Soft Derivatives

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What is a soft derivative?

- It is an asset whose payoff is a function of some other variable, but that variable is not marketed.
- Examples: options on profit, weather, or many other things.

Outline

- Properties and variations of CAPM
- Properties and variations of zero-level pricing
- Axiomatic pricing in continuous time
- The abstract Black-Scholes equation
- The extended Black-Scholes equation
- Hedging

Linear Pricing

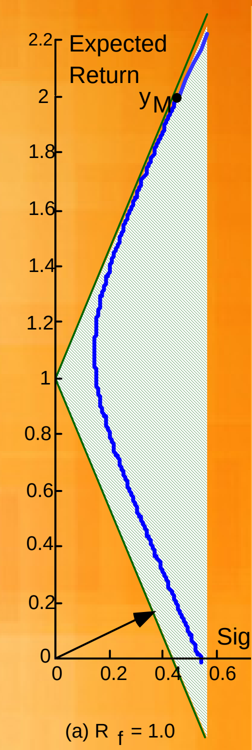
- Consider a set of n assets with payoffs (at the end of a year) $y_1, y_2, y_3, \ldots, y_n$ and prices $p_1, p_2, p_3, \ldots, p_n$.
- Linear pricing implies that a combination asset will have the combination price. That is, $\Sigma w_i y_i$ will have price $\Sigma w_i p_i$.

CAPM

$$p = \frac{1}{R}\left[E(y) - \frac{\text{cov}(y, y_M)(\bar{y}_M - p_M R)}{\sigma_M^2} \right]$$

where y_M is an efficient risky asset.

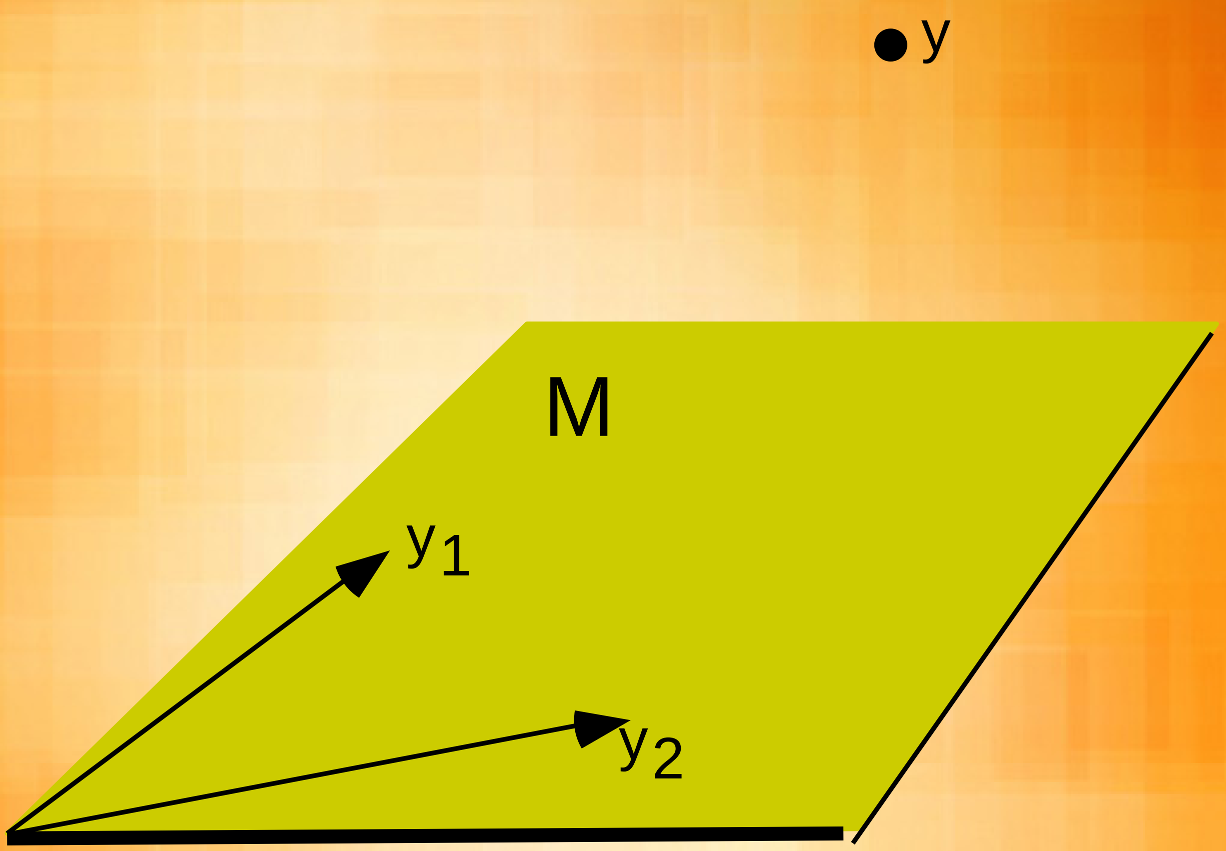
$$p = \frac{1}{R}\left[E(y) - \beta_{y,M}(\bar{y}_M - p_M R) \right]$$



CAPM

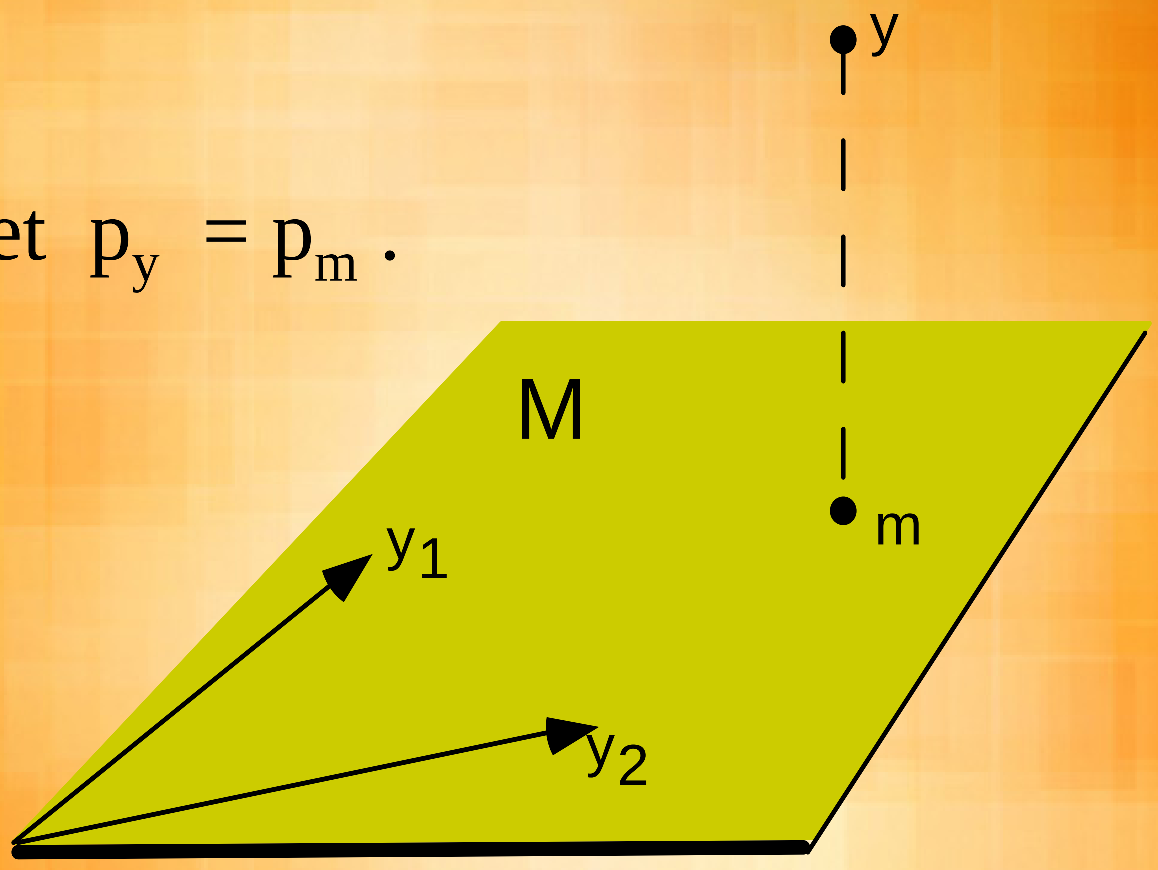
- It is always true (if it exists) for the given set of assets.

Asset Outside Span



Price by Projection

Set $p_y = p_m$.



Equivalence

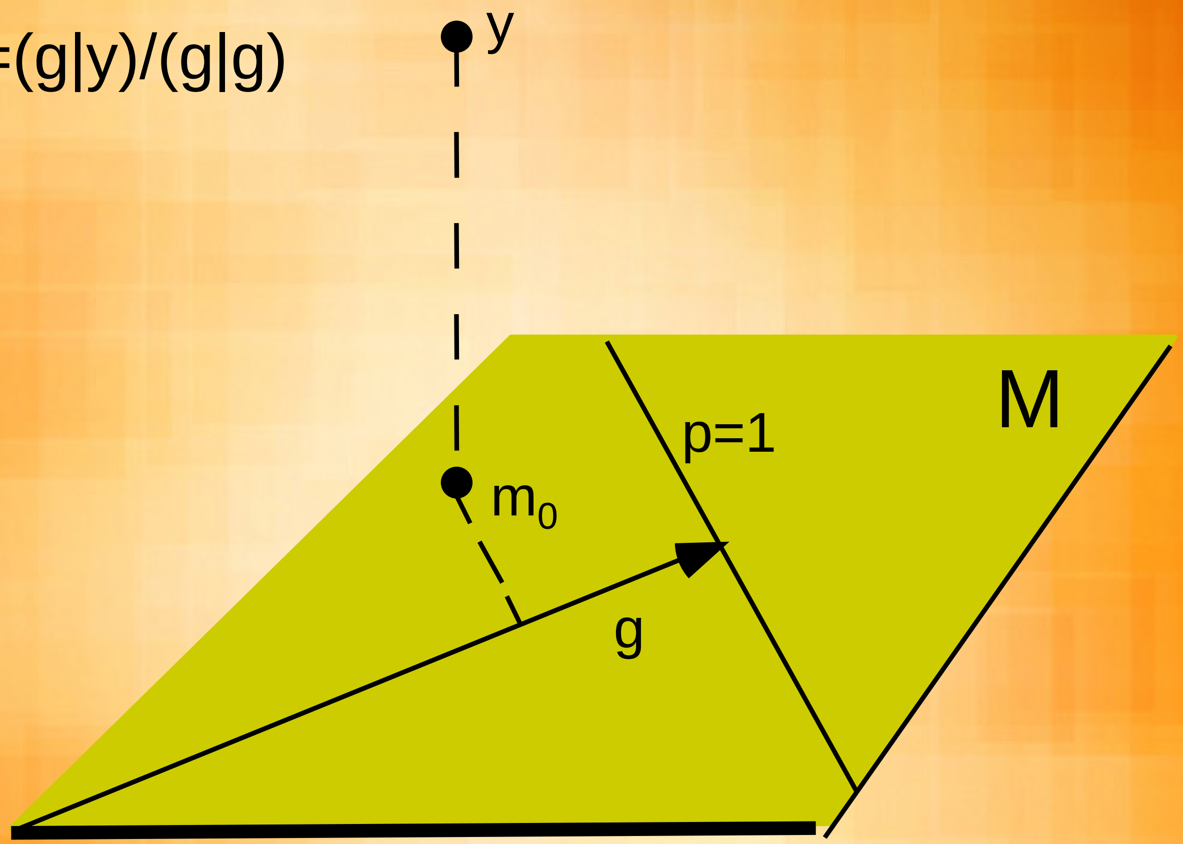
- The price assigned by the CAPM (when it exists) is equal to the projection price (which always exists).

Hilbert Space for Pricing

- H is the set of all linear combinations of asset payoffs.
- The inner product is
  $(y_1 | y_2) = E(y_1\, y_2) = E(y_1)E(y_2) + \text{cov}\,(y_1, y_2)$
- Since H is finite dimensional, all subspaces are closed.

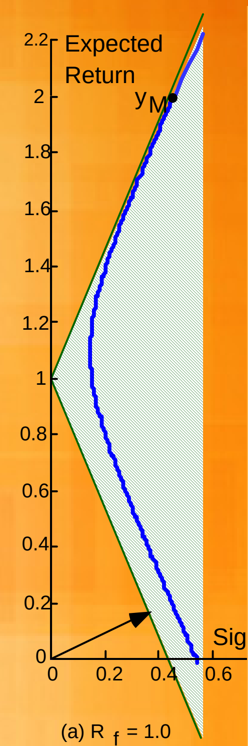
Minimum Norm Pricing Vector

p=(g|y)/(g|g)



Minimum-Norm Pricing

$$p = \frac{1}{R}\left[E(y) - \frac{\operatorname{cov}(y, y_M)(\bar{y}_M - p_M R)}{\sigma_M^2} \right]$$

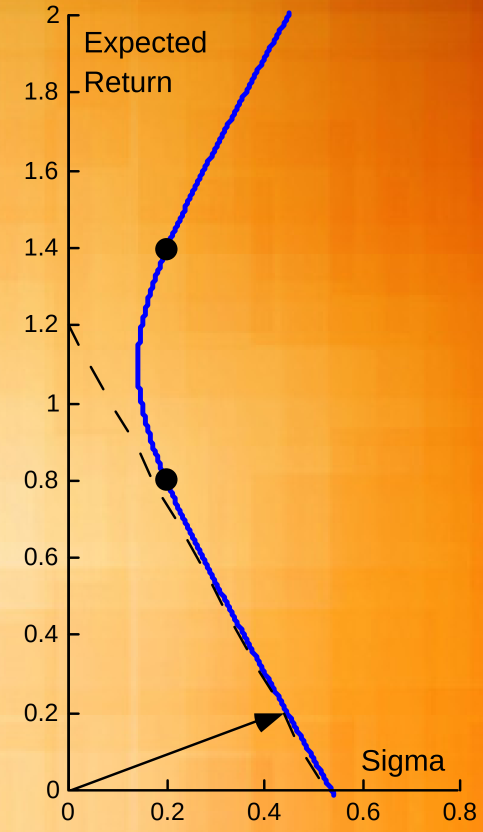


where now y_M is the traded asset with price 1 and minimum norm.

$$p = \frac{1}{R}\left[E(y) - \beta_{y,M}(\bar{y}_M - p_M R) \right]$$

Application to Pricing

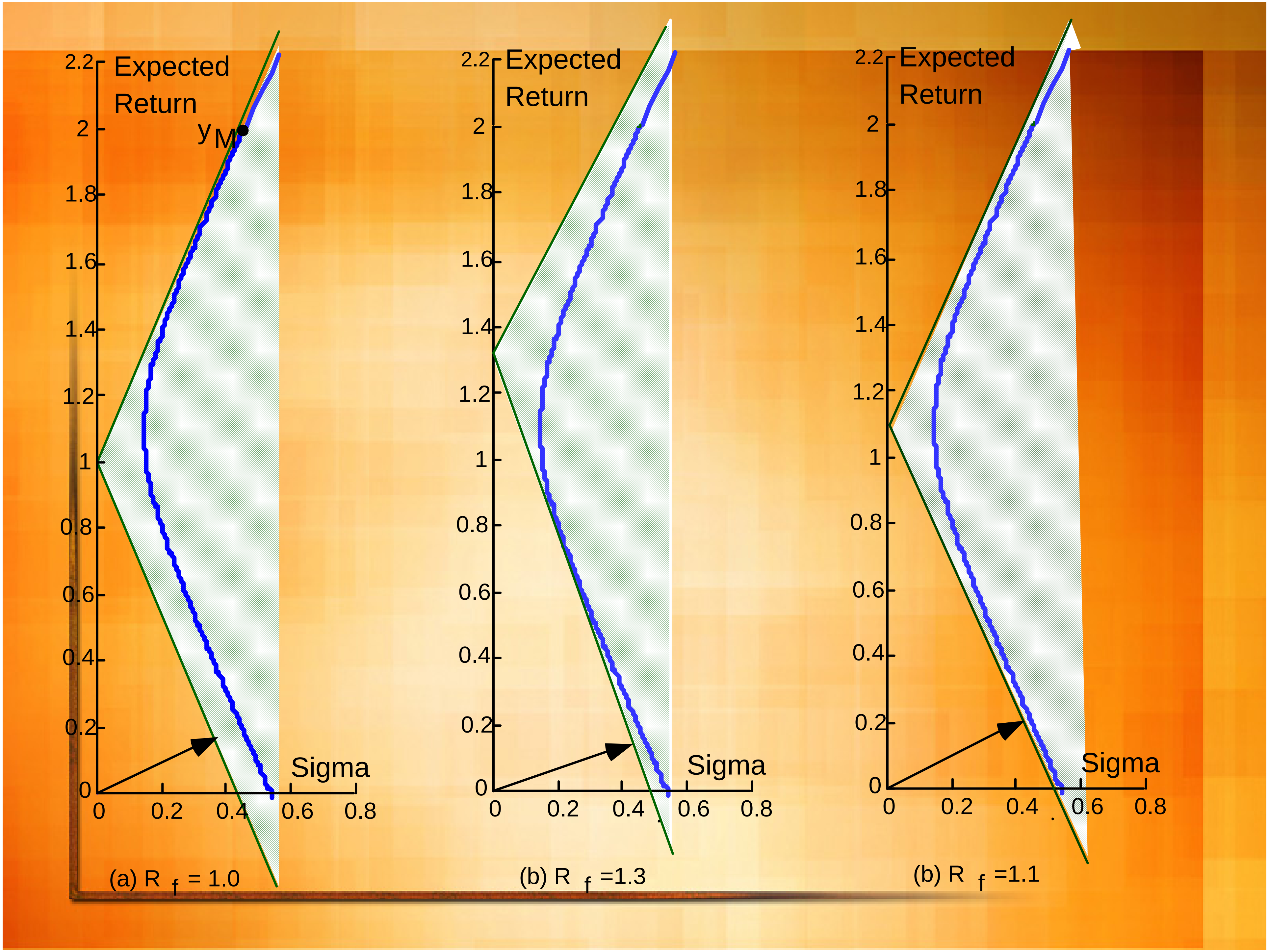
- $y_1 = R_1 = 1.4, \quad y_2 = R_2 = .8$
- $\sigma_1 = \sigma_2 = .20$
- Uncorrelated.



- Implied risk-free return = 1.2
- $p = E\{y(-y_1 + 2\, y_2\}/.24$
- $y_M = -y_1 + 2\, y_2$
- $p = (R_0)^{-1}\, [E(y) - cov(y, y_M)(E(y_M) - p_M\, R_0)/\sigma_M^2]$

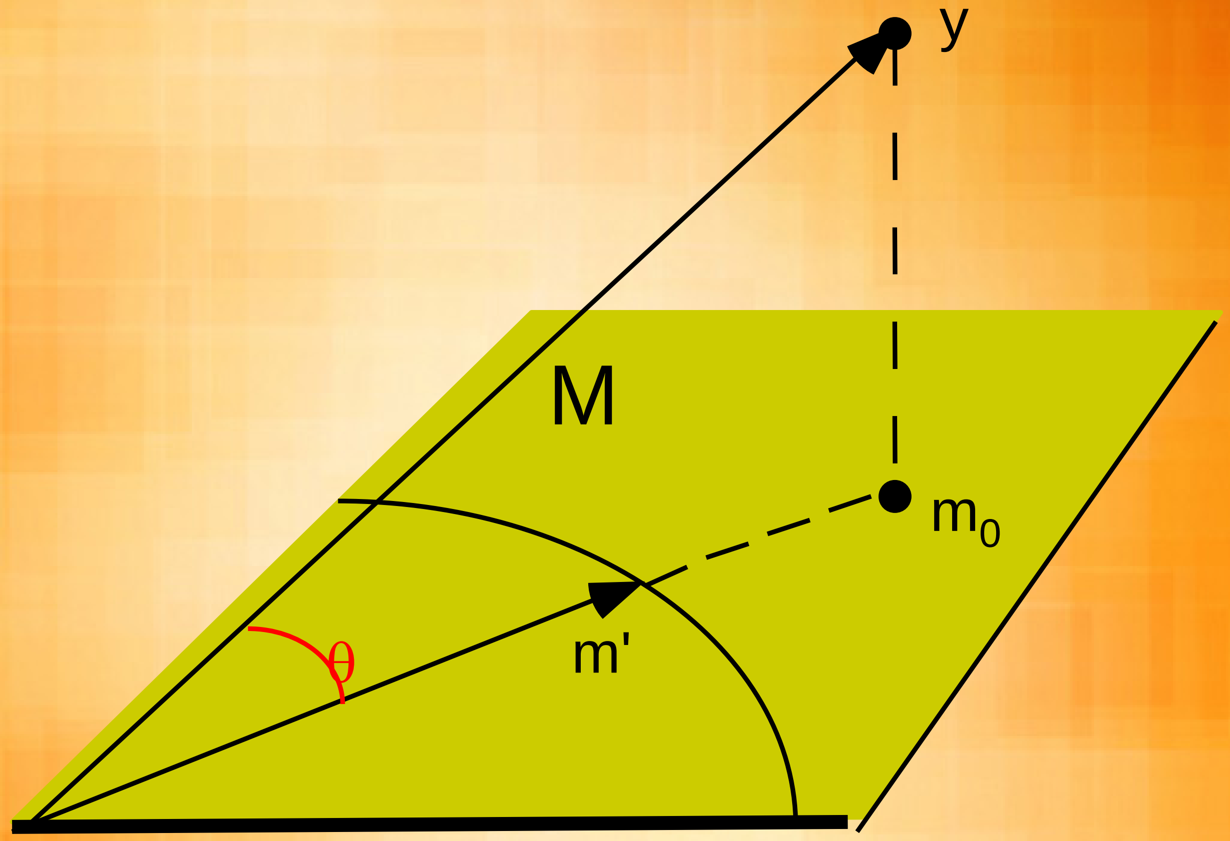
Add Risk-free Asset

- Same y_1 and y_2 as before.
- Risk-free return is R_f
- Critical value is $R_f = 1.1$
- In some cases there is no efficient risky portfolio and hence no CAPM formula. But there is always a minimum norm pricing formula.



Expected Return
y_M
Sigma
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
2.2
(a) R f = 1.0
(b) R f =1.3
(b) R f =1.1

The Dual Theorem

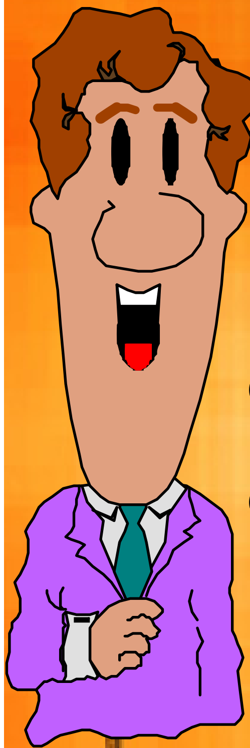


Correlation Pricing Formula

$$p = \frac{1}{R}\left[E(y) - \frac{\operatorname{cov}(y, y_M)(\bar{y}_M - p_M R)}{\sigma_M^2} \right]$$

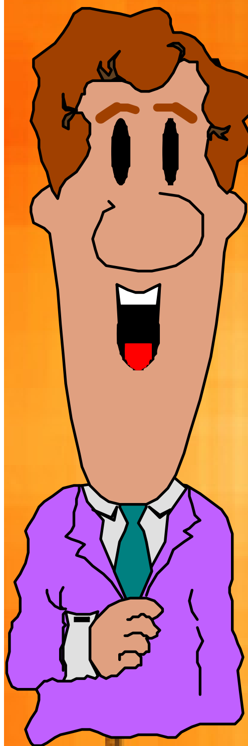
where now y_M is an asset most correlated with y.
(The magnitude and risk-free component are arbitrary.)

Try it for y in M. Then $y_M = y$ is most correlated.
We find $p = R^{-1}[\ E(y) - E(y) + p_y\ R] = p_y$.



[illegible]dvantages

- Solidly based on projection theorem.
- The correlation prici[illegible]g formula for an asset *y* uses the pri[illegible]s of assets that are similar to *y*.
- It is a rigorou[illegible] of “pricing by comparables”[illegible]



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- Solidly based on projection theorem.
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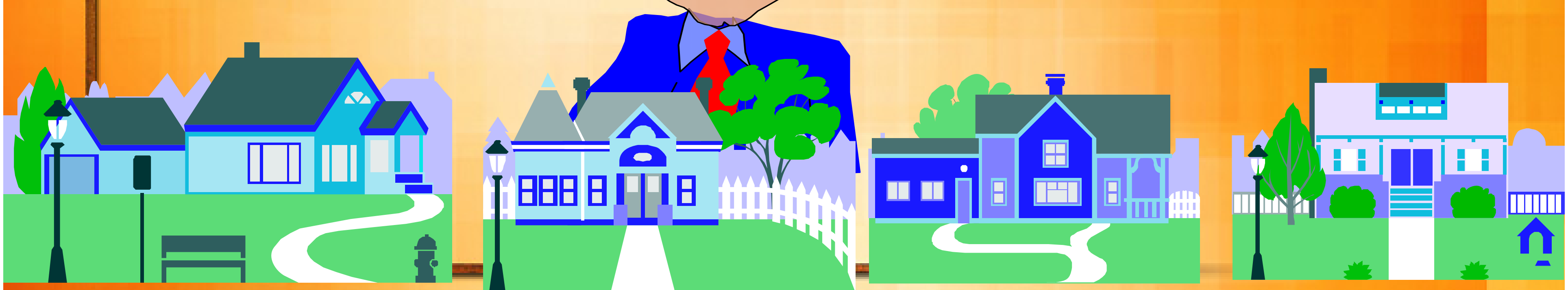
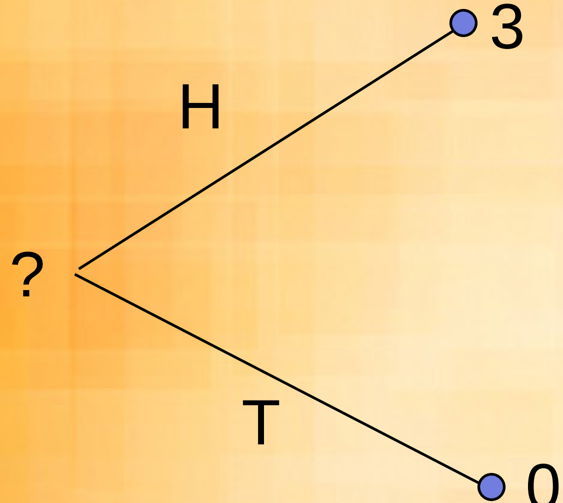


Second Topic

- Use general portfolio theory to assign prices
- Zero-level prices
- Does result depend on utility function?
- Applications

Coin Flip



Add an asset

- Suppose asset with payoff x is not in the market but is available to you. What is the logical price?
- Theorem. Assume there is no arbitrage in the original system. Then there is an open interval of the real line such that for p_x in this interval no arbitrage is possible.
- For coin flip, what is the interval?

Zero-Level

- Theorem. There is a price p_x such that x is taken in the portfolio at zero level. Thus, the optimal portfolio does not change.
- This is called the zero-level price.
- For coin flip, what is the zero-level price?

Portfolio Problem

- Maximize $E[U(y_0)]$
- Sub to $y_0 = a_1 y_1 + a_2 y_2 + \ldots + a_n y_n + a_{n+1} R_f$
- $W = a_1 p_1 + a_2 p_2 + \ldots + a_n p_n + a_{n+1}$

Necessary conditions

- $E[U'(y_0)\, y_i] = \lambda\, p_i$ for each i and some $\lambda > 0$.
- Can find $\lambda = RE[U'(y_0)]$
- $p = E[U'(y_0)\, y] / RE[U'(y_0)]$

Independent x

- If x is in dependent of all y_i's then we can find a unique zero-level price.
- Since $p=E[U'(y_0)x]/R_f E[U'(y_0)]$, it follows that $p_x = E(x)/R_f$. Price is universal zero-level price.
- There are other cases where the zero-level price is universal.

Continuous-time Framework

- Market prices follow stochastic processes, such as $dx_i(t) = \mu_i x_i(t)\,dt + \sigma_i x_i(t)\,dz_i$.
- There is a risk free asset with rate of return r.
- Frictionless trading is possible at every instant.
- Everyone is a price taker.

General Idea of Operational Calculus

- We only need to know how at time t to price payoffs at time $t + \mathrm{d}t$.
- We use only knowledge of how to value risk free payoffs and marketed payoffs.
- Only first-order terms in $\mathrm{d}t$ are relevant.

An Operational Calculus

FOUR AXIOMS:

- Pricing a constant: If C is constant, then $P\{C\}=C\cdot(1- r\,dt)$
- Pricing a marketed quantity: If x is an evolving market variable that neither pays dividends nor requires holding costs, $P\{x + dx\} = x$.
- $P\{dt\} = dt$
- P is linear.

Main Application

- Since x is a constant

  $x = P\{x + dx\} = (1- r\,dt)\,x + P\{dx\}$

  Hence $rx\,dt = P\{dx\}$

Fundamental Pricing Equation

- A value function on the span of marketed assets must satisfy

$$rV(x,t)\, dt = \mathrm{P}\{dV(x, t)\}$$

This is the general Black--Scholes Equation

Standard Black--Scholes Equation

$$\mathrm{d}V = V_t \mathrm{d}t + V_x \mathrm{d}x + \frac{1}{2} V_{xx} (\mathrm{d}x)^2$$

$$= V_t \mathrm{d}t + V_x \mathrm{d}x + \frac{1}{2} V_{xx} \sigma^2 x^2 \mathrm{d}t$$

$$\mathrm{P}\{\mathrm{d}V\} = V_t \mathrm{d}t + V_x r x \mathrm{d}t + \frac{1}{2} V_{xx} \sigma^2 \mathrm{x}^2 \mathrm{d}t$$

$$rV(x,t)\ \mathrm{d}t = \mathrm{P}\{\mathrm{d}V(x,\ t)\}$$

$$rV = V_t + V_x r x + \frac{1}{2} V_{xx} \sigma^2 x^2$$

Extend outside the marketed space

- Apply similar idea to payoffs outside the marketed space.
- Use instantaneous projection (as CAPM uses projection).

The Extended framework

$$dx_e = \mu_e x_e dt + \sigma_e x_e dz_e$$ Underlying process

$$dx_i = \mu_i x_i dt + \sigma_i x_i dz_i \quad i = 1, 2, \ldots, n$$ Stock processes

$$F(x_e(T))$$ Payoff function

Instantaneous Projection

- Definition of price

$$p_y = \mathrm{P}\{y \mid M\}$$

Price of y when projected onto M

- Pricing equation

$$V(x_e t) = \mathrm{P}\{V(x_e, t) + \mathrm{d}V(x_e, t) \mid M\}$$

$$rV(x_e, t)\mathrm{d}t = \mathrm{P}\{\mathrm{d}V(x_e, t) \mid M\}$$

General extended B-S equation

- For standard case

$$\mathrm{d}V = [V_t + V_{x_e}\mu_e x_e + \tfrac{1}{2}V_{x_e x_e}\sigma_e^2 x_e^2]\mathrm{d}t + V_{x_e}\sigma_e x_e \mathrm{d}z_e$$

Projection of dz_e

$$\{dz_e \mid M\} = a\,dt + b\,dz_m$$

$$E\left[(dz_e - a\,dt - b\,dz_m)dt\right] = 0$$

$$E\left[(dz_e - a\,dt - b\,dz_m)dz_m\right] = 0$$

$$a = 0, \quad b = \rho_{em}$$

$$\{dz_e \mid M\} = \rho_{em}\,dz_m$$

Price of dz_e

- From operational calculus:

$$rx_m dt = P\{dx_m\} = P\{\mu_m x_m dt + \sigma_m x_m dz_m\}$$

$$= \mu_m x_m dt + \sigma_m x_m P\{dz_m\}$$

- Hence

$$P\{dz_m\} = \frac{(r-\mu_m)}{\sigma_m} dt$$

- Finally

$$P\{dz_e\} = \frac{(r-\mu_m)}{\sigma_m} \rho_{em} dt$$

Beta Form

We have

$$P\{dz_e \mid M\} = \frac{\rho_{em}}{\sigma_m}(r - \mu_m)dt$$

Define $\beta_{em} = \sigma_{em} / \sigma_m^2$

Then

$$P\{dz_e \mid M\} = \frac{\beta_{em}}{\sigma_e}(r - \mu_m)dt$$

The Equation (Standard Version)

$$dx_e = \mu_e x_e dt + \sigma_e x_e dz_e$$ Underlying process

$$dx_i = \mu_i x_i dt + \sigma_i x_i dz_i \quad i = 1, 2, \ldots, n$$ Stock processes

$$F(x_e(T))$$ Payoff function

$$rV(x_e, t) = V_t(x_e, t) + V_{x_e}(x_e, t) x_e [\mu_e - \beta_{em}(\mu_m - r)] + \frac{1}{2} V_{x_e x_e}(x_e, t) x_e^2 \sigma_e^2$$ Main equation

$$V(x_e, T) = F(x_e)$$ Boundary condition

Alternate x_m's

- Most-correlated marketed asset.
  - Valid for x_e and all its derivatives
- Dual Asset
  - Valid for all assets.
  - Always exists
- Markowitz portfolio
  - Valid for all assets
  - May not exist

Universal Propery

$$\max_{\alpha} \mathrm{E}\{U(W(T)\}$$

subject to $\mathrm{d}W(t) = \alpha_x' \mathrm{d}x + \alpha_e \mathrm{d}V(x_e, t)$

$$W(0) = W_0$$

$$\alpha_x' x + \alpha_e V(x_e, t) = W(t)$$

Solution:

$$\alpha_e = 0$$

Universal Property

$$J(x_e, W, t) = U(W(T))$$

$$J(x_e, W, t) = \max_\alpha \mathrm{E}\{J(x_e + \mathrm{d}x_e, W + \mathrm{d}W, t + \mathrm{d}t)\}$$

$$\text{subject to} \quad \mathrm{d}W(t) = \alpha_x' \mathrm{d}x + \alpha_e \mathrm{d}V(x_e, t)$$

$$W(0) = W_0$$

$$\alpha_x' x + \alpha_e V(x_e, t) = W(t)$$

Solution:

$$\alpha_e = 0$$

Optimal Replication

- Can replicate at each instant so as get best fit.
- Begin with $V(x_e, 0)$ dollars and at each instant allocate fraction γ to the most-correlated asset and $1-\gamma$ in the risk free asset, where

$$\gamma = \frac{V_{x_e}(x_e, t)\, x_e \beta_{em}}{V(x_e, t)}$$

Extended Equation Properties

- Prices are linear.
- Instantaneous projection consistent.
- No arbitrage opportunities generated.
- Gives universal zero-level price.
- Agrees with B--S if pricing a derivative.
- Gives best replication (or best hedge).
- Can be based on market or most-correlated portfolio.
- Easy to implement (similar to B--S).
- Gives formula for total error after hedge.

Extensions

- Discrete time (lattice) method
- Risk-neutral form
- More complex dynamics
- Expectations of any variable
- Hedging

Risk-Neutral Process

$$dx_e = \mu_e x_e dt + \sigma_e x_e dz_e$$ Underlying process

$$dx_i = \mu_i x_i dt + \sigma_i x_i dz_i \quad i = 1, 2, ..., n$$ Stock processes

$$dx_e = \omega_{em} x_e dt + \sigma_e x_e dz_e$$ Risk-Neutral Process

$$\omega_{em} = \mu_e - \beta_{em}(\mu_m - r)$$

Example

- Consider an option on the variable $x_e(T)$ with strike price K.
- x_e may be estimated stock value of untraded company.
- x_e may be revenue or profit
- Can get closed-form expression

Option formula

$$rV(x_e,t) = V_t + V_{x_e}\omega x_e + \frac{1}{2}V_{x_e x_e}\sigma_e^2 x_e^2$$

$$\omega = \mu_e - \beta_{em}(\mu_m - r)$$

$$V(x_e,t) = x_e e^{(\omega - r)(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2)$$

$$d_1 = \frac{\ln(x_e/K) + (\omega + \frac{1}{2}\sigma_e^2)(T-t)}{\sigma_e\sqrt{T-t}}$$

$$d_2 = \frac{\ln(x_e/K) + (\omega - \frac{1}{2}\sigma_e^2)(T-t)}{\sigma_e\sqrt{T-t}}$$

Specific Example

- Standard Option
  - $K = 60$, $S = 62$, $T = 5$ mo, $r = 10\%$, $\sigma = 20\%$ Value = 5.85
- Non traded version
  - $K = 60$, $S = 62$, $T = 5$ mo, $r = 10\%$, $\sigma_e = 20\%$, $\mu_e = 8\%$, $\sigma_m = 15\%$, $\mu_m = 14\%$, $\rho = .7$
  - Value = 4.79

Projection Error
(Risk that cannot be hedged)

- Let $S(x_e, t)$ be the variance at T as seen at t. We may find S from

$$0 = E(dS) + \delta^2$$

- Or, in detail,

$$S_t + S_{x_e}\mu_e x_e + \frac{1}{2}S_{x_e x_e}\sigma_e^2 x_e^2 + [V_{x_e}\sigma_e x_e]^2(1-\rho_{em}^2) = 0$$

$$S(x_e, T) = 0$$

Summary Conclusion

- B-S pricing renders a derivative redundant, in the sense it can be constructed from existing securities.
- Extended pricing renders a soft derivative irrelevant, in the sense that no risk-averse investor will want it in a portfolio.

References
(available on Stanford website)

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