DERIVATIVES AS TRADEABLE ASSETS

TERRY J LYONS

1. Introduction

Consider a market with a heavily traded asset, where a secondary market for vanilla call options has developed, and where the voltility in the price of the fundamental asset has a stochastic fluctuations. This paper aims to develop a model and methodology for the joint behviour of the prices of the call option and underlying asset. In consequence, we are able to provide a more systematic approach to hedging and pricing other less commonly traded derivatives.

The existence and market price of the traded derivative should strongly influence the hedging and pricing behaviour of a bank or intermediary selling the *OTC derivative*[1]; it provides a new opportunity to hedge risk, it introduces a danger of arbitrage, and it changes the market price of the OTC derivative via the market practise of using implied volatilities.

Our objective can be summarised as the identification of low dimensional models, *complete in market priced assets, where the liquid derivatives and the underlying assets* are independent tradeable assets.

To be useful, a model must be consistent[2], risk adjusted, and provide a reasonably explicit description for the dynamics of the traded assets; it should allow one to reduce the hedging/pricing question to standard and computationally feasible calculations in the spirit of the classical theory of arbitrage theory. OUr models have all these properties.

1.1. Stochastic Volatility - an academic approach.

The existing academic literature approaches our question indirectly, looking at models for stochastic volatility and the market premium for risk. We are not happy with the details of such an approach, as the relationship between the hard to measure "market premium of risk" and the volatility of the liquid derivative, and the underlying assets seems too distant to be reliable. As the full matrix of joint volatilities of the traded assets is the essential quantity that influences price it seems wise to model these directly.

Date: March 21, 1999.

1991 *Mathematics Subject Classification.* Primary ; Secondary.

Key words and phrases. Finance, Black and Scholes, Arbitrage, Derivative.

The author would like to acknowledge assistance from EPSRC Senior Fellowship, and TMR Contract.

[1]For simplicity we use the term *OTC derivative* when we refer to the derivative an intermediary is interested in pricing and selling, and a *traded derivative* when we refer to the derivative with a market price.

[2]Force the pricing relationship between derivatives and the underlying that arise because of *arbitrage*. See later for the definition.

To understand this point consider a basket of underlying assets and the idealistic situation where a Markov model has been constructed which perfectly describes the random evolution of their prices and their volatilities:

$$
\begin{aligned}
dS_t &= \mu(S_t,\phi_t)\, S_t dt + \sigma(S_t,\phi_t)\, S_t dW_t \\
d\phi_t &= b(S_t,\phi_t)\, dt + \rho(S_t,\phi_t)\, dV_t
\end{aligned} \tag{1.1}
$$

where ϕ is an extra state variable that captures all that is important to the evolution of the system. In general the state variable will not be directly tradeable. Let us add to our assumption on the model (1.1) that the dimension of the additional state variable ϕ_t in this model can be matched to the number of traded options $\left\{O^i \,|i=1,...,d\right\}$. However, on it's own, and even if the model perfectly captured the behaviour of our system, (1.1) will not price any of the assets O^i; information concerning attitudes to risk must be added.

In his pioneering paper, Vasicek [5], observed that although prices of the O^i are not determined by such a model, arbitrage assumptions force the prices of different derivatives products to be mathematically interrelated, and this insight points the way to the explicit identification of the extra ingredient that must be identified making the prices unique.

1.1.1. *The market premium for risk.* The Markovian assumptions embodied in (1.1) ensure that at any time t the two state variables (S_t,ϕ_t) contain all the information that is available at time t concerning the future evolution of S_t. It is therefore at least plausible to assume that the price of any option or derivative on S should be a function of (S_t,ϕ_t) alone. The arbitrage arguments of Vasicek show how the existence of such a family of pricing functions is essentially equivalent to the assumption that the market attaches a consistent market price to risk. We will not re-derive the equivalence here, but to set up notation, we will explain how the model (1.1) leads to a huge range of different pricing model and hedging strategy, each suggesting it's own model for the joint volatility of (S_t,O_t).

Let $\widetilde{b}(S_t,\phi_t)$ be *any* bounded function, which we refer to (somewhat loosely) as the price of risk. Consider the change of measure from $\mathbb{P}$ to $\mathbb{P}^{\widetilde{b}}$ so that under the new measure, the model for (S_t,ϕ_t) satisfies the stochastic differential equation:

$$
\begin{aligned}
dS_t &= \sigma(S_t,\phi_t)\, dW_t \\
d\phi_t &= \Big(\rho(S_t,\phi_t)\, b(S_t,\phi_t) + \tilde{b}(S_t,\phi_t)\Big)\, dt + \rho(S_t,\phi_t)\, dV_t
\end{aligned} \tag{1.2}
$$

Providing σ do not become zero $\mathbb{P}^{\widetilde{b}}$ is absolutely continuous with respect to $\mathbb{P}$, so that any argument (e.g. about hedging) holds almost surely with respect to the one probability measure will also hold almost surely with respect to the other.

Suppose for simplicity that the terminal time for each option O^i is T and that $O^i_T = F^i(S_T)$. Define $O^{\widetilde{b}}_t$ by setting

$$O^{\widetilde{b}}_t = \mathbb{E}^{\tilde{b}}\left[F(S_T)\,|\mathcal{F}_t\right].$$

It follows from the Markov property, that $O^{\widetilde{b}}_t$ is a function $O^{\widetilde{b}}(S_t,\phi_t,t)$ of the state variables. Exploiting the equality of dimension between the variables $O^{\widetilde{b}}_t$ and ϕ_t it will generically the case that $O^{\widetilde{b}}(S_t,\bullet,t)$ is at least locally invertible. To make our point more simply, we further assume that it is globally invertible so that $\phi_t = \phi^{\widetilde{b}}\left(O_t^{\widetilde{b}}, S_t, t\right)$ is it's inverse.

In this case, the new variables $O_t^{\widetilde{b}}$ when taken with S_t are also a complete set of state variables. Indeed, $\left(O_t^{\widetilde{b}}, S_t\right)$ will be Markov and we can recover its joint volatility by differentiating the inverse function. Under $\mathbb{P}^{\widetilde{b}}$ the process $\left(O_t^{\widetilde{b}}, S_t\right)$ is a martingale and satisfies the SDE

$$\begin{aligned} dS_t &= 0dt + \sigma\left(S_t, \phi^{\widetilde{b}}\left(O_t^{\widetilde{b}}, S_t, t\right)\right) S_t dW_t \\ dO_t^{\widetilde{b}} &= \rho\left(S_t, \phi^{\widetilde{b}}\left(O_t^{\widetilde{b}}, S_t, t\right)\right) \frac{\partial O^{\widetilde{b}}}{\partial \phi}\left(S_t, \phi^{\widetilde{b}}\left(O_t^{\widetilde{b}}, S_t, t\right), t\right) dV_t \\ &\quad + \sigma\left(S_t, \phi^{\widetilde{b}}\left(O_t^{\widetilde{b}}, S_t, t\right)\right) \frac{\partial O^{\widetilde{b}}}{\partial S}\left(S_t, \phi^{\widetilde{b}}\left(O_t^{\widetilde{b}}, S_t, t\right), t\right) dW_t \end{aligned}$$

This system of tradeable assets is obviously complete, and any contingent claim which is functionally dependent on the underlying assets S_t (or even on the derivatives $O_t^{\widetilde{b}}$) can be expressed in a unique way as a stochastic integral against the assets $\left(O_t^{\widetilde{b}}, S_t\right)$.

Therefore our choice of b has lead to a consistent model with known volatility for the joint evolution of the prices for our fixed choices of options and the underlying. Assuming we have correctly identified market price of risk so that the volatility of the prices corresponds to reality, we can provide a consistent risk free hedging strategy to price any given contingent claim.

1.1.2. *The essential difficulty.* So what is the problem with the Vasicek approach set out above?

1. There is an obvious difficulty in choosing the function $\widetilde{b}$ sensibly, this becomes particularly tricky when one realises that stable hedging and pricing that is robust to small changes in the modelling relies on understanding the joint volatility of the traded assets $\left(O_t^{\widetilde{b}}, S_t\right)$. The riskless hedge is completely determined by the volatility of the tradeable assets, in our case (S_t, O_t). Get their volatility wrong, and the second order effects of hedging will cause the portfolio value to drift away from its planned value. The approach outlined above implicitly forces one to identify this "market price of risk function" $\widetilde{b}$, solve the PDE to find the price function $O^{\widetilde{b}}$ for this model, compute the inverse, and the derivative, to finally get the volatility of the option (S_t, O_t). Unfortunately, this relationship is not explicit or local. Changes in the values of $\widetilde{b}$ away from the current values of the state variables will change the volatility of (S_t, O_t); moreover, the effect is indirect and even obscure. The lack of a transparent relationship between parameters and the critical volatilities and lack of measures of the errors in the joint volatility arising from different price of risk models means that they cannot currently be regarded as robust.
2. A second related problem concerns the state variable ϕ. If it has an intrinsic meaning as representing some economic factors, then it is going to evolve and have a value; in theory once the market premium of risk has determined the values of (S_t, ϕ_t) each of the options O_j^i has its price completely determined.

In general, this will inevitably produce a conflict with market prices. Recalibration replaces this difficulty with an ever changing model for the volatility and hedging losses.[3]

1.2. **Practitioner approaches.** Probably the most common approach adopted by practitioners to accommodate the existence of prices for traded derivatives is calibration. One can consider models which are complete in the underlying assets but which have a number of undetermined parameters. By setting the parameters so that their model produces prices for traded derivatives that agree with market prices they hope that the Black Scholes framework will adequately reflect the interrelated movement of the OTC option and the underlying asset and permit it to be hedged.

In other words, implied volatility is used to accommodate the reality of derivatives priced independently by the market, in the hope that the market will incorporate all relevant historical pricing data relating to the volatility of the underlying.

This approach gives no protection against the implied volatility moving. An intermediary may well use some vega hedging in an attempt to minimize the impact on portfolio values when the calibration implied by the derivatives undergoes changes.

This "practitioner" approach is computationally feasible and of course this is very important, but it also has considerable limitations. Without a vega hedge, there is complete exposure to model risk and a shift in the implied volatility can result in a significant change in the value of the portfolio. But as we will see later, vega hedging is an imperfect attempt to avoid this difficulty.

Moreover, the implicit assumption that the market in the underlying assets is complete raises a serious intellectual obstruction to tackling our main question: how should we hedge when we have traded options. This difficulty arises because the implicit consequence of this assumption is that the liquid option can be synthesized as a portfolio in the underlying asset alone. One is forced to conclude that the liquid derivative adds nothing to the market and is redundant when one tries to hedge.

The stochastic volatility approach has more intellectual validity and avoids this second conflict. However, it generally lacks computational bite and cannot always be translated into a realistic hedging strategy. The risk from the need to recalibrate does not in general go away.

1.3. **This paper.** This paper aims to provide a more limited and more computationally valid mathematical framework for analyzing and determining the appropriate price for an OTC derivative, and a hedging strategy to synthesize the same, in the presence of nearby liquid derivatives. In outline one could say that our approach is guided by the principle that once a product is heavily traded in the market, its price has it's own independent volatility and (in so far as it influences the pricing of our OTC options) should be included in our vector of prices. The problem is to model that volatility reasonably accurately using models that are complete in the visible traded assets including the options.

Of course we will never be able to completely characterize the joint volatilities of the options and underlying correctly. But we are not so concerned by this for

[3]Our approach in this paper is only free of such criticism if one uses historical data to calibrate the joint volatility model as the prices of assets and traded derivatives are independently incorporated. However, if one had many traded derivatives and some were used to calibrate our model (e.g. to match smile) then our approach would be subject to the same criticism.

the following two reasons. We see the situation as parallel to the primeval days when Black and Scholes had just arrived and where historical models of volatility sufficiently accurate to price and hedge derivatives. The pricing of OTC option is now essentially a second order process as liquid derivatives can be used to offset much of the risk in a contract and really one only has to price a smaller residual contract; substantial errors in this residual price will not radically affect the price of the original contract of interest. It is for this reason that banks are willing to enter OTC contracts at all - under the surface they hope they are really entering contracts with much smaller total downside and pretty uncertain price, but with the uncertainty compensated by large profit margins to compensate for the high risks.

Although we leave the investigation of this aspect for another paper, it is quite possible to introduce non-linear pde's to take into account the uncertainty of the volatilities we model, following [2]. We expect this more robust approach to have similar features to the intuition we mention above. Residual contracts will see a big spread between buy and sell reflecting the uncertainties that cannot be hedged away. However, the "nearness" of many OTCs to existing derivatives will result in the price of a typical OTC being small. The spreads in the prices of the residuals are a reflection of uncertainties, but will always remain within arbitrage limits.

The structure of the paper can be summarized as follows.

1. We first review in a less discursive way, two standard industrial approachs to the existence of liquid derivatives: calibration with implied volatilities and vega hedging.
2. Then, we analyze the above approach for the minimalist example where there is a single stock, cash can be borrowed and lent at zero rate of interest, and there is exactly one liquid derivative with a price in the market. The liquid derivative will be a simple European call option of fixed strike price and maturity.
3. Staying with the same simple framework we propose approximate volatility models for the joint volatility of the call and the underlying. We will call such a volatility model consistent if, with probability one for the associated risk neutral measure, the paths of the call and the underlying coalesce appropriately at the maturity time. We illustrate this by showing that our main model is consistent, but that it is a delicate matter as small perturbations to it are not. The prices of contingent claims can therefore be calculated and hedged using the standard known volatility/no arbitrage paradigm. However, the price of the liquid derivative and the underlying asset are no longer deterministically related, and the hedge involves them both in a dynamic way.
4. The basic calculations in (3) do not depend on the underlying model being lognormal motion, nor are they dependent on there only being one derivative traded in the market. Combining Dupire's approach[1] with ours, we believe that the case of a market with traded call options at many different strikes and could also easily be accommodated.

2. Standard practise explored

2.1. **A simplistic example.** A very simple example should focus attention. Suppose a single security is freely traded, and has price processes S_t and that in addition there is a single option freely traded on the market with price process $O_{K,t}$, where for simplicity, the option is a European call, with payout at a fixed maturity T given by $(S_T - K)^+$. Now suppose that a client approaches an intermediary wishing to purchase or sell a similar option on the same underlying, with the same maturity time T, but with a strike price $K' \neq K$. There are a number of alternatives that can might taken in developing the price $O_{K',t}$ for this new option.

1. In the first case[4] one could collect large amounts of data relating to the volatility of the underlying security, using this historical data, to estimate volatility for the new option; then use the Black and Scholes model.
2. In the second case one could back out an implied volatility from the traded option under the assumption that it satisfies the Black and Scholes model for some choice of the volatility, and use this implied volatility as a substitute for the experienced volatility in the Black and Scholes formula to price the new option.

Hybrid's are also possible. Of course the first approach ignores the marketed option and leaves the bank open to arbitrage and is really a non-starter although it might be an excellent approach in an immature market where options are not freely traded. The second approach, effectively the industry standard, is an interpolation which ignores historical data and cannot directly accommodate more market prices than there are parameters in the model.

Hedging the resulting contract poses additional questions. Some are not easily settled in the classical theory. Common sense suggests that there is a huge difference between a hedging strategy that holds the security and numeraire alone, and one that also uses the traded option. Unfortunately, the classical theory following on from applying Black and Scholes predicts that the traded option can itself be synthesized using the underlying security and so gives no guidance about how much of the traded option the bank should hold when hedging the OTC option.

In practise this difficulty in hedging is often finessed, in this case one could purchase a unit of the traded option, and then use the underlying to dynamically hedge the residual liability using the standard theory and implied volatility. For the simple contracts studied here, this approach, with it's static hedge in the traded derivative is obviously a much more stable and lower risk strategy than hedging the original claim in terms of the underlying alone. The residual contract has a maximum value of $|K - K'|$ and if this is small, even quite substantial errors in hedging the residual will not seriously affect the price of the original contract. However, it is not obvious, even in this simple example that the static hedge is the best one.

In theory and for $K' > K$, one could develop a hedging strategy by taking note of the fact that the call option with strike $K' > K$ is an option on an option with the strike K. So that providing we could identify a model for the volatility of $O_{K,t}$ we could dynamically hedge the derivative $O_{K',t}$ in terms of $O_{K,t}$ and cash.

2.2. **The generic approach.** More generally and for more complex products, we can caricature the industry standard approach as follows:

- Identify and isolate components of the contract that can be priced separately, and also identify the products available in the market which are "financially close" to the OTC contract we are trying to price. Use the market prices of these to calibrate a model by backing out the volatility of the underlying

[4] We do not suggest this is sensible or done in practise.

assets. The calibrated model is then used to price the new contingent claim and to value the existing portfolio. The deltas predicted by this calibrated model are used to decide the mix of cash and the underlying securities required to hedge the portfolio.

- Also recognize that if the market changes its view of volatility the prices of the traded options will move independently of the underlying securities to reflect this, and that such changes are not hedged at all using the approaches outlined above. So introduce a second level of hedge (known as vega) - where traded options are introduced into the portfolio so that to first order, the portfolio is neutral to movements in the specified implied volatilities as well as the underlying.

Both approaches have problems attached to them. The former obviously leaves the intermediary wide open to movements in the market view of volatility. Vega hedging brings more subtle problems. If one were to analyze it mathematically, one would appreciate that it produces a risk free hedging strategy only if the volatility of the implied volatility is zero, for in any other case the second order effects of hedging will produce a portfolio whose value drifts away from the desired stationary value. It can also have hidden instabilities. None the less one might consider it a plausible approach in the case where the "vol of vol" was small.

2.2.1. *Vega hedging our simplistic example.* Before proceeding to the main part of the article, we briefly look at the consequences of using implied volatility and vega hedging in the simplistic example of a liquid European option with strike K and an OTC European option with strike K' introduced in Section (2.1). Without the stabilising effects of transaction costs etc. vega hedging exhibits a phase transition making the computation more interesting and acting as a general warning of instability. We take advantage of this example to set up notation that will hold throughout the article.

2.3. **Notation.** Suppose that the price process S_t for the securities is modelled by a random Markov process

$$dS_t^i = a^i\left(S_t\right)dt + \sigma_{i,j}\left(S_t\right)\,S_j^i dW_t^j$$

with risk adjusted probability measure $\mathbb{P}$ so that under this law

$$dS_t^i = \sigma_{i,j}\left(S_t\right)\,S_t^i dW_j^i$$

It is well known that for this model, and under regularity conditions to ensure integrability, and conservativeness, the price $p_F\left(s,t\right)$ of European option with payout $F\left(S_T\right)$ at a predictable maturity time T, and with the current time being t and the current stock price being S_t is given by $p_F\left(s,t\right) = \mathbb{E}\left[F\left(s_T\right)\middle|\, s_t = s\right]$. The standard Black Scholes model corresponds to the case where $\sigma_{i,j} = \sigma$ if $i = j$ and zero otherwise, and there is only one stock. In this case we call the scalar σ the volatility. A call option with strike price K corresponds to a payout $F\left(s\right) = \left(s-K\right)^+$.

Definition 2.1. *Let $p\left(s,\tau\right)$ denote the standard Black-Scholes arbitrage free price of an option based on a security with geometric Brownian trajectories, strike price 1, current price, volatility 1, and having τ units of time to run till maturity.*

Scaling arguments yield from p the price of an option under different volatility and strike assumptions. Because the price process is characterized by $ds_t =$

$\sigma s_t dW_t$ standard scaling arguments show

$$\mathbb{E}\left[(s_T - K)^+ | s_t = s\right] = Kp\left(s/K, \sigma^2 (T-t)\right)$$

and using the representation for $s_t = s_0 e^{\sigma V_t - \frac{1}{2}\sigma^2 t}$ where V is another standard Brownian motion one can integrate to obtain p in the well known closed form

$$p(s,\tau) := \frac{-1 + s - Erf\left(\frac{-r+2\log(s)}{2\sqrt{2\tau}}\right) + sErf\left(\frac{r+2\log(s)}{2\sqrt{2\tau}}\right)}{2}. \tag{2.1}$$

The closed form is useful for numerical calculations, but for much of what we do in this paper it is better to use the symbolic form $Kp\left(s/K, \sigma^2 (T-t)\right)$. The function p satisfies the well known Black Scholes pde with the following normalisation

$$\frac{1}{2} s^2 \frac{\partial^2 p}{\partial s^2} - \frac{\partial p}{\partial \tau} = 0 \tag{2.2}$$

Varying σ between 0 and ∞ we see that any market price of the option satisfying the arbitrage bounds $(s_t - K)^+ \leq O_{K,t} \leq s_t$ is attained for a unique $\tau(O_{K,t}) \in [0,\infty]$.

Definition 2.2. *We say $\tau(O_{K,t})$ is the estimated operational time to maturity associated to the current price of an option with security price s, option price $O_{K,t}$. It is the unique solution in τ to the equation.*

$$Kp(s/K, \tau) = O_{K,t}.$$

$\sigma = \sqrt{\frac{\tau(O_{K,t})}{(T-t)}}$ *is the implied volatility determined by option price.*

Definition 2.3. *The actual operational time to maturity is given by the integral of the empirical volatility between now and maturity*

$$\int_t^T \sigma^2(S_t)\, dt.$$

The justification for the term operational time should be apparent to anyone with experience of the classical Black and Scholes model. It is a (random) variable reflecting the turbulence to be faced. In the classical model, the standard discrete proof demonstrates how one can hedge perfectly by reapportioning ones assets every time the underlying share goes up or down by a set percentage. The standard continuous result is obtained by taking the limit. However, the whole approach only works if one knows, in advance how many such step changes in price there will be before maturity. One does not need to know how regularly, or irregularly these changes occur, only how many there are going to be. In our language, operational time measures the number of such intrinsic ticks and in the standard model these come regularly and operational time is proportional to time; but in the real world where the volatility of the security may change, the amount of operational time before maturity will not be known in advance. In this case, the option price can be regarded as a representation of the market view of exactly how much operational time there will be from time t maturity.

2.4. **Pricing and hedging.** The market approach of using implied volatility we described in remark 2 from section (2.1) can be summarized as saying that one would value the second option at $K'\, p\left(s/K', \tau\left(O_{K,t}\right)\right)$. There are then two standard approaches to hedging it.

The first uses only the underlying, and assumes (or hopes?) that the implied volatility will not change, and also assumes that the experienced volatility will coincide with the implied volatility. In other words, it values the portfolio of stock and the option at

$$V\left(s,\tau\right) = \psi s - K' p\left(s/K', \tau\left(O_{K,\,t}\right)\right)$$

and proposes a hedge of $\psi = p^{(1,0)}\left(s/K', \tau\left(O_{K,\,t}\right)\right)$ units of stock. This immunizes the portfolio to first order against movement of the underlying given that the implied volatility remains constant. Providing empirical volatility coincides with the implied volatility[5], the fact that p in the pricing formulae solves the Black Scholes pde ensures that second order price movements are also cancelled out on average which is enough to hedge. *Movements in the implied volatility are unhedged* as are differences between the empirical and the implied volatility.

It is a stable hedge, in the sense that the amount of the stock held is always between nothing and one unit, swinging between the two according to the extent that security is out or in the money, and becoming close to one or other of these extremities as the estimated operational time goes to zero.

A vega hedge[6] would aim to hold ψ units of the underlying, and ξ units of the option with strike K so that the portfolio is neutralised against movements in price of the underlying or of volatility. In other words so that the total derivative of the portfolio in the price of the underlying and implied volatility (keeping time fixed, is zero). Because of the way that implied volatility and time always occur together we see this is the same as asking that in (s,τ) co-ordinates, $\frac{\partial V}{\partial s}$ and $\frac{\partial V}{\partial \tau}$ are both zero where

$$V\left(s,\tau\right) = \psi s + \xi K p\left(s/K, \tau\right) - K' p\left(s/K', \tau\right)$$

whereas the conventional hedge above would demand that the derivative $\frac{\partial V}{\partial s} = 0$ and set $\xi = 0$.

The vega hedge an easily be computed from the above.

$$\begin{aligned} \psi &\rightarrow -\frac{K'\, p^{(0,1)}\left(\frac{s}{K'},\tau\right)\, p^{(1,0)}\left(\frac{s}{K},\tau\right)}{K\, p^{(0,1)}\left(\frac{s}{K},\tau\right)} + p^{(1,0)}\left(\frac{s}{K'},\tau\right) \\ \xi &\rightarrow \frac{K'\, p^{(0,1)}\left(\frac{s}{K'},\tau\right)}{K\, p^{(0,1)}\left(\frac{s}{K},\tau\right)} \end{aligned}$$

and from the closed form of the solution one can deduce the portfolio suggested by the hedge. In the following graphs we show how the mix between the underlying stock ψ and the option ξ as operational time goes to zero in the case where $K < K'$ and vice versa. Meanwhile in the case where one is in the money, but $K' < K$ one gets a much better stability in the money. However, there remains a less serious out of the money singularity which could cause some difficulty. This instability seems

[5]3 I am told that the market seems to place the implied volatility on the high side of reality, reflecting the fact that the intermediaries tend to be short call options

[6]In all that follows, we will assume that we have fully discounted and that the return on a bond is zero. This achieves the usual simplification of presentation without loss of content.

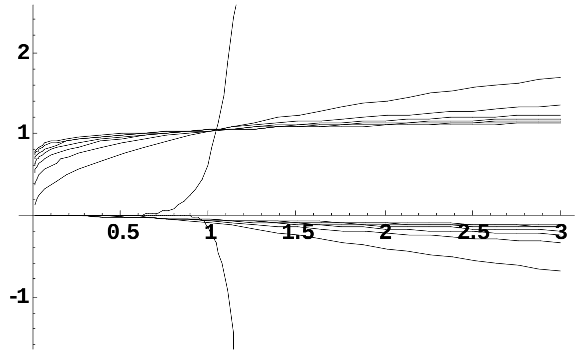

FIGURE 1. The exotic derivative has strike 1.1 while the liquid derivative has strike 1.0. Operational time ranges from .01 to 1.5 in steps of .2. The suggested hedge goes arbitrarily short in the stock as maturity approaches.

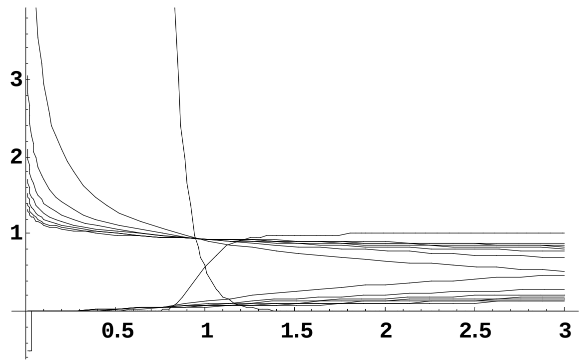

FIGURE 2. The exotic derivative has strike 0.9 while the liquid derivative has strike 1.0. Operational time ranges from .01 to 1.5 in steps of .2. The suggested hedge holds positive bounded amounts of the stock and option as maturity approaches.

to be generic and can be observed in stochastic volatility models as well.

3. The joint volatility of the call and the underlying

3.1. **Overview.** Vega hedging is a hybrid. It aims to hedge to second order in the underlying security, but only delta hedges in the direction of fluctuations in the price of the liquid option. But those fluctuations exist or one would not be interested in vega hedging!

It is the nature of markets and market makers that they impose a price on any traded security. In the absence of complete knowledge of the volatility of the underlying, the price of a liquid derivative must fluctuate in relation to any given Black Scholes predictions. The traded derivative has a price of it's own. It is not a deterministic function of the underlying assets, and in effect *it becomes a new asset* in it's own right.

If one could model the joint volatility of the underlying price vector $(S_t, O_{K,t})$ for the traded option and the underlying *together* we could use the classical Black and Scholes paradigm to price and hedge a new contingent claim using them both, however we would no longer use the classical Black Scholes formula to price the contingent claim, but a new function that comes from solving the new pde.

Although this approach seems far more natural than the vega hedging, some obvious issues need to be addressed before it can really be considered as a starter.

1. The behaviour of an option and the underlying are clearly not completely independent, and at the maturity of the option there is no independence at all. Therefore any model for the joint volatility of the pair should force this terminal relationship for the risk neutral measure without further assumptions.
2. There needs to be some basic and reasonably natural models for the volatility of the pair that can play the role of the geometric Brownian motion in the classical case.
3. Time to maturity must play an essential role in the model for the volatility, at least with respect to the volatility of the option price.

We initiate the study of these issues below, and demonstrate the existence of a workable class of models.

The approach provides a framework for pricing in a mature market where many derivatives have market prices, which takes account of those prices, and also provides rational hedging strategies, indicating the correct mix of derivatives and underlying assets. In addition, including market priced options in the hedging asset mix means that their calibration is taken into account automatically via the state of the system, rather than via the parameters. It is therefore possible, without any conflict with market prices, to use historical data to improve the model for the joint volatility.

3.2. **Connection with classical stochastic volatility papers.** Before proceeding to the details of our approach, we make one final remark. There are numerous papers on stochastic volatility, reflecting the importance of the topic. Our approach, which constructs some quite explicit models for stochastic behaviour of the implied volatility, is not primarily directed to modelling in a stochastic way, the empirical volatility of the underlying, although this is a bi-product. Our primary interest is to identify the volatility of derivatives with prices in the market, along with that of the underlying.

The volatility of our new price process $(S_t, O_{K,t})$ will still be uncertain and the methods of stochastic empirical volatility could be applied in this setting if required. However, the residual nature of most contracts, and large errors one expects, indicate that an approach based on non-linear pde and uncertain volatility might be simpler and quite adequate.

3.3. **Modelling stochastic implied volatility.** Suppose that we have two tradeable assets, a security with price S_t and a European call option O_t then we may always re-scale the units of the stock so that the strike price is 1. Our objective is to identify sensible models for volatility of the pair (S_t, O_t) and to calculate prices based on these models.

Now the process (S_t, O_t) is forced by arbitrage constraints to live in domain

$$E = \{(s, O) \,|1 + O \geq s \geq O, O \geq 0\}$$

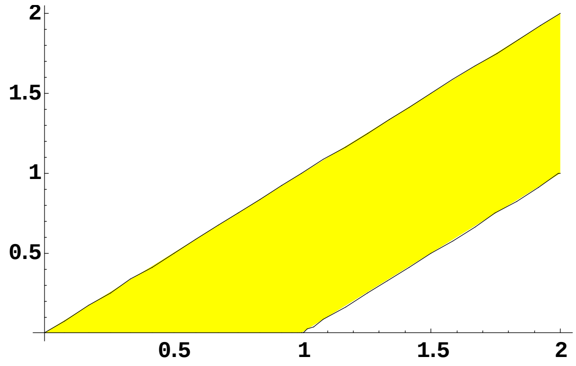

FIGURE 3. The domain E

and any value in E is potentially possible at any time although some values are obviously more likely than others. Let $p(s,\tau)$ be the function defined in (2.1), giving the Black-Scholes price of an option with current stock price s and with volatility 1, τ units of time to run. Recall that if O_t is the price of option maturing at time T, then the equation $O_t = p(s,\tau)$ always has a unique solution τ for any point in E. We regard s and τ as a new parameterisation for E, as the map $(s,\tau) \to (s, p(s,\tau))$ is one to one, taking $\mathbb{R}^+ \times \mathbb{R}^+$ onto E.

To get started, we make the following modelling assumptions:

- the volatility of the price of stock s is controlled by the value of the estimated operational time τ but is independent of the volatility of τ.
- the implied volatility determined by the option and stock prices agrees with the volatility of the stock price.

Both are only assumed to keep the mathematics relatively simple - but seem reasonable to first approximation at least. We have not had the chance to rigorously test them against market data. More complicated assumptions would not radically change the picture - only the numerical complexity of extracting a solution.

The assumptions restrict us to a model of the form

$$dS_t = \sqrt{\frac{\tau_t}{T-t}} s_t \, dV_t + \lambda_t \, dt \tag{3.1}$$

$$d\tau_t = g(T,t) \, dW_t + \mu_t \, dt \tag{3.2}$$

As both assets are assumed tradeable, one knows from the classical complete market paradigm that to obtain arbitrage prices, the next step is to choose λ_t and μ_t so as to make the measure risk neutral, in other words, chosen so that so that S_t and $O_t = p(S_t, \tau_t)$ are both martingales, and take expectations. The risk free nature of these prices make it standard that prices do not dependent on the real values of these parameters. However, there is a new feature our model must satisfy, and which we have not seen previously. It is clear that as time approaches maturity, the prices of the option and the underlying must become more closely aligned and compatible with the relationship determined by the payoff of the contingent claim.

Definition 3.1. *We say that such a model as (3.2, 3.1) is* consistent *if at the terminal time* $(S_T - 1)^+ = O_T$*. In other words if the first exit time of the process*

(O_t, S_t) from E occurs at time T and at that time the process leaves E through the lower boundary.

As an initial remark in the direction of understanding the condition of consistency we remark that our models are always "almost" consistent, in the sense of the next result.

Lemma 3.1. *In any model in the above class, (S_t, O_t) will converge to point on one of the three lines bounding E at maturity time.*

Proof. Since (S_t, O_t) is a positive martingale in each co-ordinate, it must converge as t tends to T. Suppose it converges to some point in the interior of E. It follows from standard martingale arguments that the Martingale must have finite quadratic variation along almost every sample path. Moreover, the map from $\mathbb{R}^+ \times \mathbb{R}^+$ onto E is smooth with locally bounded and invertible derivative. So the quadratic variation of (S_t, τ_t) provides a lower bound for that of (S_t, O_t) and must also be finite. However the assumption that (S_t, τ_t) converges to an interior point ensures that

$$\int_{u=t}^{\mathrm{T}} \left(\sqrt{\frac{\tau_u}{T-u}} S_u \right)^2 du = \infty$$

giving a contradiction. From this we conclude that the process (S_t, O_t) must converges to a boundary point as t tends to T. ∎

3.4. **Fixing the price.** In this section we will do the computation giving the values for λ_t, μ_t for which $(s_t, P(s_t, \tau_t))$ is martingale. Observe first that the requirement that s_t is a martingale implies that $\lambda_t \equiv 0$ since V_t is already a Brownian motion, and hence a martingale. So all the interest is in identifying the correct form for μ_t. For this it seems substantially simpler to work in a fairly general way as the explicit form of the Black and Scholes formula does not seem to play a big role. We introduce a notation for the infinitesimal generator for the diffusion $(s_, \tau_t)$:

$$L = \frac{1}{2} \left(\frac{\tau}{T-t} \right) s^2 \frac{\partial^2}{\partial s^2} + \frac{1}{2}\, g^2 \frac{\partial^2}{\partial^2 \tau} + \mu_t \frac{\partial}{\partial \tau} + \frac{\partial}{\partial t} \tag{3.3}$$

Then our requirement is to choose μ_t so that one has $Lp(s, \tau) = 0$.

By hypothesis $p(s, \tau)$ is the Black-Scholes solution and satisfies (2.2) so that

$$Lp = \left(\frac{1}{2}\, g^2 \frac{\partial^2}{\partial^2 \tau} + \left(\mu_t + \left(\frac{\tau}{T-t} \right) \right) \frac{\partial}{\partial \tau} \right) p(s, \tau)$$

and thus the condition that $Lp = 0$ translates into

$$\frac{\partial}{\partial \tau} \log \left| \frac{\partial p}{\partial \tau} \right| = -2 \frac{\left(\mu_t + \left(\frac{\tau}{T-t} \right) \right)}{g^2}$$

It remains to compute $\frac{\partial}{\partial \tau} \log \left| \frac{\partial p}{\partial \tau} \right|$ and we will have an explicit form for μ.

We could just go directly into computation using the formula (2.1) but it is better and applies more generally to observe that if p is any solution to the equation

$$\left(\frac{1}{2} s^2 \frac{\partial^2}{\partial s^2} - \frac{\partial}{\partial \tau} \right) p = 0$$

then so is $\frac{\partial p}{\partial \tau}$, and this function can be rewriten as $\frac{1}{2} s^2 \frac{\partial^2}{\partial s^2}$ aswell. However, from this it is obvious that the boundary data corresponding to the solution $\frac{\partial p}{\partial \tau}$ for our particular p is given by a delta function and is hence relatively easy to compute without computation!

We have

$$\left(\frac{-\tau}{T-t}\right)-\frac{g^2}{2}\frac{\partial}{\partial\tau}\log\left|\frac{\partial p}{\partial\tau}\right|=\mu_t$$

So far we have not used the explicit form of the call option Black and Scholes solution, but on doing this one gets the following result.

Proposition 3.2. *The value of μ making the process (S_t, O_t) risk neutral is given by*

$$\mu=\frac{1}{2}\left(\frac{1}{8}+\frac{1}{2\tau}-\frac{1}{2}\frac{\left(\ln\frac{K}{s_0}\right)^2}{\tau^2}\right)g^2-\frac{\tau}{T-t}$$

which is deduced from

$$\frac{\partial}{\partial\tau}\log\left|\frac{\partial p}{\partial\tau}\right|=-\frac{1}{8}-\frac{1}{2\tau}+\frac{\left(\ln\frac{K}{s_0}\right)^2}{2\tau^2}$$

using the identity

$$\frac{\partial}{\partial\tau}\log\left|\frac{\partial p}{\partial\tau}\right|=-2\frac{\left(\mu+\left(\frac{\tau}{T-\tau}\right)\right)}{g^2}$$

from above.

Corollary 3.3. *The risk adjusted process has the infinitesimal generator*

$$\begin{aligned}L &= \frac{1}{2}\left(\frac{\tau}{T-t}\right)^2 s^2\frac{\partial^2}{\partial s^2}\\ &+\frac{1}{2}g(\tau,T-t)^2\left[\frac{\partial^2}{\partial\tau^2}+\left(\frac{\log\left(\frac{s}{K}\right)^2}{2\tau^2}-\frac{1}{2\tau}-\frac{1}{8}\right)\frac{\partial}{\partial\tau}\right]+\left(\frac{-\tau}{T-\tau}\right)\frac{\partial}{\partial\tau}\\ &+\frac{\partial}{\partial t}\end{aligned}$$

Proof: We must be careful about normalisations to get a consistent answer. Define

$$ds_t = s_t\,dV_t \text{ or } s_t = s_0 e^{V_t-\frac{1}{2}t}$$

where V is a Brownian motion with variance t at time t, then

$$p(s,T-t)=\mathbb{E}\left[(s_T-K)^+\,|s_t=s\right]$$

and

$$\begin{aligned}\frac{\partial}{\partial t}p(s_0,T-t) &= -\frac{1}{2}s^2\frac{d^2p}{ds^2}\\ &= \mathbb{E}\left(-\frac{1}{2}s^2\frac{d}{ds^2}(s-K)^+\,|s_t=s_0\right)\\ &= -\int_0^\infty\frac{1}{2}\rho(s_0,s,T-t)\;s^2\frac{d^2}{ds^2}(s-K)^+\,ds\end{aligned}$$

so that

$$\frac{\partial}{\partial\tau}p\left(s_0,\tau\right)=\int_0^\infty\frac{1}{2}\rho\left(s_0,s,\tau\right)\ s^2\ \frac{d^2}{ds^2}\ (s-K)^+\,ds$$

where $\rho\left(s_0,s,T-t\right)$ is the density of the Log normal distribution. That is, it is the density of $s_0e^{V_t-\frac{1}{2}(T-t)}$ at s. A simple calculation shows this to be

$$\rho\left(s_0,s,\tau\right)=\frac{1}{s}\frac{1}{\sqrt{2\pi\tau}}e^{-\frac{\left(\frac{1}{2}\tau+\log s-\log s_0\right)^2}{2\tau}}$$

Suppose that f is a smooth function of compact support at infinity and bouned at zero then integrating by parts:

$$\begin{aligned}
&\int_0^\infty\frac{1}{2}\ f\left(s\right)\ s^2\ \frac{d^2}{ds^2}\ (s-K)^+\,ds\\
=&\int_0^\infty\frac{1}{2}\ (s-K)^+\ \frac{d^2}{ds^2}\left(f\left(s\right)\ s^2\right)\ ds\\
=&\int_K^\infty\frac{1}{2}\ (s-K)\ \frac{d^2}{ds^2}\left(f\left(s\right)\ s^2\right)\ ds\\
=&-\int_K^\infty\frac{1}{2}\frac{d}{ds}\ (s-K)\frac{d}{ds}\ \left(f\left(s\right)\ s^2\right)\ ds\\
=&-\frac{1}{2}\int_K^\infty\frac{d}{ds}\ \left(f\left(s\right)\ s^2\right)\ ds\\
=&\ \frac{1}{2}K^2f\left(K\right)
\end{aligned}$$

and combining these two remarks we have that

$$\begin{aligned}
\frac{\partial}{\partial\tau}p\left(s_0,\tau\right) &= \frac{1}{2}K^2\rho\left(s_0,K,\tau\right)\\
&= \frac{1}{4}K\sqrt{\frac{2}{\pi\tau}}\exp\left(-\frac{1}{8}\frac{\left(\tau+2\ln\left(K/s_0\right)\right)^2}{\tau}\right)
\end{aligned}$$

and that

$$\begin{aligned}
\frac{\partial}{\partial\tau}\ \log\left|\frac{\partial p}{\partial\tau}\right| &= -\frac{1}{2\tau}-\frac{2\tau\left(\tau+2\ln\left(K/s_0\right)\right)-\left(\tau+2\ln\left(K/s_0\right)\right)^2}{8\tau^2}\\
&= -\frac{1}{2\tau}-\frac{2\tau\left(\tau+2\ln\left(K/s_0\right)\right)-\left(\tau+2\ln\left(K/s_0\right)\right)^2}{8\tau^2}\\
&= -\frac{1}{2\tau}-\frac{\left(\tau+2\ln\frac{K}{s_0}\right)\left(\tau-2\ln\frac{K}{s_0}\right)}{8\tau^2}\\
&= -\frac{1}{2\tau}-\frac{\tau^2-4\left(\ln\frac{K}{s_0}\right)^2}{8\tau^2}\\
&= -\frac{1}{8}-\frac{1}{2\tau}+\frac{\left(\ln\frac{K}{s_0}\right)^2}{2\tau^2}
\end{aligned}$$

and the result follows by substitution.

3.5. **Models for g.** If τ represents number of ticks of "operational time" till maturity - then do we have any intuition as to how it should behave and how volatile this estimate should be. Suppose then that you regard the volatility of market as a measure of the number n of market actions till maturity, where n is large, then the number that will arrive over a time period of length t is approximately $nt \pm \sqrt{nt}$ (mean $\pm \sqrt{\text{variance}}$.) So it seems quite reasonable that the market view of the operational time should swing by this sort of amount over a unit of time, suggesting a model of the form

$$dS_t = \sqrt{\frac{\tau_t}{T-t}} s_t \, dV_t \tag{3.4}$$

$$d\tau_t = \sqrt{\tau} dW_t + \left(\frac{1}{2} \left(\frac{1}{8} + \frac{1}{2\tau} - \frac{1}{2} \frac{\left(\ln \frac{K}{s_0} \right)^2}{\tau^2} \right) \tau - \frac{\tau}{T-t} \right) dt \tag{3.5}$$

for initial experiments. The crucial point is that the volatility of the operational time is heavily dependent on the time to maturity. Moreover we have not just assumed that the volatility of the underlying security is stochastic - more subtly we have assumed that the option has its own vol depending on maturity. A heuristic example of what we have in mind comes from the process of downloading a file across the Internet. Most web browser programmes provide a continual estimate for the time till download of the file - it is often based on some compromise between the current and integrated rate of data transfer, and is remarkably frustrating as it can go up as well as down. The relative volatility can get bigger as one gets near the completion of the transfer.

The model above is obviously consistent, as the final term ensures that the process τ, for small time intervals near maturity, is bounded above with probability one; and so combining with the lemma we proved above, about the convergence of the process to one of the three boundaries of the region E, it follows that it must converge to the lower boundary as required for consistency.

3.5.1. *A second example.* A second model sets $g = \sqrt{\tau}\sqrt{\frac{\tau}{T-t}}$,and seems rather similar, as we do not expect the volatility $\sqrt{\frac{\tau}{T-t}}$ to fluctuate by orders of magnitude as the contracts approach maturity. This second model has attractive features from a mathematical perspective.

$$dS_t = \sqrt{\frac{\tau_t}{T-t}} s_t \, dV_t \tag{3.6}$$

$$d\tau_t = \sqrt{\tau}\sqrt{\frac{\tau}{T-t}} dW_t + \left(\frac{1}{2} \left(\frac{1}{8} + \frac{1}{2\tau} - \frac{1}{2} \frac{\left(\ln \frac{K}{S_t} \right)^2}{\tau^2} \right) \tau \frac{\tau}{T-t} - \frac{\tau}{T-t} \right) dt \tag{3.7}$$

In this second example, we may transform this process into a nicer one by introducing a random time change. Let

$$du = \frac{\tau}{T-t} dt$$

then we can rewrite (3.6,3.7) as

$$(3.8) \qquad dS_t = s_t\, d\tilde{V}_u$$

$$(3.9) \qquad d\tau_t = \sqrt{\tau} d\tilde{W}_u + \left(\frac{1}{2} \left(\frac{1}{8} + \frac{1}{2\tau} - \frac{1}{2} \frac{\left(\ln \frac{K}{S_t}\right)^2}{\tau^2} \right) \tau - 1 \right) du$$

The fascinating remark is that *this model is not consistent.* There is always a small chance that the process τ will escape to infinity.

4. Wider issues

4.1. Robust approaches to hedging. It is therefore open to the bank to devise consistent models for the volatility of underlying assets and derivatives at once. It goes without saying that this paper treats a very special case and is not the complete picture. There are a number of ways in which to try to improve it. The first is to note that it is not vital to get the correct volatility model for the joint behaviour. This is because of the way these approaches to pricing are used in the market. The point is that real contracts have quite small residual components after a crude hedge with derivatives, so that the price of the contract is not ultra-sensitive, and using a robust conservative hedging strategy permitting a wide possible range of volatilities need not make a large difference to the price of a contract. It is also easy to impliment.

4.2. Broader classes of model and many strikes. In general there are a number of tradeable derivatives, and those that are traded freely are often change with only those broadly on the money having liquidity. Moreover different derivatives have different maturity. We certainly do not have a clear model to choose in every example - although the principle of looking for sensible and consistent models is obviously a sensible one. However, we recall the approach of Dupire where he uses derivatives at all strikes and maturities to predict a price dependent model for the volatility of the security. This approach allows one to easily generalise the results of this paper to derivatives with different strikes.

4.3. Other relevant work. Like the present paper, Zhu and Avelleneda [6] also use a derivative as an additional stochastic state variable while explicitly retaining market completeness, although from a somewhat different standpoint. They pre-specifying a lognormal process for the instantaneous volatility of a single underlying asset, and then make the connection with traded derivatives by identifying this volatility with the implied volatility of short maturity call options, which necessarily therefore have the same implied volatility.

References

[1] Dupire, B., *Pricing and hedging with smiles*, in Mathematics of derivative securities (Cambridge, 1995), Publ. Newton Inst. (15), 103–111 Cambridge Univ. Press, Cambridge, 1997.

[2] Hobson D., Stochastic Volatility, Chapter 14 in Statistics in Finance, eds. David J Hand, Saul D. Jacka, Arnold Applications in Science.

[3] Hull J., White A. (1982), The Pricing of Options on Assets with Stochastic Volatilities, Journal of Finance, XLII 2, 281-300.

[4] Lyons, T.J. *Uncertain Volatility and the Risk-Free Synthesis of Derivatives*, Applied Mathematical Finance 2, 1995, 117-133

[5] Vasicek, O.A., *An Equilibrium Characterisation of the Term Structure,* Journal of Financial Economics 5, 1977, 177-188

[6] Zhu Y., Avellaneda M., A Risk Neutral Stochastic Volatility Model, International Journal of Theoretical and Applied Finance, 1 **2** 289-310.

(T. Lyons) Department of Mathematics, Imperial College,180 Queens Gate, London SW7 2BZ, England

E-mail address, Terry J Lyons: t.lyons@ic.ac.uk