The Black-Scholes implied volatility at extreme strikes

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Part I: The Tail-Wing Formula

Outline:
- all the background needed on one page
- statement of result
- sketch of proof
- examples with known tail asymptotics
- forthcoming in Math. Finance

- **Black-Scholes** normalized call price given by

$$c_{BS}(k,\sigma) = N(d_1) - e^k N(d_2)$$

where k is log-strike and $d_{1,2}(k) = -k/\sigma \pm \sigma/2$.

- **Notation:** F denotes the distribution function of risk-neutral returns, $\bar{F} \equiv 1 - F$ and (if $\exists$) $F' \equiv f$

- **Implied volatility** defined by

$$c_{BS}(k, V(k)) = \int_k^\infty \left(e^x - e^k\right) dF(x) \equiv c(k)$$

- **Definition:** g regularly varying, index α, $g \in R_\alpha$ iff

$$g(xt)/g(t) \to x^\alpha \qquad \text{as } t \to \infty.$$

- **Examples:** $t^2/2 \in R_2$, $t \log t \in R_1$, ...

- **Notation:** $g \sim h$ iff $g(t)/h(t) \to 1$ as $t \to \infty$.

Tail-wing formula [Benaim, F]: Assume $\alpha > 0$ and $\exists \epsilon > 0 : \mathbb{E}[e^{(1+\epsilon)X}] < \infty$ and define

$$\psi(x) \equiv 2 - 4\left(\sqrt{x^2 + x} - x\right).$$

Then

$$\text{(i)} \implies \text{(ii)} \implies \text{(iii)} \implies \text{(iv)},$$

where, always as $k \to \infty$,

$$-\log f(k) \in R_\alpha; \tag{i}$$

$$-\log \bar{F}(k) \in R_\alpha; \tag{ii}$$

$$-\log c(k) \in R_\alpha; \tag{iii}$$

and

$$V(k)^2/k \sim \psi\left[-\log c(k)/k\right]. \tag{iv}$$

If (ii) holds then $-\log c(k) \sim -k - \log \bar{F}$ and

$$V(k)^2/k \sim \psi\left[-1 - \log \bar{F}(k)/k\right], \tag{iv'}$$

if (i) holds, then $-\log f \sim -\log \bar{F}$ and

$$V(k)^2/k \sim \psi\left[-1 - \log f(k)/k\right]. \tag{iv''}$$

- There is a similar result for the left tail resp. wing.

- Note $\psi : [0, \infty] \searrow [0, 2]$ and $\psi[x] \underset{x \to \infty}{\sim} 1/(2x)$.

- If $-1 - \log f(k)/k \to p^* \in (0, \infty)$ then

$$V(k)^2 \sim \psi(p^*) \times k \quad \text{(asymptotically linear)}$$

- If $-\log f(k)/k \to \infty$ then

$$V(k)^2 \sim \frac{1}{-2 \log f(k)/k} \times k \text{ (asymptotically \textit{sub}linear)}$$

- **Sanity check:** Black-Scholes returns are Gauss with variance σ^2. Hence

$$-\log f_{BS}(k) \sim k^2/\left(2\sigma^2\right) \in R_2$$

From the tail-wing formula,

$$V(k)^2 \sim \frac{1}{-2 \log f_{BS}(k)/k} \times k \sim \sigma^2$$

in trivial agreement with the flat smile $V \equiv \sigma$.

Sketch of proof: A motivating example,

$$-\log \int_x^\infty e^{-t^2/2} dt \sim x^2/2 \text{ as } x \to \infty$$

Bingham's Lemma: $g \in R_\alpha$ with $\alpha > 0$. Then

$$-\log \int_x^\infty e^{-g(t)} dt \sim g(x) \text{ as } x \to \infty.$$

Claim 1: $-\log \bar{F} \sim -\log f$.
Apply Bingham's lemma to $g = -\log f$.

Claim 2: $-\log c(k) \sim -k - \log \bar{F}(k)$.
Apply Bingham's lemma after Integration by Parts

$$c(k) = -\int_k^\infty \left(e^x - e^k\right) d\bar{F}(x) = \int_k^\infty e^x \bar{F}(x)\, dx.$$

Claim 3: $V(k)^2/k \sim \psi(-\log c(k)/k)$.
Show that $\log c(k) = -d_1^2/2 + O(\log k)$ *so that*

$$\frac{\log c(k)}{k} = -\frac{k}{2V(k)^2} + \frac{1}{2} - \frac{V(k)^2}{8k} + O\left(\frac{\log k}{k}\right)$$

and solve for $V(k)$.

That's it!

Examples:

- **NIG Model:** $X = X_T \sim NIG(\alpha, \beta, \mu T, \delta T)$.
  We know that

  $$f(k) \sim C|k|^{-3/2} e^{-\sqrt{\beta^2+\gamma^2}|k|+\beta k} \text{ as } k \to \pm\infty$$

  Therefore $-\log f \in R_1$ and

  $$\log f(k)/k \to \left(-\sqrt{\beta^2+\gamma^2}+\beta\right) \text{ as } k \to +\infty.$$

  and the tail-wing formula gives

  $$\begin{aligned} \frac{\sigma_{BS}^2(k,T)\,T}{k} &\sim \psi\left(-1-\log f(k)/k\right) \\ &\sim \psi\left(-1+\sqrt{\beta^2+\gamma^2}-\beta\right). \end{aligned}$$

- **FMLS Model:** $X = X_T \sim L_\alpha\left(\mu T, \sigma T^{1/\alpha}, -1\right)$ with $\alpha \in (1, 2]$. Asymptotics of $\bar{F}$ known and imply

  $$-\log \bar{F}(k) \sim k^{\frac{\alpha}{\alpha-1}} \times \left[T\alpha\sigma^\alpha \left|\sec(\pi\alpha/2)\right|\right]^{-1/(\alpha-1)}.$$

  Note $-\log \bar{F} \in R_{\alpha/(\alpha-1)}$.
  From the tail-wing formula

  $$\sigma_{BS}^2(k, T)\, T \sim k^{1-\frac{1}{a-1}} \times \frac{1}{2}\left[T\alpha\sigma^\alpha \left|\sec(\pi\alpha/2)\right|\right]^{1/(\alpha-1)},$$

  consistent with Black-Scholes as $\alpha \uparrow 2$.

- **Merton:** $X.$ is Lévy with triplet (μ, σ^2, K) where K is λ (=intensity of jump) times a Gaussian measure with mean α and standard deviation $\delta > 0$ describing the distribution of jumps. $\bar{F}(k)$ equals

  $$\mathbb{P}[X > k] \leq \inf_z e^{-zk}\mathbb{E}[\exp(zX)] = e^{K(z^*)-z^*k}$$

  where $K(z) = \log \mathbb{E}[\exp(zX)]$ and $z^* = z^*(k)$ is determined from $K'(z^*) = k$. Here

  $$K(z) = T\left\{z\mu + \frac{1}{2}z^2\sigma^2 + \lambda\left(e^{z\alpha + z^2\delta^2/2} - 1\right)\right\}$$

  from which $z^* = z^*(k) \sim \sqrt{2\log k}/\delta$ and

  $$\log \bar{F}(k) \leq K(z^*) - z^*k \sim -z^*k \sim -k\sqrt{2\log k}/\delta$$

  Nice Lévy tail estimates (Albin-Bengtsson, 2005) confirm

  $$\log \bar{F}(k) \sim -\frac{k}{\delta}\sqrt{2\log k}.$$

  From the tail-wing formula,

  $$\sigma_{BS}^2(k,T)\,T \sim \delta \times \frac{k}{2\sqrt{2\log k}}.$$

Part II: Models with Known Moment Generating Functions

Outline:
- link to Roger Lee's moment formula
- Tauberian theory
- Several Criteria
- Time Changed Lévy models
- numerical examples
- forthcoming in *Journal of Applied Probability*

- What if only a moment generating function M is known? **Roger Lee's moment formula** states that

$$\limsup_{k \to \infty} \frac{\sigma_{BS}^2(k,T)\,T}{k} = \psi\left(-1 + r_T^*\right)$$

with critical exponent

$$r^* \equiv r_T^* \equiv \sup\left\{r \geq 0 : M(r) \equiv \mathbb{E}\exp(rX_T) < \infty\right\},$$

usually seen directly from explicitly known M.

- If $r^* = \infty$ the moment formula only says

$$\sigma_{BS}^2(k,T) = o(k).$$

In contrast, the tail-wing formula contains the full asymptotics (cf examples in Part I)

- Consider $r^* < \infty$. Numerical evidence that in all practical cases " $\limsup = \lim$ ", that is

$$\lim_{k \to \infty} \frac{\sigma_{BS}^2(k,T)\,T}{k} = \psi\left(-1 + r_T^*\right) \qquad (1)$$

Can one prove (1) knowning the mgf M?

- Yes! By the Tail-Wing-Formula suffices to show

  $$\log \bar{F}(k) \sim -r^* k \text{ with } r^* = \sup_{r \geq 0} \{r : M(r) < \infty\} \qquad (2$$

  and we have sufficient criteria for (2) that (seem to) cover all examples. (Remark: $\log \bar{F}(k) \lesssim -r^* k$ is easy.)

- **Criterion I:** M or one of its derivatives (i.e. $M', M'', \ldots$) blows up in a regularly varying way at r^*.
  Criterion II: $\log M$ blows up in a regularly varying way at r^*.
  Idea of proof: Esscher-type change of measure followed by an application of Karamata's resp. Kohlbecker's Tauberian Theorem.
  ($\rightsquigarrow$ fine monograph by Bingham, Goldie, Teugels.)

More Lévy Examples:

- **Variance Gamma:** $VG\,(m, g, CT)|_{T=1}$ has mgf

$$M(s) = \left(\frac{gm}{(m-s)(s+g)}\right)^C$$

See that $r^* = m$ and the M satisfies criterion I:

$$M(r^* - s) \sim \left(\frac{gm}{m+g}\right)^C s^{-C} \text{ as } s \to 0+\,.$$

- **NIG Model (again!):** $NIG\,(\alpha, \beta, \mu T, \delta T)|_{T=1}$ has mgf

$$M\,(s) = \exp\left\{\delta\left\{\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + s)^2}\right\} + \mu s\right\}$$

See that $r^* = \alpha - \beta$ and M' satisfies criterion I:

$$M'(r^* - s) \sim 2\delta\alpha\sqrt{2\alpha}s^{-1/2}M(r^*) \text{ as } s \to 0+\,.$$

- **Kou's Double Exponential model** has mgf

$$\log M(s) = \frac{1}{2}\sigma^2 s^2 + \mu s + \lambda \left(\frac{p\eta_1}{\eta_1 - s} + \frac{q\eta_2}{\eta_2 + s} - 1 \right).$$

  See that $r^* = \eta_1$ and M satisfies criterion II:

$$\log M(\eta_1 - s) \sim \lambda p \eta_1 s^{-1} \text{ as } s \to 0+.$$

- ... and in all Lévy examples $r^* \in (1, \infty)$ does not depend on T,

$$\forall T > 0 : \lim_{k \to \infty} \frac{\sigma_{BS}^2 (k, T)\, T}{k} = \psi(-1 + r^*).$$

Time-changed Lévy Process:

- Lévy process (L_t) $\leftrightsquigarrow$ cumulant generating fct of L_1 :

  $$K_L(v) \equiv \log M_L(v) \equiv \log E[\exp(vL_1)]$$

  Independent random clock $\tau = \tau(\omega, T) \geq 0$ with cgf $K_\tau = \log M_\tau \implies$ mgf of $L \circ \tau$ given by

  $$M(v) = M(v;T) = \exp[K_\tau(K_L(v))] .$$

- **Theorem:** If both M_L and M_τ satisfy one of the criteria, then M does. $r^* = \sup\{r : M(r) < \infty\}$
  Corollary: $\frac{\sigma^2_{BS}(k,T)T}{k} \sim \psi(-1 + r^*)$

- In practise, $K_\tau = K_\tau(\cdot, T), K_L$ known $\implies$ $r^* = r^*_T$ easy to determine and have full analytic understanding of term structure of smile at extreme strikes,

  $$T \mapsto \psi(-1 + r^*_T) .$$

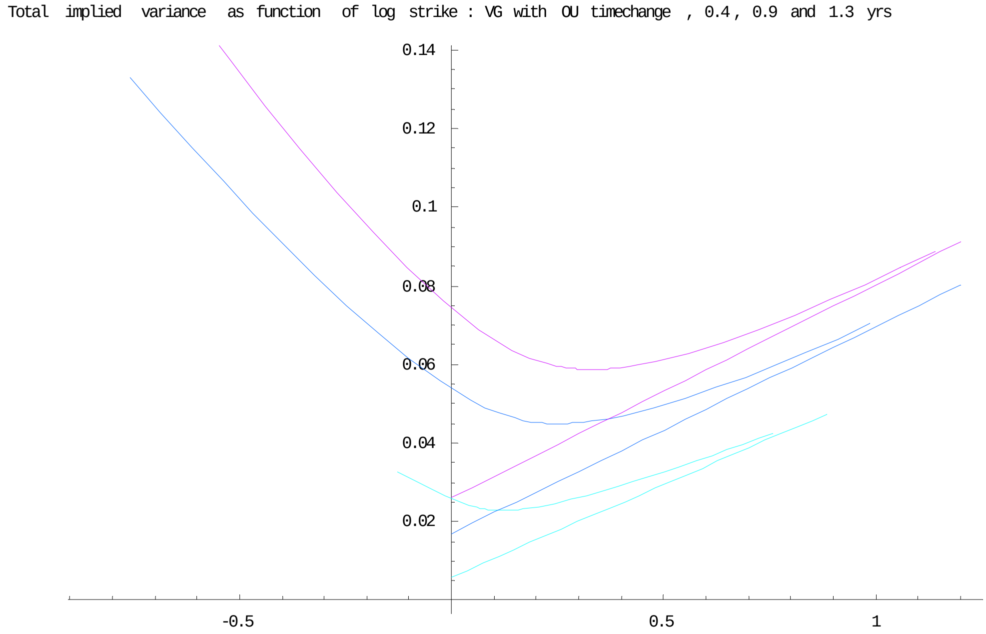

Figure 1:

- Example: **Variance Gamma with OU time change** with Schoutens et al. parameters. Plot $\sigma_{BS}^2(k,T)\,T$ for

$$T = 0.4,\ 0.9,\ 1.3.$$

The respective smile slopes $\psi\left(-1 + r_T^*\right)$ are

$$0.047,\ 0.053,\ 0.054 \in [0,2]\,.$$

Part III: Local and Stochastic Volatility

Outline:
- The CEV model
- Review of some models+their $K \uparrow \infty$ smile
- SABR
- Moment explosion in stochastic vol models
- open problems

- **CEV model:** $dS = \sigma S^{1-\beta} dW$. Density available (mod. Bessel fcts). Or exploit scaling

  $\mathrm{d}\tilde{S} \equiv \mathrm{d}(S/K) = \sigma (S/K)^{1-\beta} \epsilon \, \mathrm{d}W = \sigma \tilde{S}^{1-\beta} \epsilon \, \mathrm{d}W,$

  with $\epsilon = 1/K^{\beta} \to 0$ for $K \to \infty$. Set $S^{\epsilon} := \tilde{S}$. Freidlin-Wentzell:

  $\epsilon^2 \log \mathbb{P}(S_T^{\epsilon} > 1) \sim -\frac{1}{2T} d^2(S_0^{\epsilon}, 1) \sim -\frac{1}{2T} d^2(0,1)$

  where $d(0,1) \sim \int_0^1 \left(\sigma x^{1-\beta}\right)^{-1} \mathrm{d}x = 1/\left(\beta\sigma\right)$. It follows that

  $$\log \mathbb{P}(S_T > K) \sim -\frac{K^{2\beta}}{2\beta^2 \sigma^2 T}.$$

  Tail of "CEV-returns" $\log S_T$ decays exponentially fast $\implies$ regular-variation assumption in tail-wing formula not fullfilled!

- **Thm:** (TWF for exp decay) If $-\log \mathbb{P}(S_T > \cdot)$ is regularly varying then tail-wing formula holds.

- Forde, Labordere (08) propose general heat-kernel estimates to derive large-strike smile asymptotics for more general local models. (Challenge: in general no explicit density or MGF.)

- **Stochastic vol models:** assume $d\left\langle W, Z\right\rangle = \rho dt$

- Avellaneda-Zhu (99) study smile asymptotics of

$$dS = \sigma S dW, \quad d\sigma = -\frac{1}{2}\rho\eta\sigma^2 dt + \eta\sigma dZ.$$

  When $\rho = 0$, $\sigma_{BS}(k)\sqrt{T} \sim \sqrt{2k}$, refined by Gulisashvili-Stein (08) to

$$\sigma_{BS}(k)\sqrt{T} \sim \sqrt{2k} - \frac{\log k + \log\log k}{2\eta\sqrt{T}} + O(1).$$

  (explains term-structure of un-annualized implied vol!)

- Hagan et. al (02) introduce SABR model

$$\begin{aligned} dS &= \sigma S^{1-\beta} dW \\ d\sigma &= \eta\sigma dZ \text{ with } dW dZ = \rho dt. \end{aligned}$$

Accurate asymptotic solution for implied vol in the at-the-money region (*Hagan's formula*) but wings more problematic ... Andersen-Piterbarg (06) show

$$\mathbb{E}[S_T^p] \le \left[S_0^{2\beta} + \beta(p-1)\int_0^T \mathbb{E}\left(\sigma_t^{p/\beta}\right)^{2\beta/p} \mathrm{d}t\right]^{\frac{p}{2\beta}}$$

from which $\log \mathbb{E}[S_T^p]/p^2 \lesssim \eta^2 T/\left(2\beta^2\right)$. For $\rho = 0$, we can check $\gtrsim$ so that

$$\log \mathbb{E}\left[S_T^p\right] = \log \mathbb{E}\left[e^{p \log S_T}\right] \sim \underbrace{\frac{\eta^2 T}{\beta^2}}_{=:C} \frac{p^2}{2} \equiv C\frac{p^2}{2}$$

Kasahara's exponential Tauberian theorem relates log-asymptotics of the mgf to log-asymptotics of the tail. Apply to $\log S_T$:

$$-\log \bar{F}(k) \sim \frac{1}{C}\frac{k^2}{2}$$

and from the tail-wing formula

$$\begin{aligned}\frac{\sigma^2(k,T)\,T}{k} &\sim \psi\left[-1-\log \bar{F}(k)/k\right] \\ &\sim \psi\left[\frac{1}{2C}k\right] \sim C/k \equiv \frac{\eta^2}{\beta^2}\frac{T}{k}.\end{aligned}$$

We then have (as was conjectured by Piterbarg)

$$\sigma(k,T) \sim \frac{\eta}{\beta} \text{ as } k \to \infty.$$

- **Aside:** Hagan et al. show that the pdf of S_t is "approximately Gaussian" with respect to distance

$$\begin{aligned} d(S_0,S) &= \frac{1}{\eta}\log\frac{\sqrt{\zeta^2-2\rho\zeta+1}+\zeta-\rho}{1-\rho}, \\ \zeta &= \frac{\eta}{\sigma_0}\int_{S_0}^{S}\frac{1}{u^{1-\beta}}du \sim \frac{\eta}{\sigma_0}\frac{S^\beta}{\beta} \end{aligned}$$

With $\rho = 0$, as $S \to \infty$, $d(S_0,S) \sim \log\zeta/\eta \sim \beta\log S/\eta$ and

$$-\log\mathbb{P}[S_T \in dS] \approx \frac{1}{2T}d(S_0,S)^2 \sim \frac{\beta^2}{2\eta^2 T}(\log S)^2.$$

Let f denote the pdf of $\log S_T$. Then

$$-\log f(k) \sim \frac{\beta^2}{2\eta^2 T}k^2$$

This is consistent with $\log\bar{F}$-estimate obtained earlier using Kasahara's Tauberian theorem and the tail-wing formula gives the same asymptotic implied vol

$$\sigma\left(k,T\right)\sim\eta/\beta \text{ as } k\rightarrow\infty.$$

- Lions-Musiela [Annales de l'IHP 2008] consider

$$\begin{aligned} dS_t &= \sigma_t^\delta S dW, \ \ S_0 > 0 \\ d\sigma_t &= \eta\sigma_t^\gamma dZ_t + b(\sigma_t)\, dt, \ \sigma_0 = \xi > 0 \end{aligned} \tag{1}$$

and give essentially sharp conditions for $\mathbb{E}S_t^m < \infty$. We can use their ideas in our context: set $L^{(m)}z :=$

$$\frac{\eta^2}{2}\xi^{2\gamma}\partial_{\xi\xi}z + \left(\eta\rho m\xi^{\delta+\gamma} + b(\xi)\right)\partial_\xi z + \frac{m^2-m}{2}\xi^{2\delta}z.$$

- **Theorem:** (Applies to $\gamma+\delta=1, \rho \in (-1,1]$.) Let $\bar{z} = \bar{z}(t,\xi;m)$ be a super-solution and $\underline{z}(t,\xi;m)$ be a subsolution to

$$\partial_t z - L^{(m)}z = 0 \text{ with } z|_{t=0} \equiv 1 \text{ on } [0,\infty).$$

Then

$$\underline{z}(T,\sigma_0;m) \le \mathbb{E}[S_T^m]/S_0^m \le \bar{z}(T,\sigma_0;m).$$

If both $\underline{z}$ and $\bar{z}$ blow up at $T = T_0(m^*) < \infty$ then $\sigma_{BS}^2(k,T)T/k \lesssim \psi(m^*(T)-1)$ and $\sim$ for "regular" blowup of

$$\log \underline{z}(T,\sigma_0;m^*-\varepsilon), \log \bar{z}(T,\sigma_0;m^*-\varepsilon).$$

- **Theorem: (**Applies to $\gamma + \delta < 1$.) Assume there is no moment explosion and

$$\log \underline{z}(T, \sigma_0; m) \sim \log \bar{z}(T, \sigma_0; m) \in R_\alpha.$$

Then explicit tail-asymptotics (and hence smile asymptotics) can be obtained by Kasahara's Tauberian theorem. (This is work in progress ...)

- Open problems and future work ...

Part V: References

Benaim, S.; Friz, P.K.: Smile Asymptotics I: Regular Variation, Math. Finance (forthcoming)

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