denotes the current maximum.

We assume that the asset price is sampled on N occasions during the life of the option with maturity T. Denote by M_n the *maximum observed asset price* at the *sampling date* t_n, $n = 0, \ldots, N - 1$. For completeness define $t_N := T$ and assume that the sampling begins at time 0,[2] so that $t_0 = 0$ and $M_0 = S_0$, the initial asset value. It should be noted that unlike the continuously-sampled case the asset price is *not* bounded above by the maximum for the discretely-sampled option. The maximum M_n is a constant value throughout the period $[t_n, t_{n+1})$, since no sampling takes place until time t_{n+1}. Effectively M_n is simply a parameter in the formulation during this period, and any randomness in the model is due to

[1] Rate options may be treated by simple extensions to the methods developed here.

[2] The implementation of any sampling scheme is computationally straightforward, so that very general schemes can be solved in this manner, including *partial lookbacks* with sampling only during a subset of the options' lifetime.

the asset price process. The Black-Scholes PDE will thus be satisfied within the period with jump conditions applied at sampling dates, see [17, 32] for more details.

Across a general sampling date t_n the maximum is updated from a value M_{n-1} just prior to the date to a value M_n just at the sample date. The value M_n is simply the maximum of M_{n-1} and the asset price at time t_n, hence

$$M_n = \max\left(S, M_{n-1}\right). \tag{26}$$

To avoid arbitrage opportunities the option value must be continuous across sampling dates for any particular realisation of the asset. This leads to the jump condition

$$V\left(S, M_{n-1}, t_n^-\right) = V\left(S, M_n, t_n\right) \qquad n = 1, \dots, N-1, \tag{27}$$

where t_n^- and t_n are times immediately before and at the sampling date t_n. In the time interval $[t_n, t_{n+1})$ to the next sampling date V satisfies the *augmented Black-Scholes PDE* given by

$$\frac{\partial V}{\partial t} + \sum_{i=1}^{N} \delta(t - t_i) S^m \frac{\partial V}{\partial P} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{28}$$

where $\delta(\cdot)$ denotes the *Dirac delta function* and $\lim_{m\to\infty} P^{1/m} = M$ [32].

We consider the final period $[t_{N-1}, T]$ and use a dynamic programming algorithm to determine values for earlier periods. As in the American put case, but with increased dimension, the American lookback put valuation domain $\mathbb{R}^+ \times \mathbb{R}^+ \times [t_{N-1}, T]$ can be partitioned into a continuation region $\mathcal{C}_N$ and a stopping region $\mathcal{S}_N$ and we can establish the existence of an optimal exercise boundary. In this period we must have from arbitrage considerations

$$V(S, M_{N-1}, t) \geq \psi(S, M_{N-1}) \qquad t \in [t_{N-1}, T], \tag{29}$$

for M_{N-1} any possible value of the maximum, with V and $\frac{\partial V}{\partial S}$ continuous. The boundary at $S = 0$ is an absorbing boundary since the asset price follows GBM and if the asset has zero value it will remain zero. If the option is held until maturity in this case, then the value at exercise is equal to the payoff and so at a time $t \in [t_{N-1}, T]$ the option value is given by the discounted payoff

$$V(0, M_{N-1}, t) = e^{-r(T-t)}\psi\left(0, M_{N-1}\right) = e^{-r(T-t)} M_{N-1} \qquad t \in [t_{N-1}, T]. \tag{30}$$

This contradicts (29) and so the option must be *stopped*, i.e optimally exercised, when the asset price reaches 0.

To complete the formulation of the discretely-sampled lookback put value in the final period we require a terminal condition and boundary conditions at $S = 0$ and as $S \to \infty$. In the final period $[t_{N-1}, t_N]$ our terminal condition is that the value of the option equals the payoff at maturity, i.e.

$$V(S, M_{N-1}, T) = \psi(S, M_{N-1}) \qquad \forall S, M_{N-1} \in \mathbb{R}^+. \tag{31}$$

The boundary condition at $S = 0$ is given by (30). As $S \to \infty$ the value of the option tends to zero monotonically, since at maturity the option value is zero if $S \geq M_{N-1}$. It is sufficient for this formulation to say that the option value can grow at most linearly[3] with S as $S \to \infty$. Hence we implement the boundary condition

$$\frac{\partial^2 V}{\partial S^2} \to 0 \text{ as } S \to \infty. \tag{32}$$

Again we log-transform the primitive variables ($\xi := \ln S, \zeta_{N-1} := \ln M_{N-1}$) and formulate the valuation problem with fixed ζ_{N-1} as an OCP with respect to the transformed operator $\mathcal{L} := \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial \xi^2} + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial}{\partial \xi} - r$, defining a new partition with regions $\tilde{\mathcal{C}}_N$ and $\tilde{\mathcal{S}}_N$. Thus the American lookback put valuation problem in the final period may be formulated in terms of the transformed value function $V := V\left(e^{\xi}, e^{\zeta_{N-1}}, t\right)$ as the unique solution of the order complementarity problem

$$(\text{OCP}) \begin{cases} V(\cdot,\cdot,T) = \tilde{\psi} \\ V \geq \tilde{\psi} \\ \mathcal{L}V + \frac{\partial V}{\partial t} \geq 0 \\ \left(\mathcal{L}V + \frac{\partial V}{\partial t}\right) \wedge (V - \tilde{\psi}) = 0 \quad \text{a.e. in } \mathbb{R} \times \mathbb{R} \times (t_{N-1}, T], \end{cases} \tag{33}$$

where $\tilde{\psi}(\xi, \zeta_{N-1}) := \max\left(e^{\zeta_{N-1}} - e^{\xi}, 0\right)$ and V now denotes the option value as a function of ζ_{N-1} and ξ. This puts us in a framework equivalent to the vanilla American put in Section 2, but with the additional parameter ζ_{N-1}, and hence we can show equivalence to an abstract LP for each value of the maximum $\zeta_{N-1} \in (-\infty, \infty)$. The problem must now be solved for all possible values of the parameter ζ_{N-1}. Applying the jump conditions (27) at t_{N-1} to obtain the terminal value $V(S, M_{N-2}, t_{N-1}^-)$, the argument may be repeated for the period $[t_{N-2}, t_{N-1}]$ and, by backwards recursion, eventually for the period $[0, t_1]$.

4.2 Numerical lookback solution

We have set up the valuation problem for the final period $[t_{N-1}, T]$ for a general maximum value $\zeta_{N-1} \in (-\infty, \infty)$ in complementarity form and can formulate it as an abstract linear programme. Applying the approximations of §2.2 we solve it approximately using the tridiagonal simplex solver of Section 3. The boundary conditions (30, 31 and 32) must be transformed into the log variables resulting in the Neumann boundary condition $\partial V / \partial \xi := 0$ at the lower bound of ξ approximating the boundary at $S = 0$, i.e. as $\xi \to -\infty$.

The numerical algorithm for solving the discretely-sampled valuation problem is thus the following:

- Starting at the option maturity solve the final period LP problem in a backward time-stepping manner using the payoff of the option at maturity as a terminal condition, together with the log-transformed boundary conditions, for all values of the maximum ζ_{N-1}. This gives the value of the option until the $(N-1)^{\text{st}}$ sample date t_{N-1}.
- Apply the jump condition across sample date t_{N-1} to obtain the value at time t_{N-1}^-.
- Use this data as a terminal condition for solving the valuation problem for the penultimate period, and repeat for all remaining periods until the value at time 0 is obtained.

[3]For further discussion see [32] (pp. 212–214).

4.3 Discretely-sampled Asian options

Among the many variations of exotic options which have appeared in recent years, *Asian options* have found a significant position in the over-the-counter (OTC) market. Asian options are contracts that have payoff functions dependent on some form of average of the realised price of the underlying asset during the lifetime of the option. They are of special importance to investors in allowing an extra dimension of risk management by guarding against major fluctuations in the assets' price since any extreme movements will be averaged in the terminal payoff. This risk control could also be achieved by rebalancing a hedge portfolio, but the Asian option has the advantage of not incurring the extra transaction costs inherent in these schemes. Asian options are also less susceptible than most vanilla contracts to manipulation of the underlying spot price close to the maturity of the option.

In this paper we present results for both Asian rate and strike options. An *Asian strike* (or *floating strike) option* has a payoff in which the strike in the corresponding vanilla option payoff is replaced by the average. An *Asian rate* (or *fixed-strike*) *option* has the vanilla payoff with the asset price at exercise replaced by the average. For example, the Asian strike put has payoff $\psi(S, A) := \max(A - S, 0)$, where A is the average, and the Asian rate put has payoff $\psi(S, A) := \max(K - A, 0)$ for the fixed strike K.

The asset's average price may be calculated using either *arithmetic* or *geometric* techniques, with geometric averaging leading to closed-form solutions for certain European option pricing problems [31].

The procedure for pricing discretely-sampled Asian options is similar to that for discretely-sampled lookbacks, but with different boundary and jump conditions in the formulation. Defining the option value function $V(S, A, t) : \mathbb{R}^+ \times \mathbb{R}^+ \times [0, T] \to \mathbb{R}$, where A is the current average value – again only a parameter in the augmented Black-Scholes PDE – we have an equivalent formulation to that for the lookback options. We define the *discrete arithmetic running sum* at a general time $t \in [0, T]$ by

$$I_A(t) := \sum_{i=1}^{j(t)} S(t_i) \tag{34}$$

where $j(t)$ is the largest integer such that $t_{j(t)} \leq t$. Since there are $j(t)$ components of the sum $I_A(t)$,

$$A_A(t) := \frac{I_A(t)}{j(t)} \tag{35}$$

is the *arithmetic average at time t* after $j(t)$ samples of the asset price. The *geometric average* is similarly defined in terms of the exponential of the arithmetic average of the logarithms of the sampled asset price. The general Asian option value $V(S, A, t)$ satisfies (28) with the second term replaced by $\frac{1}{n}\sum_{i=0}^{n}(S - A_n)\delta(t - t_n)\partial V/\partial A$.

As for lookback options, for Asian options we solve the parametric Black-Scholes PDE and require a jump condition across the sampling dates together with terminal and boundary conditions to complete the theoretical and numerical formulations. Dropping the subscripts on the average and sum variables we consider here the jump condition for the arithmetic average option, with the geometric option requiring only a simple modification.

Across a sampling date t_n the running sum is updated from a value I before the sample to a value $I + S$ at the sample. So with n samples forming the sum in I, the arithmetic average is updated from $A_{n-1} = \frac{I}{n}$ to $A_n = \frac{I+S}{n+1}$ across the sampling date. Hence, we have the jump condition

$$V\left(S, A_{n-1}, t_n^-\right) = V\left(S, A_n, t_n\right) \qquad n = 1, \ldots, N-1. \tag{36}$$

Note that interpolation techniques must be used for the jump condition for Asian options since it cannot be guaranteed that a jump will land on a node of the discretized domain. Interpolation is not necessary for lookback options.

Again the terminal condition $V(S, A_{N-1}, T) = \psi(S, A_{N-1}) \quad \forall S, A_{N-1} \in \mathbb{R}^+$ and the boundary growth conditions as described in Section 4.1 are required. After logarithmic transformations we solve the corresponding OCP (33) in the new variables for each sampling period and each possible value of the logarithm of the average.

5 Numerical Results

In this section we give results for empirical tests of the tridiagonal simplex solution method applied to the vanilla American put and discretely-sampled lookback put and Asian option valuation problems. The accuracy of the LP method applied to the American put problem is documented in [5, 14] but we update this to show the accuracy and speed of this solution method for lookback and Asian options. Most significantly we highlight the speed of the method compared to commercial simplex solvers and the iterative *projected successive over-relaxation method* (PSOR) due to Cryer [11]. This method is currently the most popular amongst practitioners for the solution of complementarity problems.

All numerical computations were performed on an IBM RS6000/590 UNIX workstation with 0.25GB of memory running under AIX 4.2. All codes are written in C with double precision and compiled using the IBM *xlc* [22] compiler with optimization at level 3 (O3) and processor-specific compiler options. Unless otherwise stated all finite difference approximations were of the Crank-Nicolson type with $\theta = 0.5$. We direct the interested reader to [14] for more details of general implicit and explicit approximations and their effect on solutions. All domain discretisations were chosen for the exotic options so as to give an accuracy in the results of 0.1% of the initial asset price, for all the ranges of parameters. This coincides with the accuracy stated in [7, 35].

5.1 Vanilla American put

Table 1 illustrates the savings that the new tridiagonal simplex solver makes over PSOR and a commercial simplex solver - in this case the simplex algorithm in the IBM Optimization Subroutine Library[4] (OSL) [23]. The simplex procedure was started with a call to the OSL basis crash routine EKKCRSH at level 4, followed by a call to the dual simplex method EKKSSLV. No presolving routines were used but the simplex solver was 'hot-started' from the previous time-step's optimal basis. For further discussion on the use of the OSL library in this context see [21].

The timings in Table 1 are CPU times, including all data initialisation for the value at time 0 of an at-the-money American put with parameters $K := 1.0,\ T := 1.0,\ \sigma := 20\%$ and $r := 10\%$. The log stock price was bounded above by $U := 2$ and below by $L := -1$, giving the

[4]See [21] for comparison of this library with PSOR, interior-point and simplex methods.

range in untransformed variables as $[0.37, 7.39]$. The number of time steps M was set at 1000, and the number of space steps I varied. We see from the table that all our tridiagonal revised simplex methods are linear in space and give impressive speed-ups over OSL-simplex, with the constant-coefficient method (column 2) approximately 80 times faster. For comparison, results are included for the UL update technique (column 3) and solution times for the tridiagonal solver with re-calculation of the whole decomposition at each iteration (column 4). The slowest of our solution methods is approximately 20 times faster than OSL-simplex, and is able to value the option to high accuracy in a fraction of a second. As reported in [14] the PSOR method is faster than commercial simplex for smaller discretisations, but the OSL-simplex is almost linear in solution time as a function of the number of space steps I, whilst PSOR solution times exhibit exponential behaviour. However, two of the three tridiagonal solvers out-perform both these methods at all levels.

Space Steps	Tridiagonal Simplex: Constant coefficients	UL Update	Recalculation	IBM-OSL Simplex	PSOR
75	0.02	0.05	0.10	2.01	0.07
150	0.05	0.08	0.17	3.65	0.13
300	0.10	0.14	0.33	7.10	0.27
600	0.19	0.24	0.65	14.16	1.25
1200	0.38	0.47	1.26	28.95	6.37
2400	0.77	1.00	2.47	60.31	37.55
4800	1.61	2.24	5.12	128.52	255.06
9600	3.71	5.09	10.77	294.13	1856.91

Table 1: Comparison of tridiagonal simplex solvers, IBM-OSL simplex and PSOR. Solution CPU times in seconds for $M := 1000$

5.2 Discretely-sampled lookback strike options

In this section we compare our results for discretely-sampled American lookback options against some of those in the literature. We are somewhat limited in our ability to compare the accuracy of solution for this option type due a sparse literature; our main sources of comparison are the binomial scheme of Babbs [1] and the PSOR method of Wilmott, Dewynne and Howison [33]. The latter paper explicitly defined the sampling scheme, whilst the paper of Babbs gives approximate sampling dates.

Babbs produced the first numerical scheme which incorporated discrete sampling by using a binomial tree based on a similarity reduced problem. The results quoted in Table 2 are for a 0.5 year option, where the underlying asset price has volatility 20% and a constant risk-free rate of 10% applies throughout the option's lifetime. The option is valued at–the–money with initial stock price $S := 100$. The sampling schemes are recreated from Babbs' paper and approximately correspond to sampling quarterly, monthly, weekly and daily. Since the asset price can rise above the current maximum price it was necessary to solve the valuation problem on a domain with greater range in the asset than in the maximum. The domain was defined such that the maximum was bounded by $[e^{-1}S, e^{\frac{1}{2}}S]$ and the stock by $[e^{-1}S, eS]$. By choosing $J := \frac{3}{4}I$, we may take $\Delta x = \Delta y$ – a desirable property to avoid unnecessary approximation in the application of the jump conditions.

Table 2 recreates Babbs' results and compares them with the PDE method solved with the constant coefficient tridiagonal simplex method for varying discretisations $\{M, I, J\}$. Table 3 shows the CPU timings for the tridiagonal solver and the PSOR method applied to the same problems and demonstrates the almost perfect linearity of the simplex method in individual variable discretizations observed in [14, 15].

	Binomial (steps)			Tridiagonal		
sampling	200	2000	6000	$200 \times 200 \times 150$	$400 \times 400 \times 300$	$800 \times 800 \times 600$
quarterly	5.18	5.19	5.19	5.17	5.18	5.19
monthly	6.91	6.94	6.94	6.92	6.93	6.94
weekly	8.42	8.49	8.50	8.45	8.49	8.49
daily	9.13	9.49	9.51	9.40	9.48	9.50

Table 2: Comparison of results for the discretely-sampled American lookback: PDE method versus the binomial scheme of Babbs.

	Solution Times (s)					
	$200 \times 200 \times 150$		$400 \times 400 \times 300$		$800 \times 800 \times 600$	
Sampling	Tridiagonal	PSOR	Tridiagonal	PSOR	Tridiagonal	PSOR
quarterly	1.86	13.48	13.57	161.93	116.44	2688.30
monthly	1.87	16.60	13.92	173.93	118.08	2146.79
weekly	2.07	16.72	13.83	157.46	121.56	1649.72
daily	3.43	18.54	19.65	158.84	136.81	1455.27

Table 3: Timings for PDE method applied to Babbs sampling schemes.

It can be seen that even with the slower of the two fast tridiagonal schemes (variable-coefficients) the solutions are calculated in two dimensions to within 1% for a large discretisation in less than 4 seconds. No timings are available for the binomial scheme, see [1].

Our second comparison is with the results of [33] whose authors use a similarity reduction of the PDE to one state variable and then solve using a PSOR method. The sampling schemes used are as follows:

- Sampling scheme A: times $0.5,\ 1.5,\ 2.5,\ \ldots,\ 10.5,\ 11.5$ months
- Sampling scheme B: times $1.5,\ 3.5,\ 5.5,\ 7.5,\ 9.5,\ 11.5$ months
- Sampling scheme C: times $3.5,\ 7.5,\ 11.5$ months.

The lookback strike option is of one year maturity with $\sigma := 20\%$ and $r := 10\%$. These results are easier to compare with ours because of the explicit description of the sampling schemes employed, but are quoted in terms of the similarity reduced variable, and so to only one decimal place. Table 4 shows the CPU timings and solutions for the sampling schemes, with the same domain discretisations as for the Babbs' comparison. Again we have agreement in the results to the accuracy quoted in solution times of less than 3 seconds, using the constant-coefficients tridiagonal simplex solver, which far out-performs the PSOR method. The higher-order discretisations for the former are given to show convergence.

Scheme	Tridiagonal						Similarity Transform
	$200 \times 200 \times 150$		$400 \times 400 \times 300$		$800 \times 800 \times 600$		
	Solution	Time	Solution	Time	Solution	Time	Solution
A	10.53	2.21	10.54	16.36	10.55	138.71	10.5
B	9.45	2.20	9.45	16.65	9.45	135.72	9.5
C	8.11	2.16	8.11	16.37	8.11	133.05	8.1

Table 4: Comparison of results for the tridiagonal method against the similarity transformed method.

5.3 Discretely-sampled Asian rate options

In this section we show numerical option valuation results for discretely-sampled arithmetic Asian rate options. The results agree with those in the literature and we obtain convergence on grids with coarser discretisations than for the lookback options. It should be noted that the framework used in this paper is the only one which uses the parametric Black-Scholes PDE with jump conditions, whereas all others solve the augmented PDE directly, with the number of time-steps denoting the number of samples.

The only Asian option with a known closed-form solution is the zero-strike European Asian rate call, and we first give results for this option to suggest that our framework is accurate. Table 5 shows that the general framework is accurate for this benchmark option whose analytic results are quoted from [3].

σ	T	Tridiagonal		
		Analytic	$300 \times 100 \times 100$	
			Value	Time
10%	0.25	98.763	98.761	4.16
	0.50	97.547	97.541	4.23
	1.00	97.175	95.163	4.34
20%	0.25	98.763	98.761	4.23
	0.50	97.547	97.542	4.26
	1.00	95.175	95.163	4.44
40%	0.25	98.763	98.747	4.46
	0.50	97.547	97.514	4.58
	1.00	95.175	95.099	5.10

Table 5: Analytic and numerical solution of the 2D PDE model for the zero-strike European Asian Call - varying discretisations.

In Table 6 we quote results for European and American Asian rate call options of varying maturities. The parameters have been chosen to correspond to the results in Barraquand and Pudet (B&P) [3]. Our PDE valuation is based on a regular mesh of size $\{M, I, J\} = \{300, 200, 200\}$, which gave the same accuracy for all parameter values as Zvan *et al.* (ZVF) [35], namely 0.1% of the strike price K. We also quote results from (ZVF) for the lower bound and 1D PDE solution technique of Rogers and Shi [29] for comparison. Again it should be noted that the sampling schemes on which the results are based on were not explicitly defined in [3]. Hence, our results are based on 90 samples of the average, with 0.25 years represented by 91 days and 0.5 years by 182 days. Discrepancies in the exact maturity of the option and in the number of samples can be highly significant. The solution time for the PDE method is less than 30 seconds for the 'update method' of the UL factorization. The convergence of the results for a subset of the parameters in Table 6 is presented in Table 7.

			European					American		
σ	T	K	Value	Lower	1D	ZVF	B&P	Value	B&P	ZVF
		95	6.114	6.118	6.114	6.133	6.132	6.622	6.546	6.646
	0.25	100	1.847	1.851	1.841	1.793	1.869	1.969	1.967	1.903
10%		105	0.155	0.148	0.162	0.162	0.151	0.158	0.152	0.161
		95	7.212	7.220	7.216	7.244	7.248	7.683	7.632	7.687
	0.50	100	3.056	3.104	3.064	3.052	3.100	3.208	3.212	3.180
		105	0.714	0.714	0.718	0.726	0.727	0.730	0.735	0.733
		95	7.212	7.220	7.216	7.244	7.248	7.542	7.371	7.521
	0.25	100	3.056	3.104	3.064	3.052	3.100	3.248	3.219	3.224
20%		105	0.714	0.714	0.718	0.726	0.727	0.998	1.001	1.009
		95	7.888	7.891	7.890	7.921	7.793	8.876	8.805	8.908
	0.5	100	4.499	4.505	4.502	4.511	4.548	4.845	4.893	4.901
		105	2.215	2.211	2.206	2.229	2.241	2.276	2.337	2.337

Table 6: Results corresponding to the European and American fixed strike arithmetic average-rate call option with $r = 10\%$. $\{M, I, J\} = \{455, 200, 200\}$

			Constant Coefficients							
			$182 \times 50 \times 50$		$273 \times 100 \times 100$		$546 \times 200 \times 200$		$1092 \times 400 \times 400$	
σ	T	K	Value	Time	Value	Time	Value	Time	Value	Time
		95	8.958	1.53	8.894	7.14	8.877	43.28	8.874	294.53
20%	0.50	100	4.861	1.57	4.850	7.05	4.845	43.05	4.844	290.24
		105	2.294	1.55	2.277	7.03	2.277	42.67	2.276	286.86

Table 7: Convergence of the LP method for the American Asian fixed-strike call option.

5.4 Discretely-sampled Asian Strike options

In Table 8 we quote results for European and American Asian strike put options. Again, we compare with the results in [3, 35] for the American options and also with the results of [29] for the European options and obtain agreement to within 0.1% of strike. The regular grid was of size $I = J = 200$ and all results were computed in less than 30 seconds. The option was valued at-the-money for varying maturity with 90 samples for both the 0.25 (daily) and 0.5 (bi-daily) year options, and 120 samples for the 1-year (3-daily) option.

		European					American		
σ	T	Value	Lower	1D	ZVF	B&P	Value	B&P	ZVF
10%	0.25	0.627	0.628	0.636	0.582	0.632	1.271	1.359	1.194
	0.50	0.666	0.666	0.668	0.640	0.671	1.601	1.601	1.494
	1.00	0.600	0.598	0.598	0.589	0.614	1.930	1.952	1.799
20%	0.25	1.722	1.714	1.719	1.679	1.724	2.900	2.876	2.773
	0.50	2.131	2.147	2.123	2.093	2.135	3.842	3.806	3.684
	1.00	2.457	2.457	2.449	2.423	2.498	4.972	4.932	4.812
40%	0.25	3.984	3.971	3.971	3.926	3.981	6.181	6.111	5.996
	0.50	5.262	5.242	5.244	5.185	5.260	8.450	8.361	8.223
	1.00	6.758	6.674	6.678	6.603	6.785	11.362	11.352	11.229

Table 8: Results for the at-the-money Asian strike option. Parameters: $r = 10\%,\ S = K = 100$.

6 Conclusions

In this paper we have demonstrated the efficiency of the linear programming (LP) approach to American option valuation [14, 15] when it is combined with time-stepping decomposition and basis factorization which takes account of the tridiagonal structure of the Crank-Nicolson mixed finite difference approximations to the Black-Scholes partial differential operator. Specifically, the LP approach produces exotic option valuations to the accuracy of alternative methods in computing times which are at most several seconds and significantly less than alternatives such as multinomial trees, convolution methods and Monte Carlo [6]. This speed-up stems largely from two sources:

1. the *linear complexity* of revised simplex method iterations in the size of the basis matrix (i.e. in the spatial discretization) at each time step due to efficient basis factorization making use of its tridiagonal structure and

2. the *near-full reduction of the time dimension* promised in [15] due to a trivial pricing scheme based on the exercise boundary. The pricing scheme results in at most a single column optimal basis change in successive (backwards) time-steps after the first few steps away from option maturity.

It should be noted that since the order complementarity problem / linear programme equivalence upon which the LP method is based has been established in the *abstract* [14], a more sophisticated discretization scheme than was used in this paper – such as multigrid [7], wavelet and other basis expansions (for the underlying Hilbert space) and Douglas-type renormalizations [30] – are possible. This is the theme of current research aimed at extending the LP approach to American derivatives in several dimensions – specifically three, five or seven. In this situation versions of both sources of speed-ups are available – although the first may be obviated by using explicit methods [16]. Note that the exotic algorithms are amenable to coarse-grained parallelization on a multi-processor computing system or workstation array. Since at each time step we are simply searching for the optimal basis in a restricted set of feasible bases for each value of the path-dependent parameter, the optimal basis problems at each time step can be solved in parallel after supplying the appropriate data to each processor.

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