Computational Aspects of Alternative Portfolio Selection Models in the Presence of Discrete Asset Choice Constraints

N.J. Jobst, M.D. Horniman, C.A. Lucas and G. Mitra
Department of Mathematical Sciences
Brunel University
Uxbridge, Middlesex UB8 3PH, United Kingdom

May 2000

Abstract

We consider the mean-variance (M-V) model of Markowitz and the construction of the risk-return efficient frontier. We examine the effects of applying buy-in thresholds, cardinality constraints and transaction roundlot restrictions to the portfolio selection problem. Such discrete constraints are of practical importance but make the efficient frontier discontinuous. The resulting quadratic mixed-integer (QMIP) problems are NP-hard and therefore computing the entire efficient frontier is computationally challenging. We propose an efficient approach for computing this frontier and provide insight into its discontinuous structure. Computational results are reported for a set of benchmark test problems.

KEYWORDS: Portfolio optimisation, mean variance model, efficient frontier, buy-in threshold, cardinality constraint, roundlot restriction.

1 Introduction

We consider the portfolio selection model of Markowitz (1952, 1959, 1987) that laid the foundations of Modern Portfolio Theory (see Constantinides and Malliaris (1995) for a survey). Markowitz shows how rational investors can construct optimal portfolios under conditions of uncertainty. The mean and variance of a portfolio's return represent the benefit and risk associated with the investment.

Markowitz shows that for a rational investor, maximising expected utility, their chosen portfolio is optimal with respect to both expected return and variance of return. He defines such a non-dominated portfolio as *efficient*, that is, it offers the highest level of expected return for a given level of risk and the lowest level of risk for a given level of return. His normative mean-variance rule for investor behaviour both implies and justifies the observable phenomenon of diversification in investment. Determining the efficient set from the investment opportunity set, the set of all possible portfolios, requires the formulation and solution of a parametric quadratic program (QP). Plotted in risk-return space the *efficient set* traces out the *efficient frontier*.

Hanoch and Levy (1969) show that the M-V criterion is a valid efficiency criterion, for any individual's utility function, when the distributions considered are Gaussian Normal. A study comparing alternative utility functions appears in Kallberg and Ziemba (1983). They show that portfolios with "similar" absolute risk aversion indices have "similar" optimal compositions, regardless of the functional form and parameters of the utility function. Hence, M-V analysis is justified for any general concave utility function of the Von Neumann-Morgenstern type (Von Neumann and Morgenstern (1944)).

The estimation of the large number of parameters (returns, variances and covariances) which are required as the input to M-V analysis is an important modelling step. Small changes in the inputs can have a large impact on the optimal asset weights. Chopra and Ziemba (1993) found that, for a typical investor's risk tolerance level, errors in the forecast means are more than ten times as important as errors in the variances and about twenty times as important as errors in covariances. For practical aspects of portfolio analysis see Hensel and Turner (1998) and Grinold and Kahn (1995). MPT has developed in tandem with simplifications to the QP required by M-V analysis. These simplifications centre around linearising the quadratic objective function or reducing the number of parameters to be estimated. Both approaches involve either the approximation or decomposition of the covariance matrix.

Tobin (1958) developed the separation theorem which states that, in the presence of a risk-free asset, the optimal risky portfolio can be determined without any knowledge of investor preferences. Ziemba, Parkan and Brooks-Hill (1974) show that the solution to the portfolio problem involving a risk-free asset can be obtained by a two-stage process; firstly solving a deterministic linear complementarity problem and then an univariate stochastic program.

Sharpe (1963) proposed that the single-index, or 'market', model was a sufficient model of covariance. Subsequently, Sharpe (1964), Lintner (1965) and Mossin (1966) independently developed the Capital Asset Pricing Model. This linear model of equilibrium asset prices explains the covariance of asset returns solely through their covariance with the market. King (1966) presented evidence of the influence of industry factors that the market model did not take into account. Rosenberg (1974) presented a multi-factor model that incorporated industry and other factors. Ross (1976) using factor analysis, developed the Arbitrage Pricing Theory which is a multi-index equilibrium model.

Index or factor models allow a simplification of the underlying QP. The covariance matrix can be expressed in a diagonal form such that the quadratic objective function is a weighted sum of squares that is easily represented in a linear form (see Sharpe (1971)). In the case that an index or factor model is not employed, the nature of the covariance matrix also permits a natural decomposition of the quadratic objective term into a weighted sum of squares (see Vanderbei and Carpenter (1993)). Standard techniques also exist for the piecewise linear approximation of quadratic objective functions.

A number of researchers have introduced alternative measures of risk for portfolio planning. In many cases these measurements are linear, leading to a corresponding simplification in the computational model. Konno and Yamazaki (1991) show that the mean absolute deviation of returns is a risk measure equivalent to variance, under the assumption of multi-variate normal returns. Speranza (1996) considers only the mean absolute value of negative deviations. Markowitz (1959) suggested that semi-variance is the real cause for concern but variance is employed as the risk measure as it is more tractable computationally and reveals the same information. Downside risk measures are typically used in dynamic asset allocation problems (see Cariño and Ziemba (1998)). Multi-objective goal programming approaches have been proposed by Lee (1972) and Lee and Chesser (1980). Young (1998) employs a minimax investment rule measuring risk as the minimum return (maximum loss) that the portfolio would have achieved over all of the past observation periods. A survey of alternative portfolio selection models appears in Horniman et al. (2000).

Simplifications to the basic quadratic programming problem have allowed the models to be extended to perform more realistic analysis, incorporating market imperfections. Rudd and Rosenberg (1979) consider linear transactions costs. Konno and Yamazaki (1991) suggest that an advantage of the MAD model is that it limits the number of stocks held, allowing control of transaction costs. Adcock and Meade (1994) combine a modulus function for transaction costs with the usual quadratic objective. Young (1998) describes how the minimax model can be adapted to include linear transaction costs.

To capture the realism of portfolio planning we introduce the following restrictions:

(i) A *Buy-in thresholds* is defined as the minimum level below which an asset is not purchased. This requirement eliminates the unrealistically small trades that can otherwise be included in an optimised portfolio.

(ii) *Cardinality constraints*: Investors may wish to specify the number of stocks in their portfolio because of monitoring and control issues.

(iii) *Roundlots* are defined as discrete numbers of assets which are taken as the basic unit of investment. Investors are restricted to making transactions only in multiples of these roundlots. This overcomes the assumption of the infinite divisibility of assets required in the development of the M-V rule.

Cardinality constraints are inherently linked with buy-in thresholds. For example, a buy-in threshold of 5% of the value of a portfolio ensures that there can be no more than 20 stocks purchased. Also, to model cardinality constraints requires a buy-in threshold to be applied. Imposing these types of constraints on the portfolio selection problem, necessitates the introduction of binary and integer variables. The basic QP becomes a QMIP and, as a result, both the size and complexity of the models are increased accordingly. Mansini and Speranza (1999) have shown that finding a feasible solution to the portfolio selection problem with roundlots is NP-complete.

Bienstock (1996) and Lee and Mitchell (1997) use QMIP techniques to solve the portfolio selection problem with an upper limit on the size of the portfolios. Chang et al. (1999) use heuristic algorithms (genetic algorithm, tabu search and simulated annealing) to solve cardinality constrained problems with specified portfolio sizes. Linear programming based heuristics are used by Speranza (1996), considering the negative semi-MAD model with cardinality constraints. This model is extended in Mansini and Speranza (1999) to incorporate roundlots and then in Kellerer, Mansini and Speranza (1997) incorporating both roundlots and fixed costs. Young (1998) also shows how fixed transaction costs can be applied to the linear minimax model.

The rest of this paper is organised as follows. In section 2 we introduce the underlying portfolio planning model and discuss the construction of the efficient froniter using two different QP models. The algebraic formulations of the M-V model including buy-in thresholds, cardinality constraints and roundlot restrictions are introduced in section 3. We also discuss some theoretical issues of discontinuities and missing sections of the efficient frontier in the presence of discrete constraints (DCEF). In section 4 we discuss our solution methods and present the computational results for datasets taken from five global stock markets. In addition, we also investigate the effects of the discrete constraints in the context of the portfolio rebalancing problem. In section 5 we discuss our research findings and present our conclusions. Appendix A contains the dataset used to illustrate the shape of the DCEF in section 4.2.

2 Mean-Variance Model

The classical M-V model and an alternative approach to computing the 'Markowitz Efficient' Frontier (MEF) are set out below. The basic notation is:

Indices:
$i, j = 1, \ldots, N$: denotes the different risky assets

Parameters:
μ_i: the expected return of asset i
σ_{ij}: the covariance between asset i and asset j
($\sigma_{ii} = \sigma_i^2$ is the variance of asset i)
ρ: the desired level of return for the portfolio

Decision variables:
x_i: the fraction of the portfolio value invested in asset i

QP1:

$$\text{Min} \quad Z_{QP1} = \sum_{i=1}^{N} \sum_{j=1}^{N} x_i x_j \sigma_{ij}$$

subject to

$$\begin{aligned} \sum_{i=1}^{N} x_i \mu_i &= \rho \\ \sum_{i=1}^{N} x_i &= 1 \\ x_i &\geq 0 \qquad i = 1, \ldots, N \end{aligned}$$

Varying the desired level of return, ρ, in QP1 and repeatedly solving the quadratic program identifies the minimum variance portfolio for each value of ρ. These are the efficient portfolios that compose the efficient set. Plotting the corresponding values of the objective function and ρ, variance and return respectively, for the efficient set traces the MEF in the mean-variance plane. Markowitz (1956) describes a 'critical line' solution algorithm tracing out the efficient frontier by identifying 'corner' portfolios - points at which a stock either enters or leaves the current portfolio. It it is common to use standard deviation rather than variance as the risk measure because the frontier is linear if a risk-free asset exists, see Tobin (1958) and Ziemba et al. (1974).

An alternative formulation of QP1 explicitly trades risk against return in the objective function using the Arrow-Pratt absolute risk aversion index R_A (see Kallberg and Ziemba (1983)). R_A is defined as

$$R_A = -\frac{u''(w)}{u'(w)}$$

where w is the portfolio wealth and u', u'' are the first and second derivatives of a Von Neumann-Morgenstern utility function u.

QP1 can be stated alternatively as

QP2:

$$\text{Max} \quad Z_{QP2} = \sum_{i=1}^{N} x_i \mu_i - \frac{R_A}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} x_i x_j \sigma_{ij}$$

subject to

$$\sum_{i=1}^{N} x_i = 1$$
$$x_i \geq 0 \qquad i = 1, \ldots, N$$

Solving for different values of R_A traces out the efficient frontier. Empirical results by Kallberg and Ziemba (1983) show that $R_A \geq 6$ leads to very risk averse portfolios, $2 \leq R_A \leq 4$ represents a moderate absolute risk aversion and $R_A \leq 2$

leads to risky portfolios. $R_A = 4$ corresponds approximately to pension fund management (typically, holdings of 60% stocks and 40% bonds). In practice it is common to plot the MEF modelling the risk-return trade-off using a parameter λ, $0 \leq \lambda \leq 1$, with the objective function

$$\text{Min} \quad Z = \lambda \sum_{i=1}^{N} \sum_{j=1}^{N} x_i x_j \sigma_{ij} + (1 - \lambda) \sum_{i=1}^{N} x_i \mu_i.$$

Setting $\frac{R_A}{2} = \frac{\lambda}{(1-\lambda)}$ shows equivalence with the objective function in QP2.

3 Efficient Frontier with Discrete Constraints

3.1 Discontinuities in the DCEF

Discrete constraints, representing practical trading requirements, introduce discontinuities into the otherwise efficient frontier. To illustrate the appearance of the discontinuities, we consider the small 4-stock example from Chang et al. (1999)) with the following expected returns, standard deviations and correlations.

	Correlation Matrix				*Exp. Return*	*Std. Deviation*
Stock No.	*1*	*2*	*3*	*4*		
1	1				0.004798	0.014635
2	0.118368	1			0.000659	0.030586
3	0.143822	0.164589	1		0.003174	0.030474
4	0.252213	0.099763	0.083122	1	0.001377	0.035770

Figure 1 plots the MEF for this dataset. From the 4 stocks we are to choose a portfolio containing only 2 stocks. We can identify our opportunity set, Figure 2, by considering the 6 pairwise combinations of stocks. Ordering by risk and return, we can eliminate the inefficient portfolios to reveal the discontinuous DCEF shown in Figure 3.

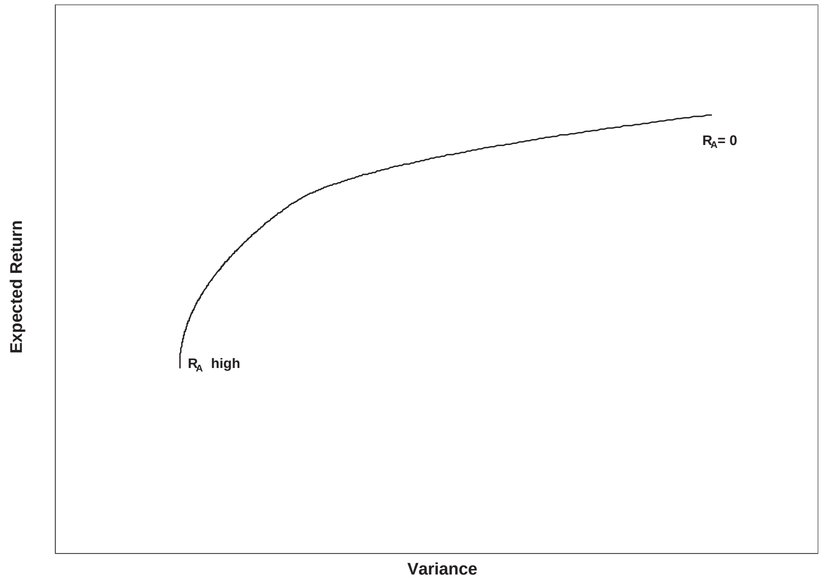

Figure 1: 4 stock example: MEF

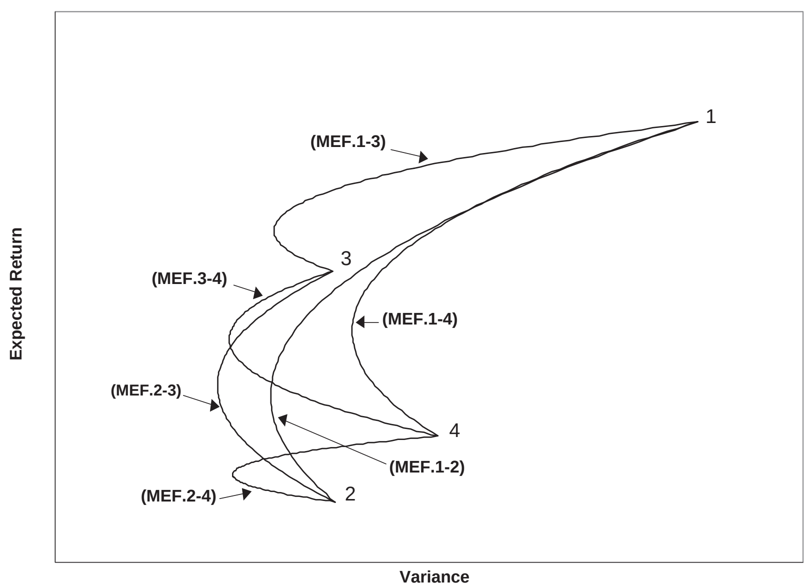

Figure 2: 4 stock example: Investment Opportunity Set

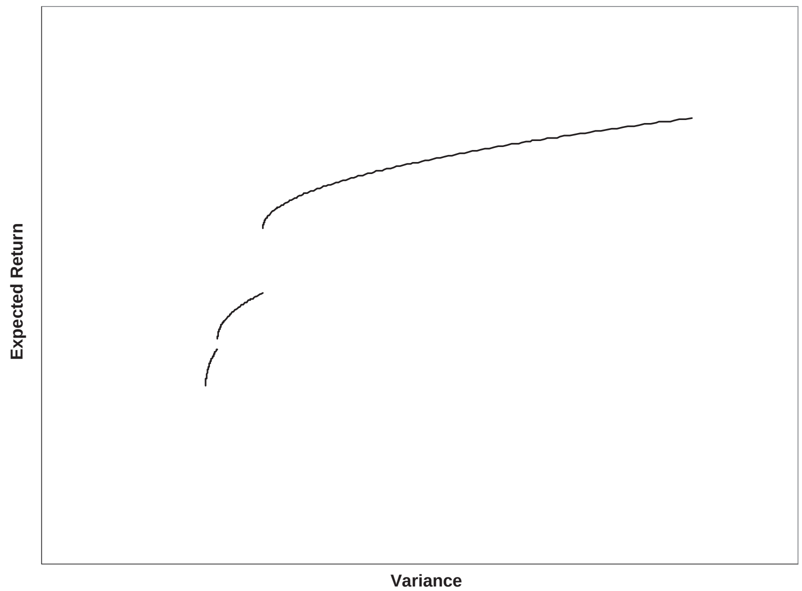

Figure 3: 4 stock example: DCEF

3.2 Representation of Discrete Constraints

In the presence of threshold constraints the portfolio weights are semi-continuous variables (see Beale and Forrest (1976)) modelled using variable upper and lower bounds. A binary variable, δ_i, and finite upper and lower bounds, l_i and u_i, respectively, are associated with each asset $1 = 1, \ldots, N$. The buy-in thresholds are represented by the constraint pair

$$l_i\delta_i \leq x_i \leq u_i\delta_i \qquad \text{and} \qquad \delta_i = 0, 1 \qquad\qquad i = 1, \ldots, N.$$

We refer to model BUY-IN as QP1 with the above constraints added. Cardinality constraints are simply modelled by constraining the sum of the binary variables to be equal to, k, the number of assets required to be in the portfolio.

$$\sum_{i=1}^{N} \delta_i = k$$

Model CARD is model BUY-IN with this additional constraint.

CARD:

$$\text{Min} \quad Z_{CARD} = \sum_{i=1}^{N}\sum_{j=1}^{N} x_i x_j \sigma_{ij}$$

subject to

$$\begin{aligned}
\sum_{i=1}^{N} x_i \mu_i &= \rho \\
\sum_{i=1}^{N} x_i &= 1 \\
x_i &\leq u_i \times \delta_i \\
x_i &\geq l_i \times \delta_i \\
\sum_{i=1}^{N} \delta_i &= k \\
\delta_i &= 0 \text{ or } 1 \qquad\qquad i = 1, \ldots, N
\end{aligned}$$

Transaction roundlots, described as either numbers of stocks, or as a cash value, are expressed as a fraction, f_i, of the portfolio wealth. The portfolio weights are then defined in terms of f_i and an integer number of roundlots, y_i. Thus, $x_i = y_i f_i$, $i = 1, \ldots, N$. Applying roundlot constraints, it may not be possible to exactly satisfy the budget requirement $\sum_{i=1}^{N} x_i = 1$. Therefore, this restriction is made 'elastic' using undershoot and overshoot variables, ϵ^- and ϵ^+, respectively, which are penalized in the objective function with a high cost γ. In an optimum solution ϵ^- and ϵ^+ are made as small as possible so that the fractional stock holdings x_i sum to a value 'as close as possible' to 1.

LOT:

$$\text{Min} \quad Z_{LOT} = \sum_{i=1}^{N}\sum_{j=1}^{N} y_i f_i y_j f_j \sigma_{ij} + \gamma\epsilon^- + \gamma\epsilon^+$$

subject to

$$\begin{aligned}
\sum_{i=1}^{N} y_i f_i \mu_i &= \rho \\
\sum_{i=1}^{N} y_i f_i + \epsilon^- - \epsilon^+ &= 1 \\
l_i \leq \; y_i f_i &\leq u_i \qquad\qquad i = 1, \ldots, N
\end{aligned}$$

$$\begin{array}{rcll} y_i & \text{integer} & & i = 1, \ldots, N \\ \epsilon^-, \epsilon^+ & \geq & 0 & \end{array}$$

Buy-in thresholds and cardinality constraints can also be applied to model LOT.

3.3 Invisible Sections of the DCEF

To generate the DCEF for the example in section 3.1 we are able to use complete enumeration. The same frontier can be generated by repeatedly solving model CARD with $k = 2$. Model CARD is based on QP1, however, using QP2 as the underlying model prevents the full DCEF from being generated. In order to explain the 'missing' sections, consider the objective function of QP2

$$\text{Max} \quad Z_{QP2} = \sum_{i=1}^{N} x_i \mu_i - \frac{R_A}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} x_i x_j \sigma_{ij}.$$

This can be rewritten as the equation of the straight line, $e = mv + c$, where e is the expected return, v is the variance, $m = \frac{R_A}{2}$ and Z_{QP2} is the e-intercept, c. Maximising Z_{QP2}, for any given value of R_A, corresponds to maximising the e-intercept, for a specified gradient. Drawing, over the feasible region, the family of lines described by any non-negative value of R_A, the e-intercept is maximised, uniquely, at the point of tangency with the upper-left border of the region. Systematically varying R_A, from zero upwards, changes the point of tangency and traces out the entire frontier. See Figure 4.

Applying this method to the non-convex region in Figure 2, we begin on the curve MEF 1-3 and slide continuously down until we find tangency with the curve MEF 2-4. The result is that our point of tangency 'jumps' to the lower curve maintaining smoothness in the increasing gradient but missing out some efficient points. See Figure 5. The missing parts of the DCEF are also shown in Figure 6.

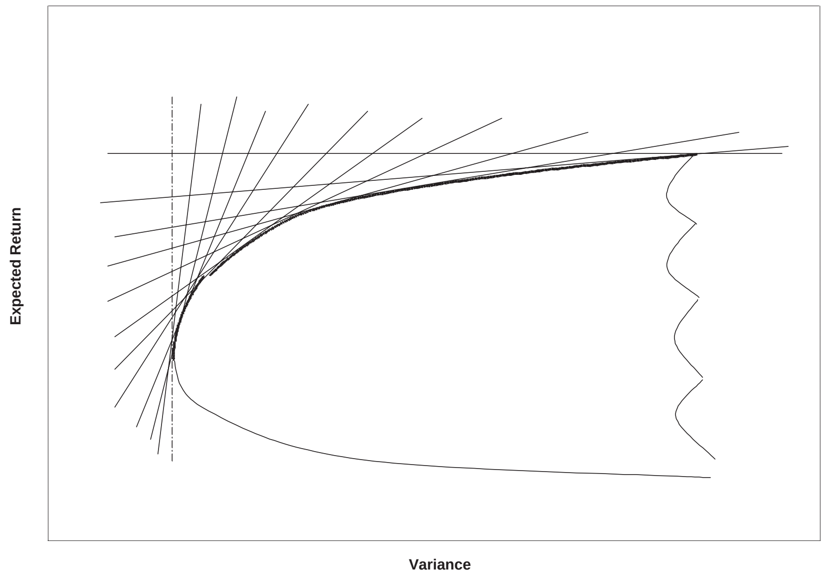

Figure 4: Tracing Out The Efficient Frontier

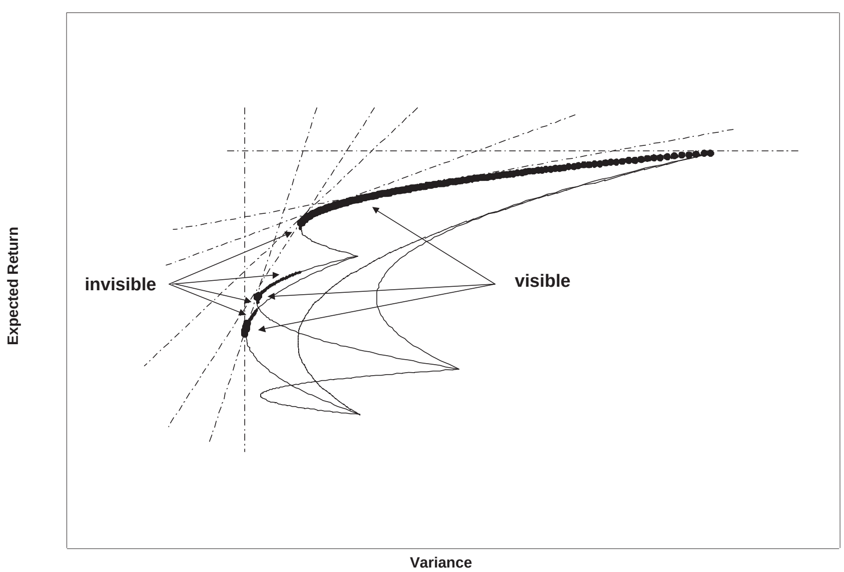

Figure 5: The gradient of the objective function of the 'lambda' formulation, model QP2

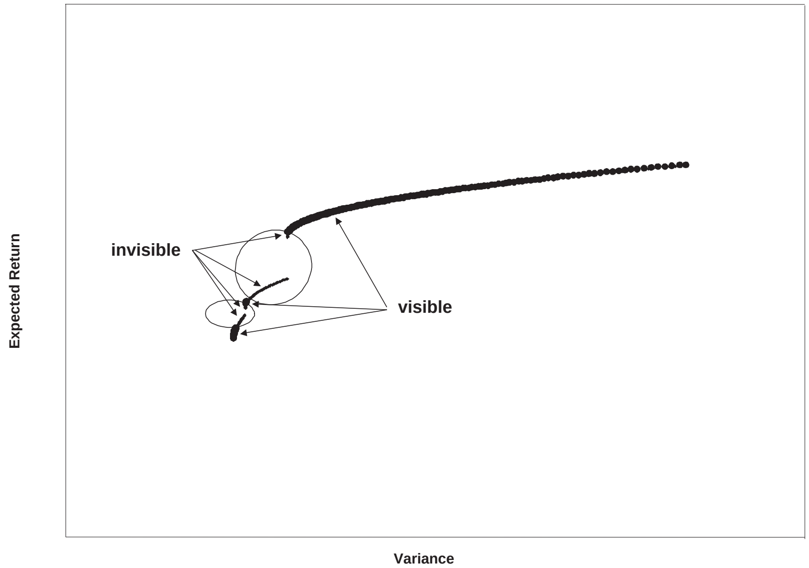

Figure 6: 'Invisible' sections of the DCEF

4 DCEF - A Computational Study

We investigate the shape of the DCEF for the BUY-IN, CARD and LOT models using 60 monthly returns (October 1994 to September 1999) for 30 stocks drawn randomly from the FTSE 100. We consider model CARD in detail for 5 datasets drawn from the Hang Seng, DAX, FTSE, S&P and Nikkei indices with 31, 85, 89, 98 and 225 stocks respectively (Beasley (1999), Chang et al. (1999)). To compute the solutions to these models within an acceptable time frame we use two heuristic solution procedures;'integer restart' and 'reoptimisation' on a reduced set of stocks. We compare our results to those of Chang et al. (1999), obtained using modern heuristic methods (genetic algorithm, tabu search and simulated annealing).

4.1 Implementation

Software Tools
The models are implemented using MPL, a mathematical programming language (Maximal (1999)). Alternatively, AMPL (Fourer et al. (1993)) can be used. The

solver system used is FortQP (Mitra et al. (2000)). A VBA routine drives the system from EXCEL where the input and output data is stored. See Figure 7. The system is run on a Pentium III PC, 500 MHz with 128 MB RAM using Windows NT.

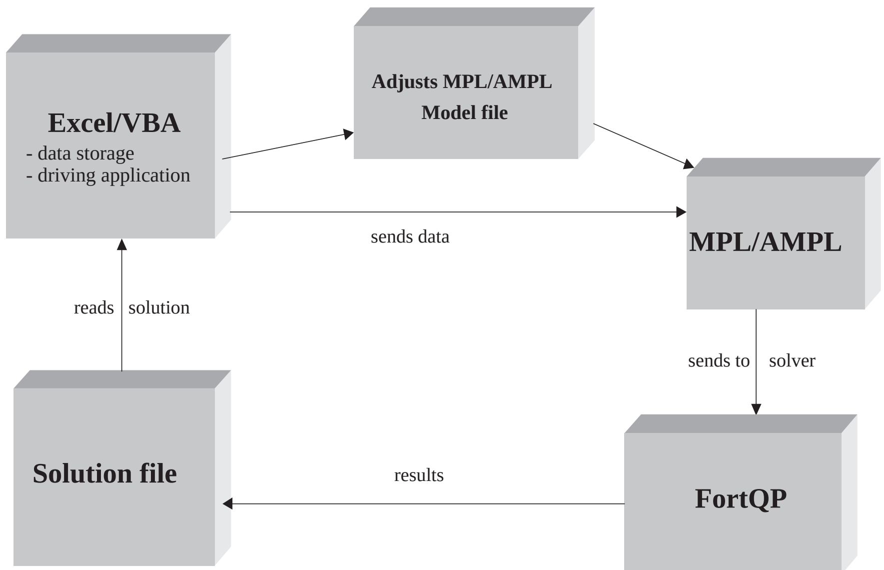

Figure 7: Data - Modeling - Solver Architecture

Solution Methods

Each point of the DCEF curve represents the global optimum solution of a 'discrete non-convex' optimisation problem. Given that the quadratic form for the minimisation problem is positive semi-definite, relaxing the discreteness restriction on the variables leads to a convex programming problem. This continuous variable QP relaxation of the problem provides a lower bound and is easily embedded (see Mitra (1976) and Lawler and Wood (1966)) in a branch-and-bound tree search paradigm.

The FortQP system implemented within the FortMP (Ellison et al. (1999)) solver has both interior point method (IPM) and sparse simplex (SSX) solution capabilities. The system is extensively tested using QLIB test data (Maros and Meszaros (1997)) and models from the finance industry. For the given family of QMIP problems at hand the branch-and-bound algorithm has been specially constructed taking into consideration the following design issues:

SSX versus IPM

In medium to large test problems IPM performs better than SSX. Yet as an embedded solver of sub-problems within branch-and-bound it is not well-suited since the 'warm start' property of IPM is extremely poor. We have therefore chosen SSX as our embedded 'optimisation engine' for solving sub-problems.

Information Sharing and Algorithm Choice
In solving the sub-problems in the child node we share (reuse) the optimum basis information (basis list and the basis factors) of the parent node. We also apply the dual algorithm which reduces the total number of pivotal steps for reoptimisation. These features also justify our choice of algorithm and vindicates the useful 'warm start' properties of the SSX.

Integer Restart Heuristic
In the construction of the DCEF involving, say, 500 points we are unlikely to solve all of these models to QMIP optimality. As a consequence, we are likely to lose the 'pareto efficient' property of the frontier and our experiments confirm this. We do, however, adopt a scheme of computing the DCEF from the highest return, and its corresponding risk, to lower return and reduced risk. We use the previous integer solution in this sequence as the 'first feasible and upper bounding QP value' for the next point (problem). This has the effect that 'within the band of sub-optimality' the DCEF points are 'efficient'.

We believe, and our experimental results vindicate (see section 4.2), that this approach is preferable to applying modern heuristics to this discrete non-convex programming problem.

Reoptimisation Heuristic
To reflect common practice (in the absence of a QMIP solver) we employ a simple heuristic solving 2 continuous QP problems for each QMIP CARD problem. The portfolio size restriction and buy-in thresholds are initially ignored and the equivalent version of QP1 is solved. If there are at least k stocks in the optimal portfolio, QP1 is solved again using only the k stocks with the largest weights. Imposing the buy-in thresholds as explicit lower bounds in the reoptimisation results in a portfolio with exactly k stocks above the appropriate buy-in thresholds.

4.2 Computational Results

Shapes of DCEFs
The discrete efficient frontiers corresponding to the three models, BUY-IN, CARD and LOT, for the 30 stocks from the FTSE 100, are shown in Figure 8. Figure 8a) shows the DCEF for model BUY-IN with a 20% threshold. Figure 8b) shows the DCEF for model CARD with $k = 4$ and a threshold of 10%. Figure 8c) shows the DCEF for model LOT with a uniform lot size of 5% of the portfolio value. In each instance there are clear discontinuities in the frontiers.

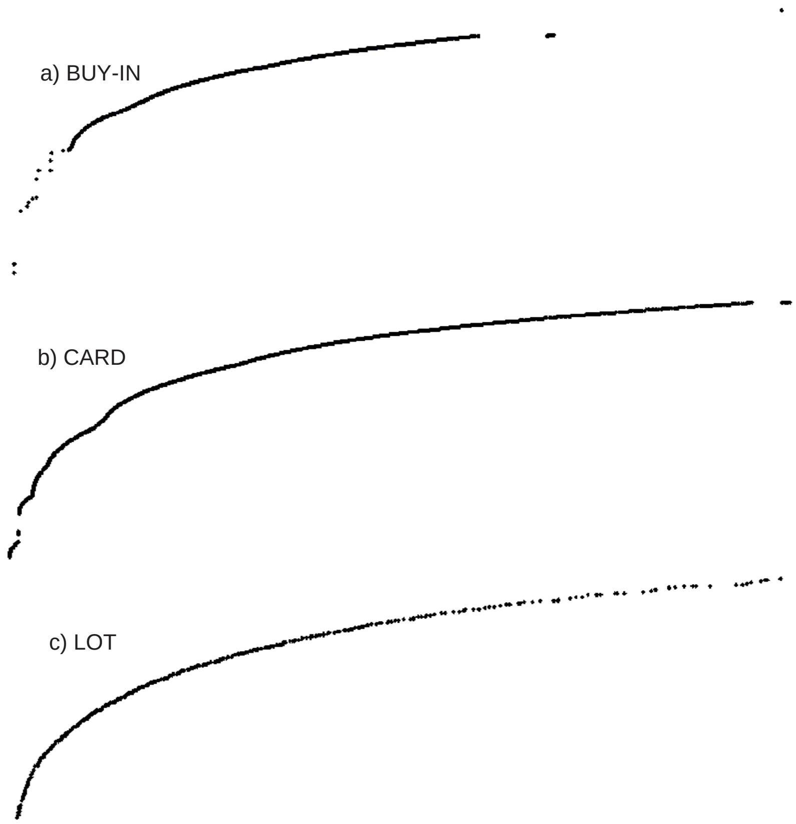

Figure 8: Shapes of DCEF frontiers

'Integer Restart' and 'Reoptimisation' Heuristics

We investigate our heuristic approaches using model CARD for the 5 datasets drawn from the Hang Seng, DAX, FTSE, S&P and Nikkei indices. We set $l_i = 0.01, \quad i = 1, \ldots, N$ and use the cardinality restriction $k = 10$. To analyse the experimental results we follow the metric used in Chang et al. (1999). Error values for points on the heuristically obtained DCEF are measured as the minimum absolute distance (vertical or horizontal) from the MEF as the *exact* DCEF is not calculated. Therefore, the reported 'errors' mainly reflect the systematic deviations due to the discrete constraints. Using the same metric allows a comparison with the modern heuristic results of Chang et al. (1999).

Integer Restart Heuristic
The QMIP problems are solved to the second, improving, feasible integer solution subject to a limit of 500 nodes in the branch-and-bound algorithm. Figure 9 shows the DCEF for the S&P dataset plotted against the MEF.

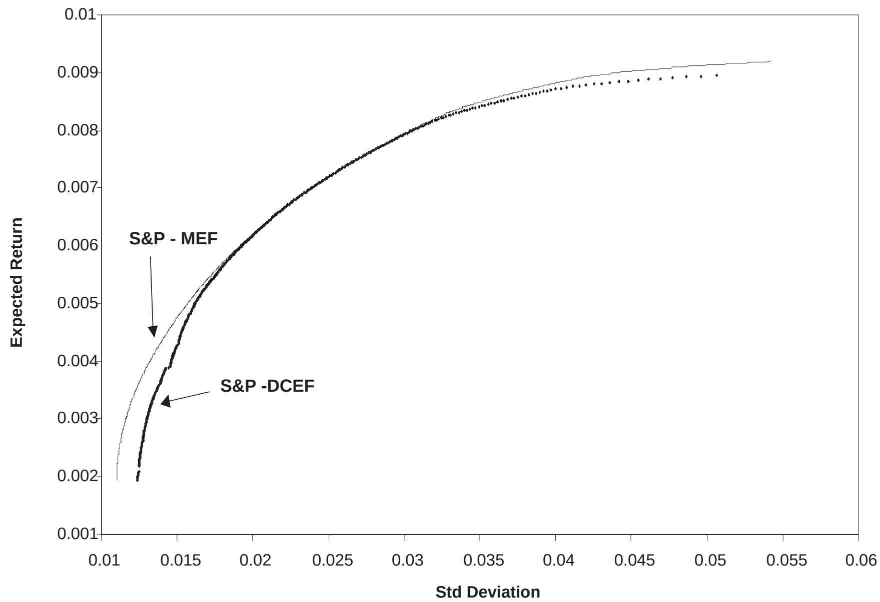

Figure 9: DCEF: S&P - 98 stocks

Table 4.1 presents the results for the integer restart method applied to the five datasets. The table includes the mean and median percentage errors, the total number of DCEF points computed, the number of integer optimal points and the total solution time in seconds. The number of optimal points obtained does not appear to influence the size of the errors observed, suggesting that when optimality is not reached, the second integer solution is a good approximation of the optimal solution.

For each dataset the mean error is below 0.02% with the median error below 0.015%. In all instances, the mean is greater than the median indicating positively skewed error distributions. The size of the errors reported indicate that the DCEFs obtained are very close to the corresponding MEFs. This is borne out by a mean error of 0.008% (median error 0.006%) for the DCEF solved to optimality (3000 points) for the Hang Seng.

In order to establish the computational advantage of the integer restart heuristic

Index	No. of Stocks	Total no. of DCEF pts	No. of integer optimal pts	Solution time *	Mean Error	Median Error
Hang Seng	31	500	492	57.55	0.01415	0.00997
DAX	85	500	228	8405.33	0.01399	0.01159
FTSE	89	500	244	10978.12	0.01141	0.00860
S & P	98	500	192	15831.97	0.01586	0.01325
Nikkei	225	500	486	18345.56	0.00618	0.00252
Hang Seng	31	3000	3000	382.21	0.00826	0.00628

Table 4.1: Results for the integer restart heuristic

we also calculate the DCEF without starting with the previous solution vector. The integer restart heuristic finds more non-dominated points and more optimal points with a smaller mean deviation in less time. To achieve similar error and optimality results the number of nodes to be searched in the B&B algorithm needs to be increased. For example, for the S&P dataset the number of nodes has to be increased from 500 to 2500 but the solution time also increases fivefold.

Reoptimisation Heuristic
The results of the reoptimisation heuristic are displayed in Table 4.2. For each dataset we consider 500 points. The discrepancy between this and the number of discrete points obtained corresponds to those portfolios with less than 10 stocks after the initial optimisation and the infeasible solutions (not achieving the desired level of return) from the reoptimisation. The number of DCEF points refers to the number of efficient discrete points.

Index	No. of Stocks	Total no. of MEF pts	No. of discrete pts	No. of DCEF pts	Solution time *	Mean Error	Median Error
Hang Seng	31	500	104	103	10	0.00021	0.00051
DAX	85	500	356	349	37.53	0.01444	0.01155
FTSE	89	500	375	355	36.18	0.01014	0.00715
S & P	98	500	356	278	44.93	0.01652	0.01356
Nikkei	225	500	376	374	280.92	0.00316	0.00151

Table 4.2: Results for the reoptimisation heuristic

When the reoptimisation heuristic can be implemented it appears to offer a good approximation of the true solution. Again the mean errors are all below 0.02%

with the median errors all below 0.015%. The reoptimisation also reproduces the positive skewness in the error distributions.

Integer Restart versus Reoptimisation Heuristic
The reported errors are similar for both methods although the reoptimisation heuristic is faster. However, the reoptimisation method cannot generate the entire frontier if any of the portfolios on the MEF contain less than k stocks. Also, reoptimising can generate inefficient points. Ignoring these inefficient portfolios leads to a coarser approximation of the true DCEF. Figure 10 shows the DCEF computed by the reoptimisation heuristic plotted against the DCEF for the restart method.

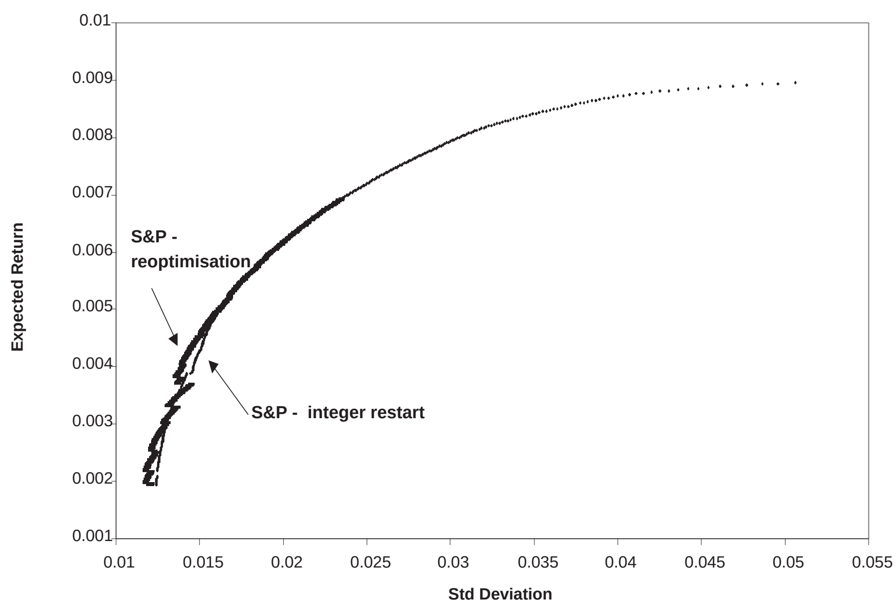

Figure 10: Restart and Reoptimisation DCEF's: S&P - 98 stocks

Comparison with Modern Heuristic Methods
Both the integer restart and reoptimisation heuristics outperform the modern heuristic methods of Chang et al. (1999) who report average mean and median deviations in excess of 1% (see Table 4.3). Clearly this makes both of our heuristic schemes very attractive, from the point of view of the quality of the discrete solution. The computational times are difficult to compare. Unfortunately, it is not possible to further compare the results since their full DCEFs are not available (Beasley (2000)).

Index	No. of Stocks	Solution Method	No. of efficient points	Mean Error	Median Error
Hang Seng	31	Integer restart heuristic	500	0.01415	0.00997
			3000	0.00826	0.00628
		Rounding heuristic	103	0.00021	0.00051
		GA heuristic	1317	0.94570	1.18190
		TS heuristic	1268	0.99080	1.19920
		SA heuristic	1003	0.98920	1.20820
		pooled (GA, TS, SA)	2491	0.93320	1.18990
DAX	85	Integer restart heuristic	500	0.01399	0.01159
		Rounding heuristic	349	0.01444	0.01155
		GA heuristic	1270	1.95150	2.12620
		TS heuristic	1467	3.06350	2.53830
		SA heuristic	1135	2.42990	2.46750
		pooled (GA, TS, SA)	2703	2.19270	2.46260
FTSE	89	Integer restart heuristic	500	0.01141	0.00860
		Rounding heuristic	355	0.01014	0.00715
		GA heuristic	1482	0.87840	0.59600
		TS heuristic	1301	1.39080	0.71370
		SA heuristic	1183	1.13410	0.63610
		pooled (GA, TS, SA)	2538	0.77900	0.59380
S & P	98	Integer restart heuristic	500	0.01586	0.01325
		Rounding heuristic	278	0.01652	0.01356
		GA heuristic	1560	1.71570	1.14470
		TS heuristic	1587	3.16780	1.14870
		SA heuristic	1284	2.69700	1.12880
		pooled (GA, TS, SA)	2759	1.31060	1.06860
Nikkei	225	Integer restart heuristic	500	0.00618	0.00252
		Rounding heuristic	374	0.00316	0.00151
		GA heuristic	1823	0.6431	0.6062
		TS heuristic	1701	0.8981	0.5914
		SA heuristic	1655	0.637	0.6292
		pooled (GA, TS, SA)	3648	0.569	0.5844

Table 4.3: Comparison with modern heuristic approaches

4.3 Investigation of a Portfolio Rebalancing Problem

We apply cardinality constraints to the portfolio rebalancing problem. The aim is to identify the trades required to adjust the initial asset holdings such that the optimised portfolio tracks (in terms of variance) a target portfolio or index. The cardinality constraint restricts the number of trades that can be made.

The optimal portfolio weights, x_i, are now defined in terms of the initial holdings, n_i, and the amounts bought, b_i, and sold s_i. Thus, $x_i = n_i + b_i - s_i$. We introduce binary variables, δ_i^b and δ_i^s, to indicate if asset i bought or sold. Constraining the sum of each pair to be at most 1 prevents an asset being both bought and sold:

$$\delta_i^b + \delta_i^s \leq 1 \qquad\qquad i = 1, \ldots, N.$$

Buy-in thresholds, LB_i^b and LB_i^s, and upper bounds, UB_i^b and UB_i^s, apply to the buying and selling variables respectively:

$$\begin{aligned} \delta_i^b LB_i^b \leq b_i \leq \delta_i^b UB_i^b & \qquad\qquad i = 1, \ldots, N \\ \delta_i^s LB_i^s \leq s_i \leq \delta_i^s UB_i^s & \qquad\qquad i = 1, \ldots, N. \end{aligned}$$

The cardinality constraint restricts the sum of all the binary variables, the number of trades made, to be no greater than k:

$$\sum_{i=1}^{N} (\delta_i^b + \delta_i^s) \leq k.$$

In practice it is common to employ a factor model to describe asset returns. Tracking a target portfolio then involves replicating the risk profile (the vector of factor sensitivities) of the target portfolio. For C factors, with f_c being the level of the cth factor, β_{ic} the sensitivity of asset i to factor c, α_i the mean return of asset i and ϵ_i the specific return of asset i, asset returns r_i, are given by

$$r_i = \alpha_i + \sum_{c=1}^{C} \beta_{ic} f_c + \epsilon_i.$$

The factors are constructed such that there is no correlation between the factors, no correlation between the factors and specific returns and it is assumed that the specific returns are uncorrelated. The variance of returns is given by

$$Var(r_i) = \sigma_i^2 = \sum_{c=1}^{C} \beta_{ic}^2 \sigma_{f_c}^2 + \sigma_{\epsilon_i}^2$$

Denoting the sensitivity of the index to factor c by I_c, the initial the portfolio rebalancing model can be stated as

REBALANCE:

$$\text{Min} \quad Z_{REB} = \sum_{c=1}^{C} y_{P,c}^2 \sigma_{f_c}^2 + \sum_{i=1}^{N} x_i^2 \sigma_{\epsilon_i}^2$$

subject to

$$\begin{aligned}
y_{P,c} &= (\sum_{i=1}^{N} x_i \beta_{ic}) - I_c & c = 1, \ldots, C \\
\sum_{i=1}^{N} x_i \mu_i &\geq \rho \\
\sum_{i=1}^{N} x_i &= 1 \\
x_i &= n_i + b_i - s_i \\
\delta_i^b LB_i^b \leq \; & b_i \leq \delta_i^b UB_i^b \\
\delta_i^s LB_i^s \leq \; & s_i \leq \delta_i^s UB_i^s \\
\delta_i^b + \delta_i^s &\leq 1 \\
\sum_{i=1}^{N} (\delta_i^b + \delta_i^s) &\leq k \\
x_i,\, b_i,\, s_i &\geq 0 \\
\delta_i^b,\, \delta_i^s &= 0 \text{ or } 1 & i = 1, \ldots, N.
\end{aligned}$$

We implement this model for 2 fixed income datasets. The first problem is to rebalance a given 20 bond portfolio, using at most 6 trades, to track an index of 330 bonds. The second problem involves a portfolio of 49 bonds tracking an index of 391 bonds, with a limit of 10 trades. For each problem we plot 200 points on the DCEF using the same integer restart and reoptimisation heuristics as before. 3 factors (explaining 90% of the total variance) are used in the model to describe returns. In the optimisations, residual risk is ignored making the factor risk of the target indices the focus of the model. Figures 11 and 12 show the DCEFs for these problems. For the smaller of the two datasets all of the restart solutions are integer optimal. For the larger dataset most solutions are integer feasible.

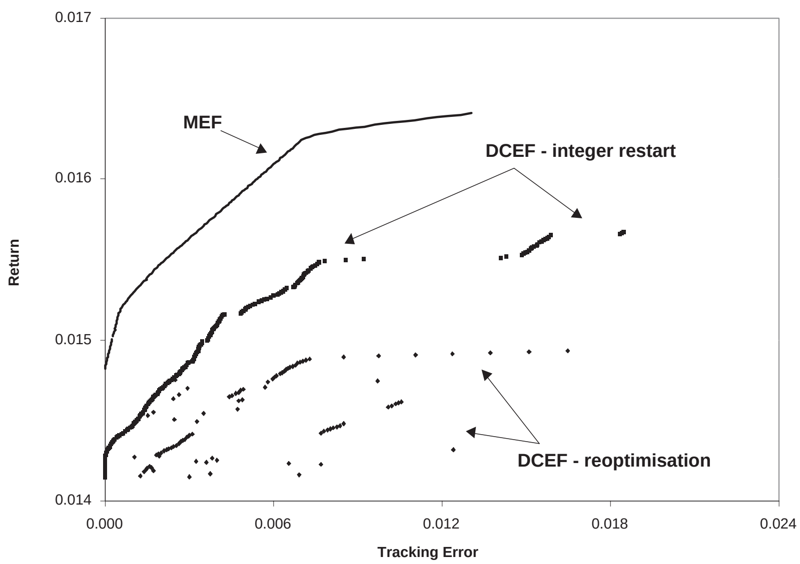

Figure 11: Rebalancing: 20 bonds, 6 bonds traded

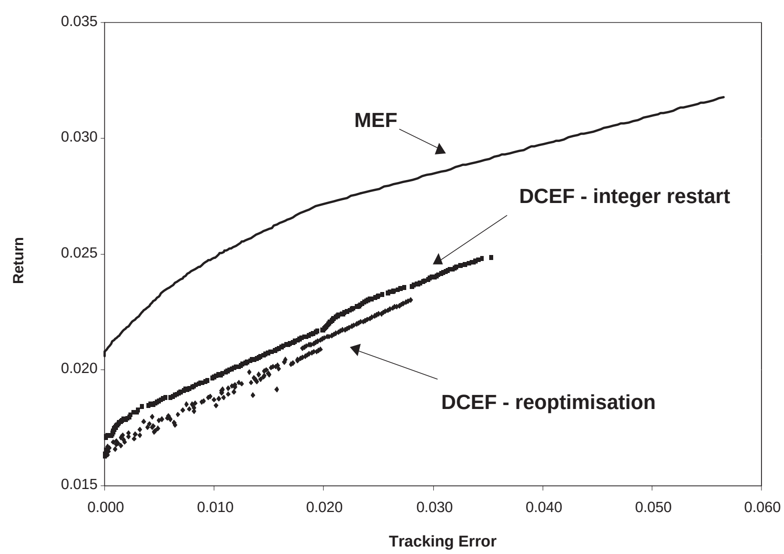

Figure 12: Rebalancing: 49 bonds, 10 bonds traded

When constructing a new portfolio, the relative positions of the MEF and DCEF, are determined by the number of stocks in a portfolio on the MEF and the size of the cardinality constraint. For the rebalancing problem it is the number of trades identified by a portfolio on the MEF, the cardinality constraint and also the initial holding that affect the relative positions of the frontiers.

5 Discussion and Conclusions

5.1 Comparative results

The use of QMIP enables us to use discrete constraints and capture important features of real-world problems. For the DCEF we highlight the discontinuities which follow as a consequence of imposing discrete constraints. We also explain why the risk-return trade-off cannot be used to construct the entire DCEF. Computing the entire 'DCEF to optimality' for even a reasonable size model remains a computationally intractable task We show that by 'integer restarting' the QMIP with the previous solution we are able to generate a reasonable number of optimal and near optimal points within a restricted branch-and-bound search. We introduce a simple 'reoptimisation' heuristic which proves to be computationally very efficient in constructing parts of the DCEF. Both methods outperform modern heuristic approaches.

The integer restart heuristic appears to be a valid method for investigating the problem of portfolio rebalancing. The reoptimisation heuristic performs relatively poorly as the restart DCEF dominates the reoptimisation DCEF in both cases studied. We also observe that in real applications the interest is not so much to construct the entire frontier accurately but to identify (and 'zoom in' to) an appropriate 'risk-return' segment where a number of alternative exact portfolios can be constructed.

5.2 Solution Systems: current state-of-the-art

The solution of convex quadratic programs by sparse simplex (Mitra (1976)) or by the interior point method (see Vanderbei (1994)) are now well established. Although there exist large sets of test data for QP problems they are not immediately relevant in the context of portfolio planning. It is also possible to apply any nonlinear solver such as the ones provided by NAG (NAG 1999) or that found within EXCEL. There are many commercially available solvers, such as those provided by MathSoft (NUOPT (1998)), Operations Research Systems

(GIANO 1999), Advanced Portfolio Technology (APT (2000)) amongst others. Commercial LP systems such as CPLEX and OSL also provide QP solver capability. Chang et al. (1999) provide heuristics for obtaining good sub-optimal solutions but they are not able to solve a discrete portfolio problem to optimality for a given data set. The authors are aware of only one other QMIP system (GIANO) but no scientific description or performance figures are available in the open literature.

Our experience vindicates that a branch-and-bound solver which uses the SSX for solving QPs is the most attractive avenue. In this approach the 'warm start' feature of the SSX is exploited which, unfortunately, rules out using IPM since its restart properties are relatively poor. Finally, we would like to highlight that our relative success in computing the 'near optimal' DCEF is due to the use of 'warm start' and dual SSX and 'integer restart' using the previous solution which speeds up the computation of the next efficient point.

5.3 Future Directions

The success of the very simple reoptimising heuristic suggests that there may be some benefit in investigating a refined version that can overcome some of the drawbacks. Although the integer restart method produces good sub-optimal results there is some scope of improving solver performance using pre-processing techniques.

Our study shows that the portfolio rebalancing problem with discrete constraints should be investigated more thoroughly. Particularly, the relationship between the cardinality constraint and the initial holding requires further exploration.

The QMIP capability will also allow the discrete models to be extended to incorporate short sales and transaction costs. It also be interesting to observe the effect these extensions have on the shape of the efficient frontier.

A natural extension is to implement the practical, discrete, constraints in dynamic multi-period models (Ziemba and Vickson (1975), Ziemba and Mulvey (1998)).

6 Acknowledgements

Mr. M. Horniman has been supported by an EPSRC case PhD studentship in which Advanced Portfolio Technology (APT) is the industrial partner. Mr. N.

Jobst's PhD research is supported by Fidelity Investments. The authors gratefully acknowledge the support provided by these organisations. The authors would like to thank W.T. Ziemba reviewing and arranging the paper for publication.

7 References

Adcock, C.J. and N. Meade (1994): "A simple algorithm to incorporate transaction costs in quadratic optimisation", *European Journal of Operational Research*, 79, 85-94.

APT (2000): Advanced Portfolio Technology Inc, 17 State Street, New York, NY10004, USA, *http://www.aptltd.com.*

Beale, E.M.L. and J.J.H. Forrest (1976): "Global optimization using special ordered sets", *Mathematical Programming*, 10, 52-69.

Beasley, J.E. (1999): *http://mscmga.ms.ic.ac.uk/jeb/orlib/portinfo.html*

Beasley, J.E. (2000): private communication.

Bienstock, D. (1996): "Computational study of a family of mixed-integer quadratic programming problems", *Mathematical Programming*, 74, 121-140.

Cariño, D.R. and W.T. Ziemba (1998): "Formulation of the Russel-Yasuda Kasai Financial Planning Model", *Operations Research*, 46, no 4, 1-17.

Chang, T-J., N. Meade, J.E. Beasley and Y.M. Sharaiha (1999): "Heuristics for cardinality constrained portfolio optimisation", to appear in *Computers and OR.*

Chopra, V.K. and W.T. Ziemba (1993): "The Effect of Errors in Means, Variances and Covariances on Optimal Portfolio Choice", *Journal of Portfolio Management*, (Winter), 6-11.

Constantinides, G.M. and A.G. Malliaris (1995): "Portfolio Theory", in *Finance*, ed. by R.A. Jarrow, V. Maksimovic and W.T. Ziemba. Elsevier, Amsterdam, pp. 1-30.

Ellison, E.F.D., M. Hajian, R. Levkovitz, I. Maros and G. Mitra (1999): "A Fortran based Mathematical Programming System, FortMP", Brunel University, London and NAG Ltd., Oxford.

Fourer, R., D.M. Gay and B.W. Kernighan (1993): "AMPL: A Modeling Language for Mathematical Programming", Duxbury Press/Brooks/Cole Publishing Company.

GIANO (1999): "GIANO for Asset Allocation", Operational Research Systems, Corso Nino Bixio 58, 12051 Alba (CN), Italy.

Grinold R.C. and R.N. Kahn (1995): "Active Portfolio Management: Quantitative Theory and Applications", Probus Press.

Hanoch, G. and H. Levy (1969): "The Efficiency Analysis of Choices Involving Risk", *The Review of Economic Studies*, 36, 335-346.

Hensel, C.R. and A.L. Turner (1998): "Making Superior Asset Allocation Decisions: A Practitioner's Guide", in *Worldwide Asset and Liability Modeling*, ed. by W.T. Ziemba and J.M. Mulvey. Cambridge University Press.

Horniman, M.D., N.J. Jobst, C.A. Lucas and G. Mitra (2000): "Constructing Efficient Portfolios with Discrete Constraints - A Computational Study", Technical Report TR/06/00, Department of Mathematical Sciences, Brunel University.

Kallberg, J.G. and W.T. Ziemba (1983): "Comparision of Alternative Utility Functions in Portfolio Selection Problems", *Management Science*, 29, 1257-1276.

Kellerer, H., R. Mansini and M.G. Speranza (1997): "On selecting a portfolio with fixed costs and minimum transaction lots". Working paper, Dipartimento di Metodi Quantitativi, Universita di Brescia, Italy.

King, B.F. (1966): "Market and industry factors in stock price behavior", *Journal of Business*, 39, 139-190.

Konno, H. and H. Yamazaki (1991): "Mean-Absolute Deviation Portfolio Optimization Model and its Applications to Tokyo Stock Market", *Management Science*, 37, 519-531.

Lawler, E.L. and D.E. Wood (1966): "Branch-and-Bound Methods: A Survey", *Operations Research*, 14, 699-719.

Lee, E.K. and J.E. Mitchell (1997): "Computational Experience of an Interior-Point SQP Algorithm in a Parallel Branch-and-Bound Framework", *Proceedings of High Performance Optimization Techniques*, Springer Verlag.

Lee, S.M. (1972): "Goal Programming for Decision Analysis", Auerback, Philadelphia.

Lee, S.M. and D.L. Chesser (1980): "Goal Programming for Portfolio Selection", *Journal of Portfolio Management*, 7, 23-25.

Lintner, J. (1965): "The valuation of risk assets and the selection of risky investments in stock portfolios and capital budgets", *Review of Economics and Statistics*, February, 13-37.

Mansini, R. and M.G. Speranza (1999): "Heuristic algorithms for the portfolio selection problem with minimum transaction lots", *European Journal of Operational Research*, 114, 219-233.

Markowitz, H.M. (1952): "Portfolio Selection", *Journal of Finance*, 7, 77-91.

Markowitz, H.M. (1956): "The optimization of a quadratic function subject to linear constraints", *Naval Research Logistics Quarterly*, 3, 111-133.

Markowitz, H.M. (1959, 1991): "Portfolio Selection: Efficient Diversification of Investments", Wiley, New York, NY (1959), Second Edition, Basil Blackwell, Cambridge, MA (1991).

Markowitz, H.M. (1987): "Mean-Variance Analysis in Portfolio Choice and Capital Markets", Basil Blackwell.

Maros, I. and Cs. Meszaros (1997): "A Repository of Convex Quadratic Programming Problems", Departmental Technical Report DOC 97/6 Department of Computing, Imperial College, London, UK.

Maximal (1999): "MPL Modeling System, User's Guide 1999", Maximal Software, Inc., 2111 Wilson Boulevard, Suite 700, Arlington, VA 22201, U.S.A., www.maximal-usa.com

Mitra, G. (1976): "Theory and application of Mathematical Programming", Academic Press.

Mitra, G., Gürtler, M. and E.F.D. Ellison (2000): "Solution of Quadratic Programming by the Interior Point Method and Sparse Simlex: A computational study", Technical Report, Department of Mathematical Sciences, Brunel University.

Mossin, J. (1966): "Equilibrium in a capital asset market", *Econometrica*, 34, 768-783.

NAG (1999): Numerical Libraries, Numerical Algorithms Group Ltd., Oxford, UK.

NUOPT (1998): "NUOPT for S-Plus", MathSoft International, Surrey, UK.

Rosenberg, B. (1974): "Extra-market components of covariance in security returns", *Journal of Financial and Quantitative Analysis*, 9, 263-273.

Ross, S.A. (1976): "The arbitrage theory of capital asset pricing", *Journal of Economic Theory*, 13, 341-360.

Rudd, A. and B. Rosenberg (1979): "Realistic portfolio optimisation", in *Portfolio Theory - Studies in the Management Sciences*, 11, ed. by E.J. Elton and M.J. Gruber. North-Holland, Amsterdam, pp. 21-46.

Sharpe, W.F. (1963): "A Simplified Model for Portfolio Analysis", *Management Science*, 9, 277-293.

Sharpe, W.F. (1964): "Capital Asset Prices: A Theory of Market Equilibrium Under conditions of Risk", *Journal of Finance*, 19, 425-442.

Sharpe, W.F. (1971): "A Linear Programming Approximation for the General Portfolio Analysis Problem", *Journal of Financial and Quantitative Analysis*, 6, 1263-1275.

Speranza, M.G. (1996): "A heuristic algorithm for a portfolio optimization model applied to the Milan stock market", *Computers and Operations Research*, 23, 433-441.

Tobin, J. (1958): "Liquidity preference as behaviour towards risk", *The Review of Economic Studies*, 25 (2), 65-86.

Vanderbei, R.J. and T.J. Carpenter (1993): "Symmetric indefinite systems for interior point methods", *Mathematical Programming*, 58, 1-32.

Vanderbei, R.J. (1994): "LOQO: An Interior Point code for Quadratic Programming", Technical Report SQR-94-15, Statistics and Operations Research, Princeton University.

Von Neumann, J. and O. Morgenstern (1944): "Theory of Games and Economic Behaviour", Princeton University Press.

Young, M.R. (1998): "A Minimax Portfolio Selection Rule with Linear Programming Solution", *Management Science*, 44, 673-683.

Ziemba, W.T. and J.M. Mulvey (editors) (1998): "Worldwide Asset and Liability Modeling", Cambridge University Press.

Ziemba, W.T., Parkan, C. and R. Brooks-Hill (1974): "Calculation of Investment Portfolios with Risk Free Borrowing and Lending", *Management Science*, 21, 209-222.

Ziemba, W.T. and R.G. Vickson (editors)(1975): "Stochastic Optimization Models In Finance", Academic Press.

A Illustrative Dataset

Average monthly returns and covariances from 5 years' data for 30 FTSE stocks.

Expected Return

Stock no.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Expected Return	1.9231	0.3950	0.9231	2.1234	0.5822	2.4004	2.3603	0.7932	1.1049	0.7388	1.1954	0.5775	2.5019	0.3152	1.5001

Stock no.	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
Expected Return	1.8989	1.7460	1.2366	1.2509	0.7493	1.2468	2.4568	2.3528	0.6651	3.4346	0.6886	1.3782	2.2393	1.8350	1.4522

Covariance Matrix

	Stock 1	Stock 2	Stock 3	Stock 4	Stock 5	Stock 6	Stock 7	Stock 8	Stock 9	Stock 10	Stock 11	Stock 12	Stock 13	Stock 14	Stock 15
Stock 1	44.3591														
Stock 2	12.5813	55.4118													
Stock 3	5.9673	0.4586	41.0694												
Stock 4	13.1064	5.3978	-1.1127	38.8121											
Stock 5	4.8330	11.8629	3.8800	9.3633	40.1504										
Stock 6	31.8484	15.7188	2.3163	7.4252	10.3679	61.8480									
Stock 7	36.4109	23.3858	-9.0213	21.4845	11.4553	45.0002	89.0782								
Stock 8	15.4496	10.2141	8.5785	5.3165	7.5599	26.1761	28.0091	44.2418							
Stock 9	3.3790	-3.9218	11.2122	-0.8193	-4.1408	3.8793	2.3141	-2.3792	49.2997						
Stock 10	14.4884	14.0008	0.3506	1.7166	25.1077	16.9762	27.0829	19.7745	-4.6766	60.4822					
Stock 11	4.8061	15.6431	-8.1843	1.6400	8.6543	14.2046	20.1257	10.4054	-4.0006	23.8506	37.8261				
Stock 12	6.3673	-1.7339	9.3644	-0.0434	7.7353	13.6164	8.3009	13.2075	8.7405	1.2429	-6.3669	31.9311			
Stock 13	17.5273	19.4895	10.0226	13.3435	21.5076	19.8401	26.1493	21.5748	-13.9164	31.2236	16.4499	-1.6659	77.1391		
Stock 14	20.3714	21.4225	-0.9602	-0.9443	17.0215	35.5623	51.5795	31.6692	-13.5125	35.2425	19.1531	11.9854	33.9238	82.2687	
Stock 15	4.6626	7.7249	13.9976	-3.2001	7.0223	11.9670	3.3668	-1.0326	11.2959	9.3192	6.4534	-4.7699	11.4253	3.9857	64.0410
Stock 16	6.9818	-4.8496	9.8336	-0.1453	7.4180	4.3640	5.3660	12.1749	9.4513	19.1732	3.8130	8.1015	8.9752	4.5364	-4.8198
Stock 17	9.2554	-0.9581	4.8525	-1.8828	-0.6854	11.8507	12.8461	9.9904	9.7687	-3.3282	-13.3500	9.8185	-10.6832	6.7398	5.3612
Stock 18	9.9860	9.5638	15.6003	11.0239	20.3285	20.9279	22.5954	25.0349	11.5507	21.2028	4.1351	0.5254	28.4416	24.1109	20.8041
Stock 19	10.8073	21.7486	0.5542	11.4746	9.7190	14.1490	14.9537	8.9985	0.0905	7.1867	8.3512	3.0824	7.2041	9.2856	1.0705
Stock 20	6.3428	-1.3307	-7.7636	11.6846	8.2683	11.3679	22.5338	20.3737	-8.8334	12.1690	5.5548	3.6094	17.8511	22.7147	0.1103
Stock 21	17.4619	10.1502	10.6081	3.9556	9.6999	19.9152	23.9791	19.4550	5.5415	14.0175	-1.3558	8.5733	15.4246	19.6629	14.3351
Stock 22	0.9183	2.6363	3.9260	12.6739	17.1122	12.3284	9.5956	11.0185	-0.2397	7.5793	-5.9742	3.5035	7.5169	8.8428	13.3204
Stock 23	6.2159	7.2150	5.8835	4.5602	15.4986	23.7688	27.1578	25.5365	3.2994	14.9532	7.0658	13.5514	18.1989	22.4374	4.8242
Stock 24	8.0790	15.9699	5.8752	3.5692	14.7254	14.1447	19.9735	16.9202	1.1718	19.2443	7.8282	4.9043	19.0235	28.8589	8.9305
Stock 25	-2.7174	-9.7647	-11.1966	-8.4836	10.5996	11.2000	-4.4236	12.7476	-9.8964	13.8600	7.3210	3.6247	13.0249	-0.2227	-4.4157
Stock 26	17.9332	1.1008	-5.1822	1.5520	4.8314	30.4307	31.6643	29.9974	-2.5327	21.1830	20.5619	11.7268	22.7388	48.3054	-7.4816
Stock 27	-0.5185	2.0338	-0.3383	-4.4157	3.6868	4.7662	6.2970	15.4424	-0.2073	19.4279	5.7230	-4.7336	20.7437	11.8125	10.9219
Stock 28	12.0854	28.2566	-8.7473	1.9600	23.9601	19.8818	29.4014	15.2403	-16.0066	34.3006	17.7291	3.7221	26.8055	36.4593	-8.8789
Stock 29	13.8039	3.9858	5.1420	16.6629	2.9495	17.1986	19.2041	3.3403	4.7296	2.7432	0.3909	8.1764	4.9850	8.1445	6.0819
Stock 30	10.3900	16.6060	3.5069	6.4344	18.8809	25.5144	34.9891	32.0006	-12.3490	26.4902	9.0255	9.7111	19.1408	39.2370	1.7386

	Stock 16	Stock 17	Stock 18	Stock 19	Stock 20	Stock 21	Stock 22	Stock 23	Stock 24	Stock 25	Stock 26	Stock 27	Stock 28	Stock 29	Stock 30
Stock 16	38.4227														
Stock 17	-0.7394	40.6584													
Stock 18	7.0093	14.5095	79.4799												
Stock 19	-7.2643	0.9579	6.7248	37.7253											
Stock 20	6.4249	5.3646	18.4547	3.0081	44.6980										
Stock 21	11.1846	12.1726	16.6755	7.1091	3.6814	39.3485									
Stock 22	-7.5054	4.7289	23.5466	9.2018	15.2742	6.4136	48.9834								
Stock 23	2.7644	14.6693	27.5381	6.5745	14.7446	12.5524	16.4556	60.0240							
Stock 24	5.2109	6.3461	28.5846	10.7705	18.1423	12.5216	9.8386	9.2266	34.4239						
Stock 25	14.0825	-4.7384	10.5420	-1.0552	13.0105	0.4526	-7.3964	5.3785	1.1923	84.3889					
Stock 26	17.5917	3.7492	12.1592	-0.0140	30.0701	7.5221	-3.0085	17.4780	18.4134	21.2663	88.3961				
Stock 27	12.5256	-4.0824	18.8931	-3.6541	14.9177	11.7879	10.2016	6.4771	12.9987	2.3873	14.2432	42.6852			
Stock 28	13.6664	-3.0415	12.5306	17.3668	19.5915	11.2792	7.0790	8.6710	22.0286	15.9701	26.1601	15.5715	65.1371		
Stock 29	0.0887	4.8677	2.7280	11.7927	5.9487	9.7819	6.5569	3.8675	9.8203	-7.4017	9.4172	1.2209	2.0285	33.3388	
Stock 30	6.9190	5.4667	22.9398	17.0197	21.6440	14.5880	23.7345	27.0986	20.3379	10.7548	19.9260	12.9230	30.2109	6.6499	51.9320