
Static and Dynamic Credit Correlation Models: Jump-diffusion of the Market Factor and its Implications

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A Brief Market Overview

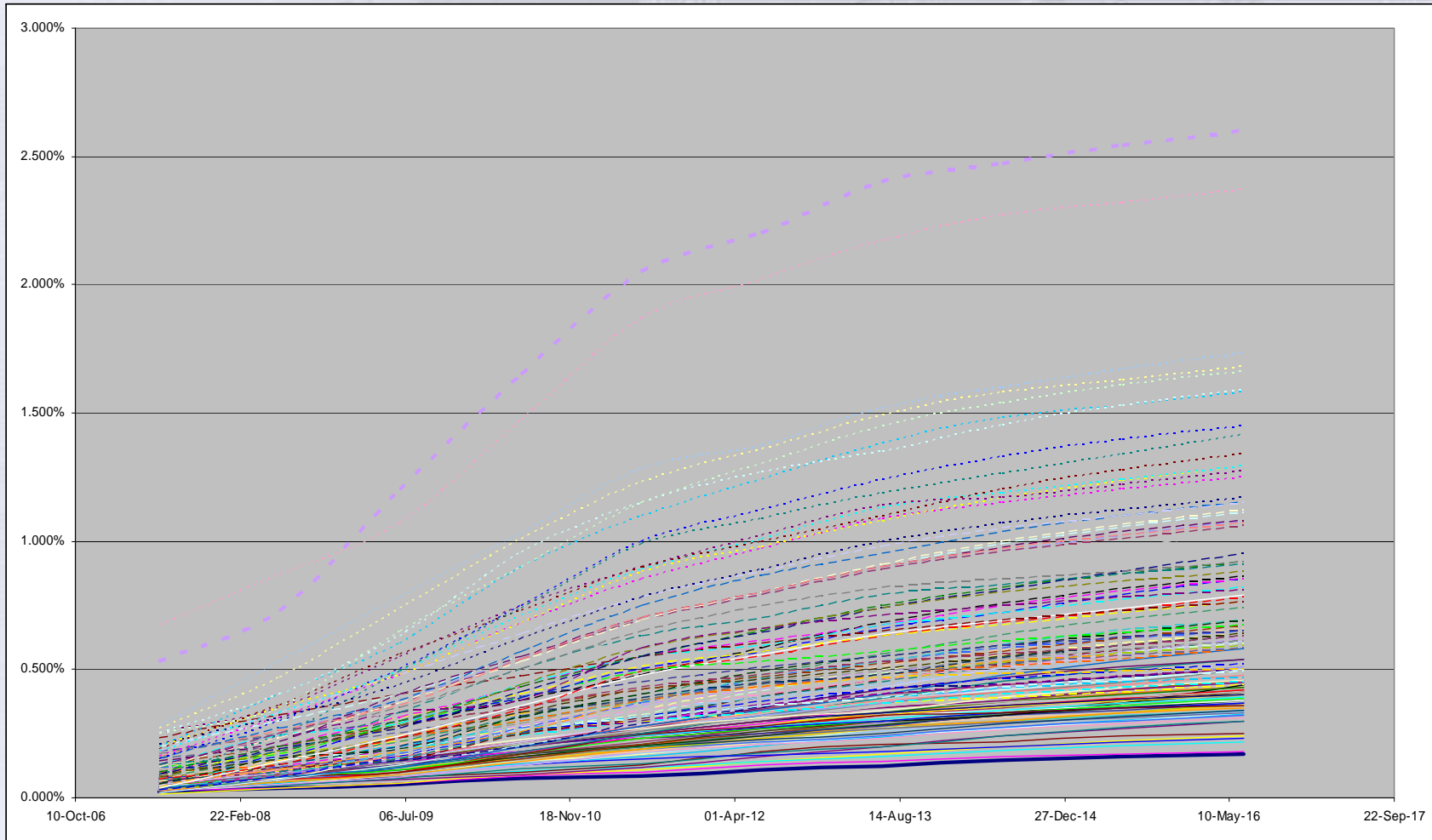
- According to a recent BBA survey, by the end of 2006 the size of the market will be \$30 trillion
- Rapid growth in CDSs, Indices, synthetic CDOs
- Key market participants: banks (trading) (35%), hedge funds (32%), banks (loans) (9%), mono-line insurers (8%), others (16%)
- Main products: CDSs (33%), Full Index Trades (30%), synthetic CDOs (over 10 names, tranches) (12.5%), Index Tranches (7.6%), synthetic CDOs (over 10 names, full capital structure) (3.7%), CLNs (3.2%), others (10%)



What kind of a model do we need?

- First, we need to explain how to price CDSs.
- Then we need to extend our theory to cover indices, tranches, baskets, etc.
- Two complementary approaches to pricing CDSs: structural and reduced form. Both have pros and contras. It was realized early on that without jumps (or/and curvilinear or uncertain barriers) it is impossible to explain short end of the CDS curve within the structural framework.
- A typical snapshot of 125 CDS curves for CDX6 on June 20th 2006 is shown in Figure 1.

What kind of a model do we need?



A typical structural model

- A typical structural model for the evolution of the log-value of the firm has the form

$$dx = vdt + \sigma(t)dW(t) + j^{(+,p)}dN^{(+)} + j^{(-,q)}dN^{(-)}$$

$$x = \ln(V / V_0)$$

$$v = r - \sigma^2 / 2 - \lambda / p + \mu / q$$

- This value is governed by a combination of a Wiener process and a Poisson process with exponentially distributed jumps. The firm defaults if the value $x(t)$ crosses a (generally time-dependent) barrier $b(t)$. When all the relevant parameters are constant, the problem can be solved analytically via the Laplace transform. However, in general this approach does not work.



A typical structural model

- Hence we need to use numerical methods. A judicious combination of finite differences for partial and ordinary differential equations allows one to build a fast and accurate numerical scheme. (This is a special feature of exponentially distributed jump sizes and does not work for more familiar Gaussian jumps.)
- We can solve the barrier problem by using the forward Fokker-Planck equation t.p.d.f. and putting probabilities below the barrier to zero. This equation has the form

$$f_t - \frac{1}{2}\sigma^2 f_{xx} + \nu f_x - \lambda p \int_0^\infty f(x-j)e^{-pj} dj - \mu q \int_{-\infty}^0 f(x-j)e^{qj} dj + (\lambda + \mu)f = 0$$

$$f(0, x) = \delta(x)$$

$$f(t, x) = 0 \text{ if } x < b(t)$$

A typical structural model

- When barrier is absent, f can be found via the following recursion (knowing analytical solution is useful for benchmarking purposes)

$$\sigma_p = p\sigma\sqrt{T}, \quad \sigma_q = q\sigma\sqrt{T}$$
$$\alpha = -\frac{p(x-vT)}{\sigma_p\sqrt{T}} + \sigma_p\sqrt{T}, \quad \beta = \frac{q(x-vT)}{\sigma_q\sqrt{T}} + \sigma_q\sqrt{T}$$

$$f_{0,0} = \frac{1}{\sigma} n\left(\frac{x-vT}{\sigma\sqrt{T}}\right)$$

$$f_{1,0} = p \exp(\sigma_p \alpha - \sigma_p^2 / 2) N(-\alpha), \quad f_{0,1} = q \exp(\sigma_q \beta - \sigma_q^2 / 2) N(-\beta)$$

$$f_{k,0} = \frac{\sigma_p (-\alpha f_{k-1,0} + \sigma_p f_{k-2,0})}{k-1}, \quad f_{0,l} = \frac{\sigma_q (-\beta f_{0,l-1} + \sigma_q f_{0,l-2})}{l-1}$$

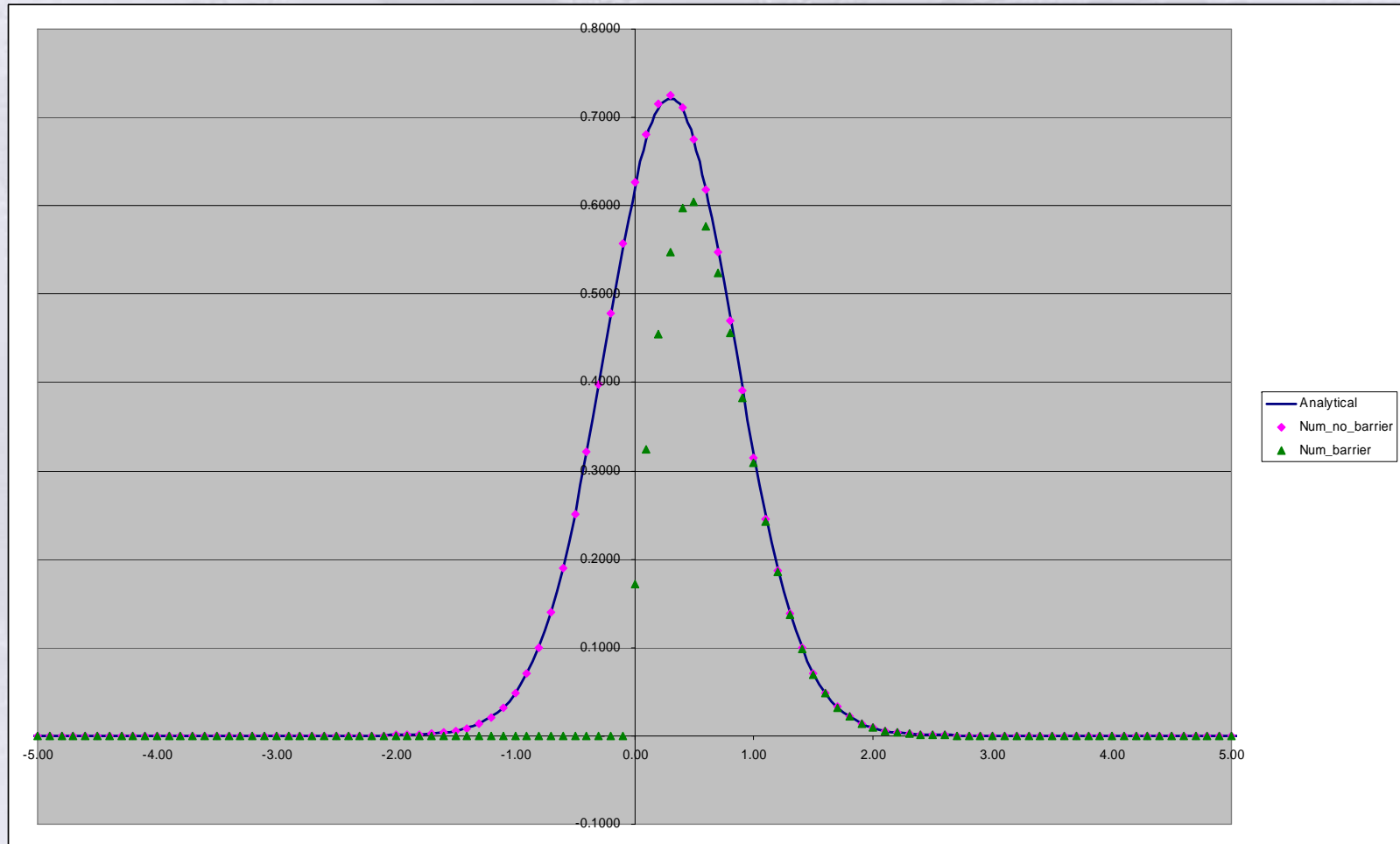
A typical structural model

$$f_{k,l} = \frac{pf_{k-1,l} + qf_{k,l-1}}{p + q}, \quad w_{k,l} = \frac{e^{-(\lambda+\mu)T} (\lambda T)^k (\mu T)^l}{k!l!}$$

$$f(x) = \sum_{k=0, l=0}^{\infty} w_{k,l} f_{k,l}(x)$$

- Vis-à-vis Gaussian distribution, f has fat tails and narrow peak.
- Comparison of numerical and analytical t.p.d.f. as well as the solution of the barrier problem with non-constant barrier is given in Figure 2.
- By bootstrapping the barrier, we can reproduce term structure of CDS for most names. In addition, we can price equity derivatives and produce a respectable volatility skew.

A typical structural model



What to do for baskets?

- We just saw that for individual names structural approach works quite well.
- People tried to extend it to baskets of different names for some time but these attempts were not entirely successful.
- In general, dimensionality of the problem is too high, and cannot be reduced in a computationally efficient way in a pure dynamic framework.
- Hence we need to use some sort of a reduced form model. In order to reduce dimensionality and generate necessary correlation among individual names, it has to be developed in a factor framework. One more traditional ingredient in the mix is analytical solvability, usually achieved via affinity.
- We propose to drop affinity altogether, and modify other ingredients as appropriate.

Motivation for our choice of model

- In order to motivate our subsequent choice, we consider short-rate models in the interest rate context.
- Classical short-rate models (Vasicek, CIR, Black-Karasinski) have the form

$$dr = \kappa(\mathcal{G}(t) - r)dt + \sigma(t)dW(t)$$

$$dr = \kappa(\mathcal{G}(t) - r)dt + \sigma(t)\sqrt{r}dW(t)$$

$$d \ln r = \kappa \ln\left(\frac{\mathcal{G}(t)}{r}\right)dt + \sigma(t)dW(t)$$

- These models deal with the evolution of the short rate directly and, apart from the last one, are analytically solvable.

Motivation for our choice of model

- We prefer to introduce a standard OU factor x , and represent r as an appropriate function of x (success or otherwise of a short-rate interest rate model crucially depends on an appropriate choice of f)

$$dx = -\kappa x dt + \sigma(t) dW(t)$$

$$r = f(x + \mathcal{G}(t))$$

- To calibrate our model, we need to price bonds and swaptions (say). The corresponding pricing equation for bonds can be written in the form

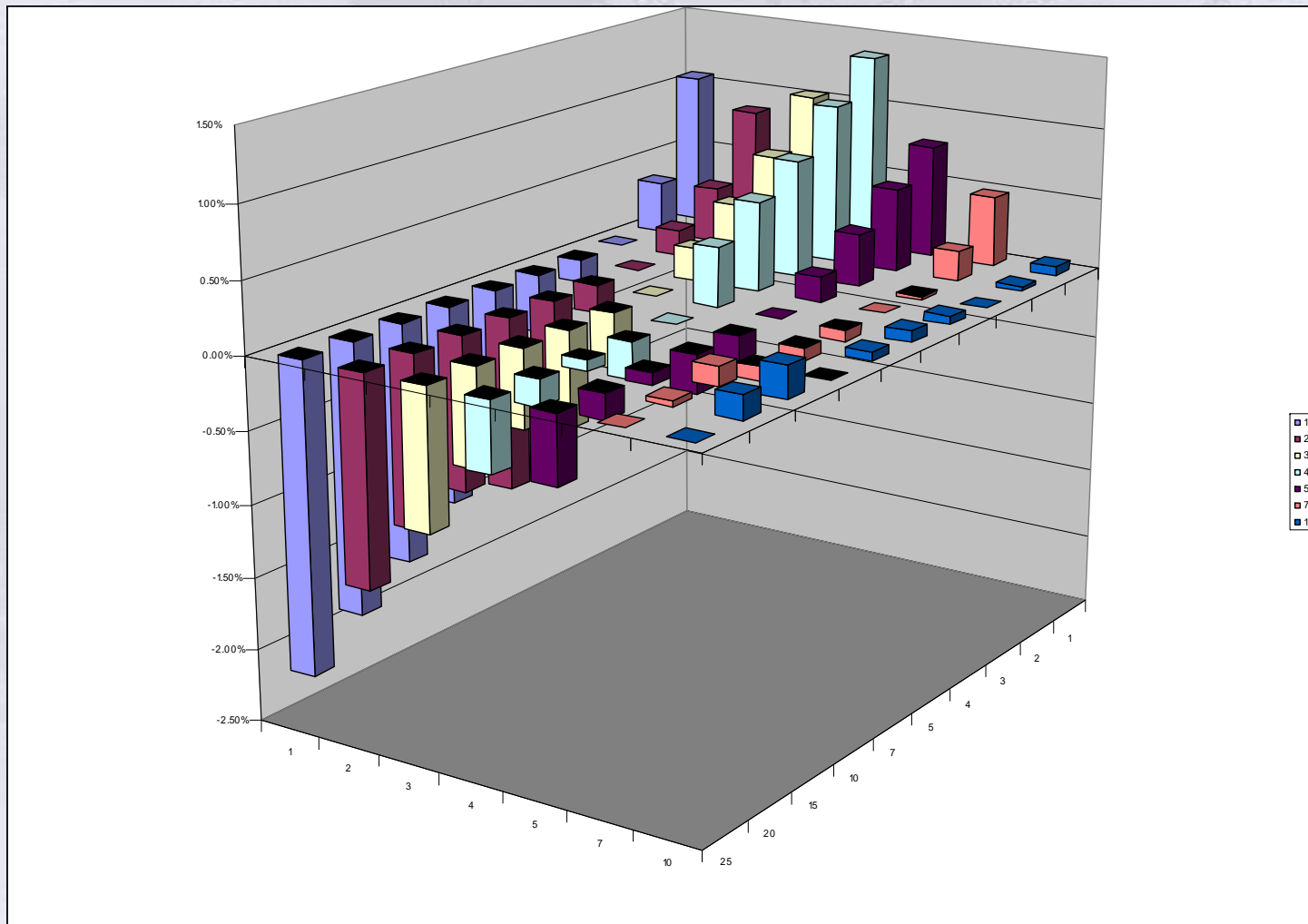
$$B_t^{(T)} - \kappa x B_x^{(T)} + \frac{1}{2} \sigma^2(t) B_{xx}^{(T)} - f(x + \mathcal{G}(t)) B^{(T)} = 0$$

$$B^{(T)}(T, x) = 1$$

- It has to be solved numerically (forward rather than backward for efficiency). Calibration to the US market is shown in Figure 3. In principle, it can be improved if extra factors are introduced.



Motivation for our choice of model



Motivation for our choice of model

- We want to extend the above model in a different direction and augment it as follows

$$dx = -\kappa x dt + \sigma(t) dW(t)$$

$$dy = F(x) dt$$

$$B^T = E\{G(y(T)) + \Theta(T)\}$$

$$F(x) = \frac{1}{1 + e^{-0.2x}} \quad (\text{say}) \quad G(y) = \frac{1}{1 + e^y}$$

- Then we can represent r in the form

$$r = \frac{e^{y+\Theta(t)}}{1 + e^{y+\Theta(t)}} [F(x) + \Theta'(t)]$$

Motivation for our choice of model

- Hence we can write the pricing equation for bonds as follows

$$B_t^{(T)} - \kappa x B_x^{(T)} + \frac{1}{2} \sigma^2(t) B_{xx}^{(T)} + F(x) B_y^{(T)} - \frac{e^{y+\Theta(t)}}{1 + e^{y+\Theta(t)}} [F(x) + \Theta'(t)] B^{(T)} = 0$$

$$B^{(T)}(T, x, y) = 1$$

- This equation certainly looks (and is) more complicated than the original one. However, we can also write it in a different form, namely,

$$\bar{B}_t^{(T)} - \kappa x \bar{B}_x^{(T)} + \frac{1}{2} \sigma^2(t) \bar{B}_{xx}^{(T)} + F(x) \bar{B}_y^{(T)} = 0$$

$$\bar{B}^{(T)}(T, x, y) = \frac{1}{1 + e^{y+\Theta(T)}}$$

- This equation is similar to the standard pricing equation for Asian options and can be solved along similar lines (as a sequence of inhomogeneous one-factor equations).



Motivation for our choice of model

- Since the above equations are fairly unusual, it is useful to solve them in the special case when $F = \text{const}$ (of course, in this case the rate is not stochastic). The corresponding solutions are

$$B^{(T)}(t, y) = \frac{1 + e^{y + \Theta(t)}}{1 + e^{y + F(T-t) + \Theta(T)}}$$

$$\bar{B}^{(T)}(t, y) = \frac{1}{1 + e^{y + F(T-t) + \Theta(T)}}$$

$$FT + \Theta(T) = \ln\left(\frac{1}{B^{(T)}} - 1\right)$$

Our model

- Inspired by the above arguments, we propose the following model for a credit basket

$$dx = (\nu - \kappa x)dt + \sigma(t)dW(t) + j^{(+,p)}dN^{(+)} + j^{(-,q)}dN^{(-)}$$

$$dy = F(x)dt$$

$$\nu = -\lambda / p + \mu / q$$

$$Q^{(i)} = E\{G(y(T) + \Theta^{(i)}(T))\}$$

- Here Q 's are survival probabilities of individual names until time T . To calibrate the model, we need to solve the following pricing PDE:

$$Q_t^{(i)} + (\nu - \kappa x)Q_x^{(i)} + \frac{1}{2}\sigma^2(t)Q_{xx}^{(i)} + \lambda p \int_0^\infty Q^{(i)}(x+j)e^{-pj}dj$$

$$+ \mu q \int_{-\infty}^0 Q^{(i)}(x+j)e^{qj}dj + F(x)Q_y^{(i)} - (\lambda + \mu)Q^{(i)} = 0$$

$$Q^{(i)}(T, x, y) = \frac{1}{1 + e^{y + \Theta^{(i)}(T)}}$$



Our Model

- Now our choice becomes more apparent. Indeed, because of our ansatz for survival probability, we can solve the same pricing equation for all names

$$Q_t + (\nu - \kappa x)Q_x + \frac{1}{2}\sigma^2(t)Q_{xx} + \lambda p \int_0^\infty Q(x+j)e^{-pj}dj$$

$$+ \mu q \int_{-\infty}^0 Q(x+j)e^{qj}dj + F(x)Q_y - (\lambda + \mu)Q = 0$$

$$Q(T, x, y) = \frac{1}{1 + e^y}$$

- and calibrate them to CDS spreads by solving an algebraic equation rather than a PDE

$$Q(0, 0, \Theta^{(i)}(T)) = Q^{(i)}(T)$$

Our Model

- Once calibration to individual names is performed, we can apply the usual recursion and calculate the loss distribution conditional on y . After that we can solve the pricing equation backward and find the expected losses for individual tranches at time 0. The overall output of the direct numerical scheme is shown in Figures 4a-4d. In order to price junior tranches rare but large jumps are necessary. (More on that below.)

- If need occurs, we can consider a multi-factor extension of the above model, namely

$$dx_i = (v_i - \kappa_i x_i)dt + \sigma_i(t)dW_i(t) + j_i^{(+,p_i)}dN_i^{(+)} + j_i^{(-,q_i)}dN_i^{(-)}$$
$$dy = [F_1(x_1) + F_2(x_2)]dt$$

- We note that our construction can be used for a totally different purpose, namely, for constructing consistent dynamics for the fractional loss of a portfolio:

$$dx = (v - \kappa x)dt + \sigma(t)dW(t) + j^{(+,p)}dN^{(+)} + j^{(-,q)}dN^{(-)}$$
$$dy = F(x)dt$$
$$L(t) = H(y(t))$$



Our Model

- For illustrative purposes we use the following set of parameters

$$T = 10$$

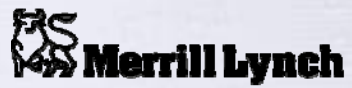
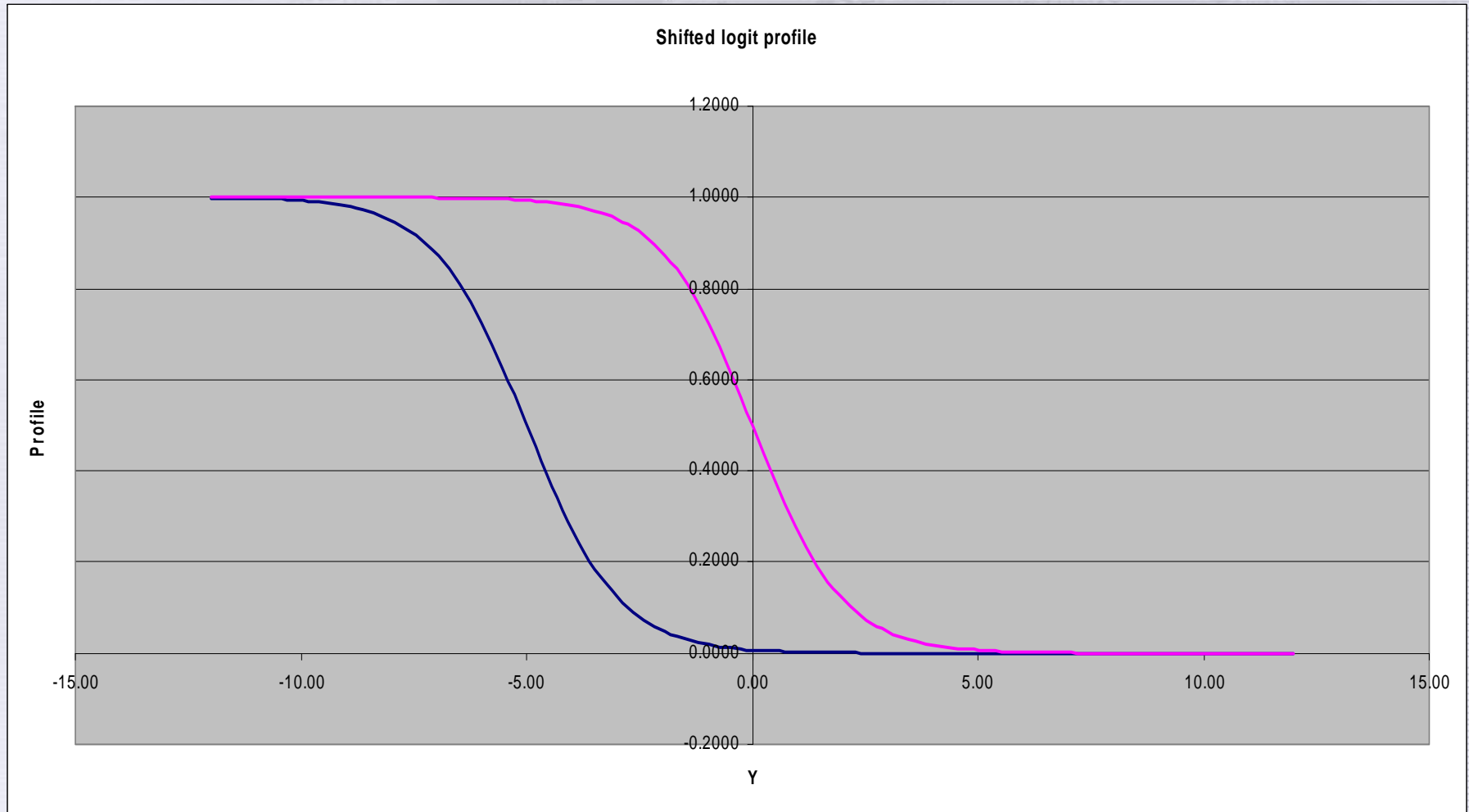
$$\lambda = 0.2\%, \quad p = 0.2$$

$$\mu = 0.2\%, \quad q = 0.2$$

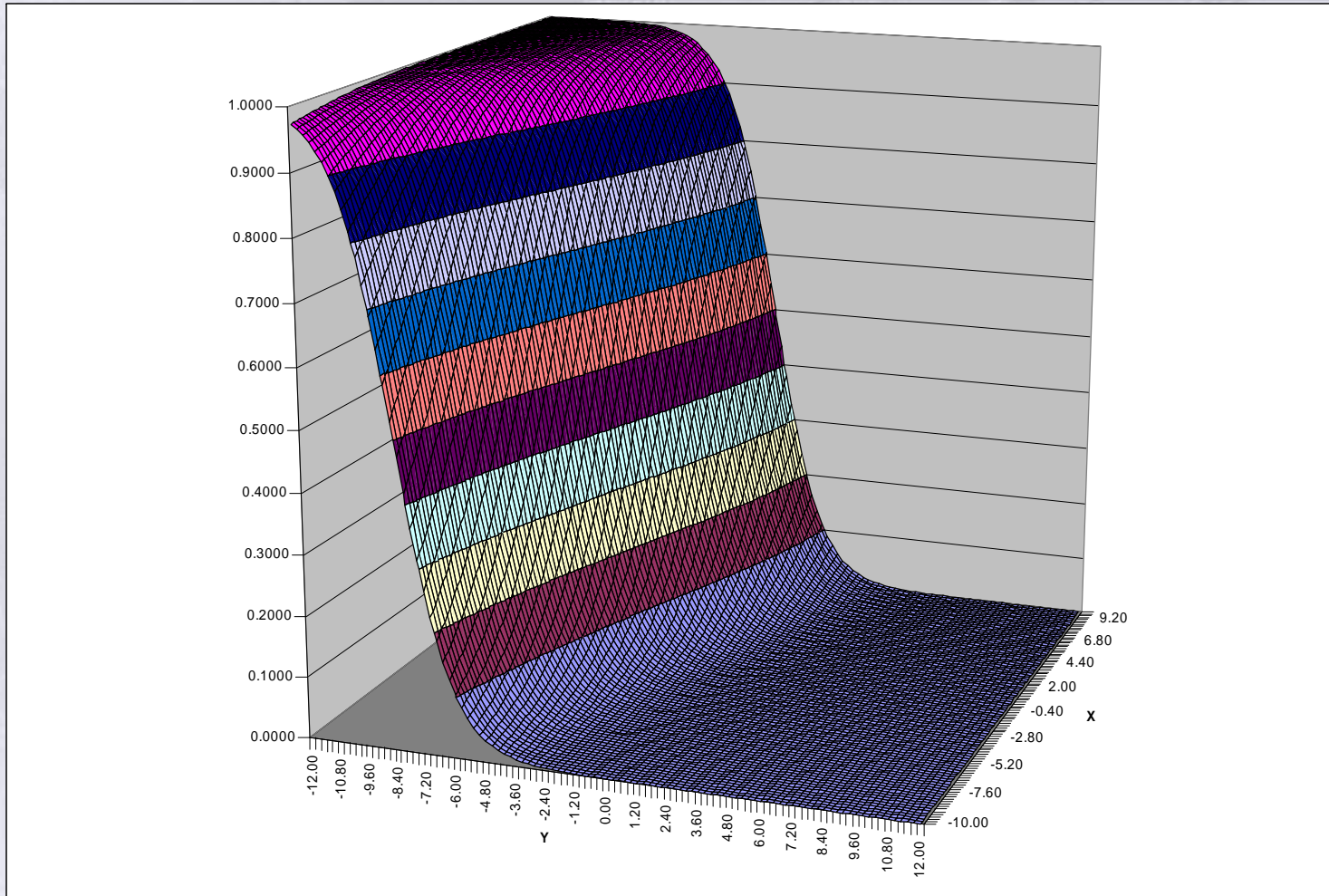
$$\nu = 5\%$$

$$\sigma = 20\%$$

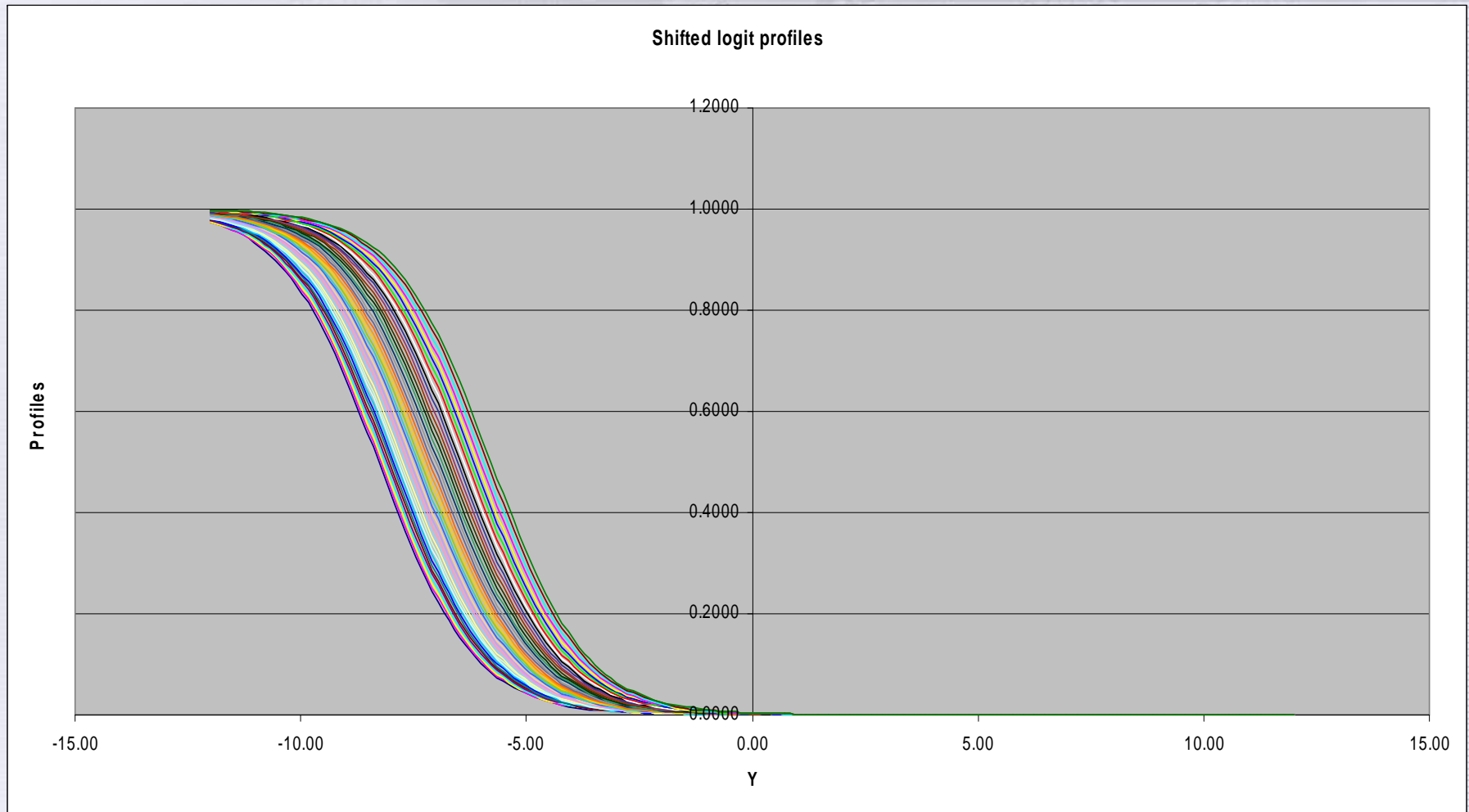
Our model



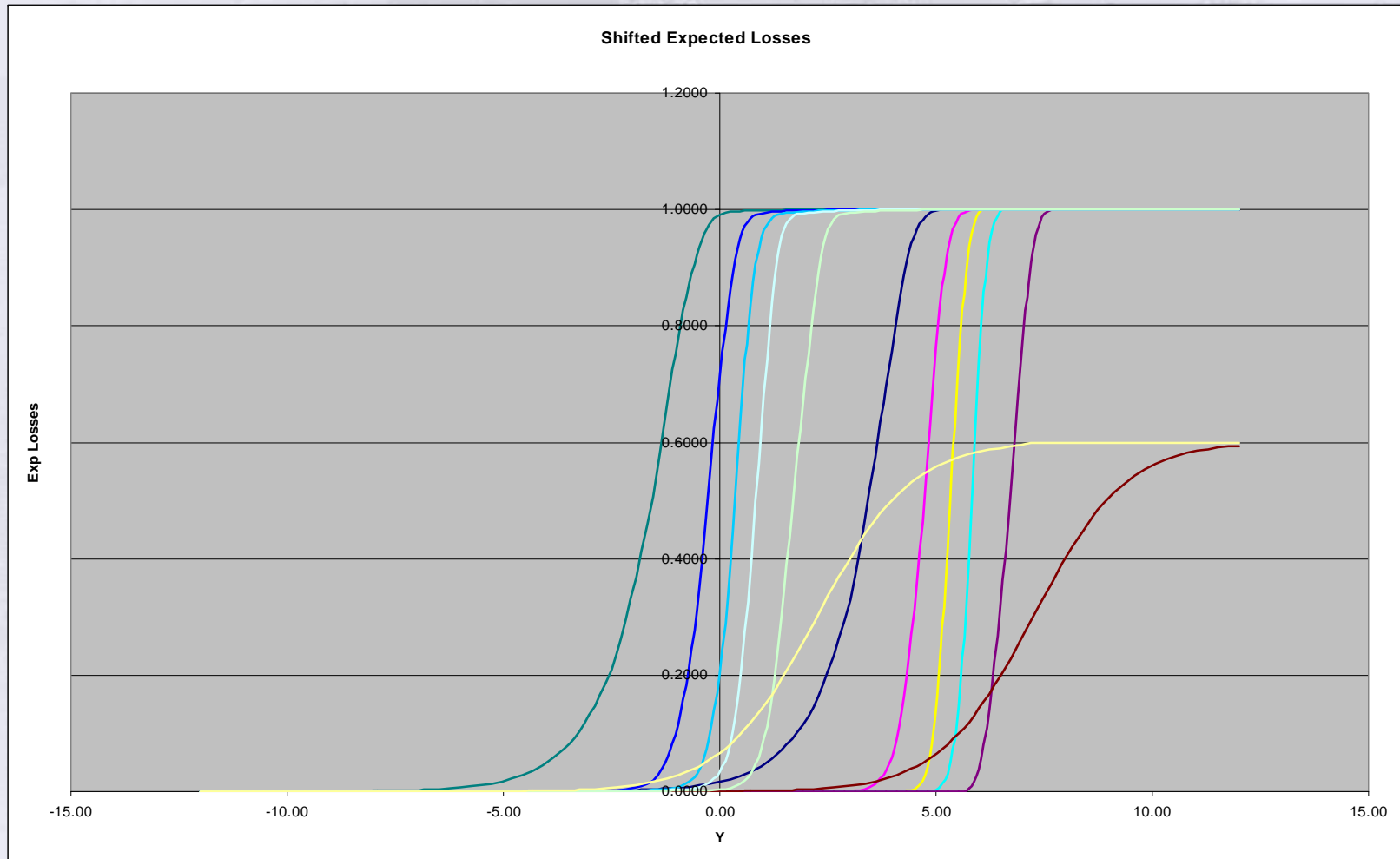
Our model



Our model



Our model



Our model

- We are in an unusual situation when it is more efficient to do the calibration via the backward induction, but the pricing via the forward induction. Of course, for this approach to be consistent, we need to design numerical schemes, which guarantee that backward and forward equations agree with machine accuracy.

- The pricing equation for the conditional t.p.d.f. has the form

$$\Phi_T + ((\nu - \kappa X)\Phi)_X - \frac{1}{2}\sigma^2(T)\Phi_{XX} - \lambda p \int_0^\infty \Phi(X - j)e^{-pj} dj$$

$$- \mu q \int_{-\infty}^0 \Phi(X - j)e^{qj} dj + F(X)\Phi_Y + (\lambda + \mu)\Phi = 0$$

$$\Phi(0, X, Y) = \delta(X)\delta(Y)$$

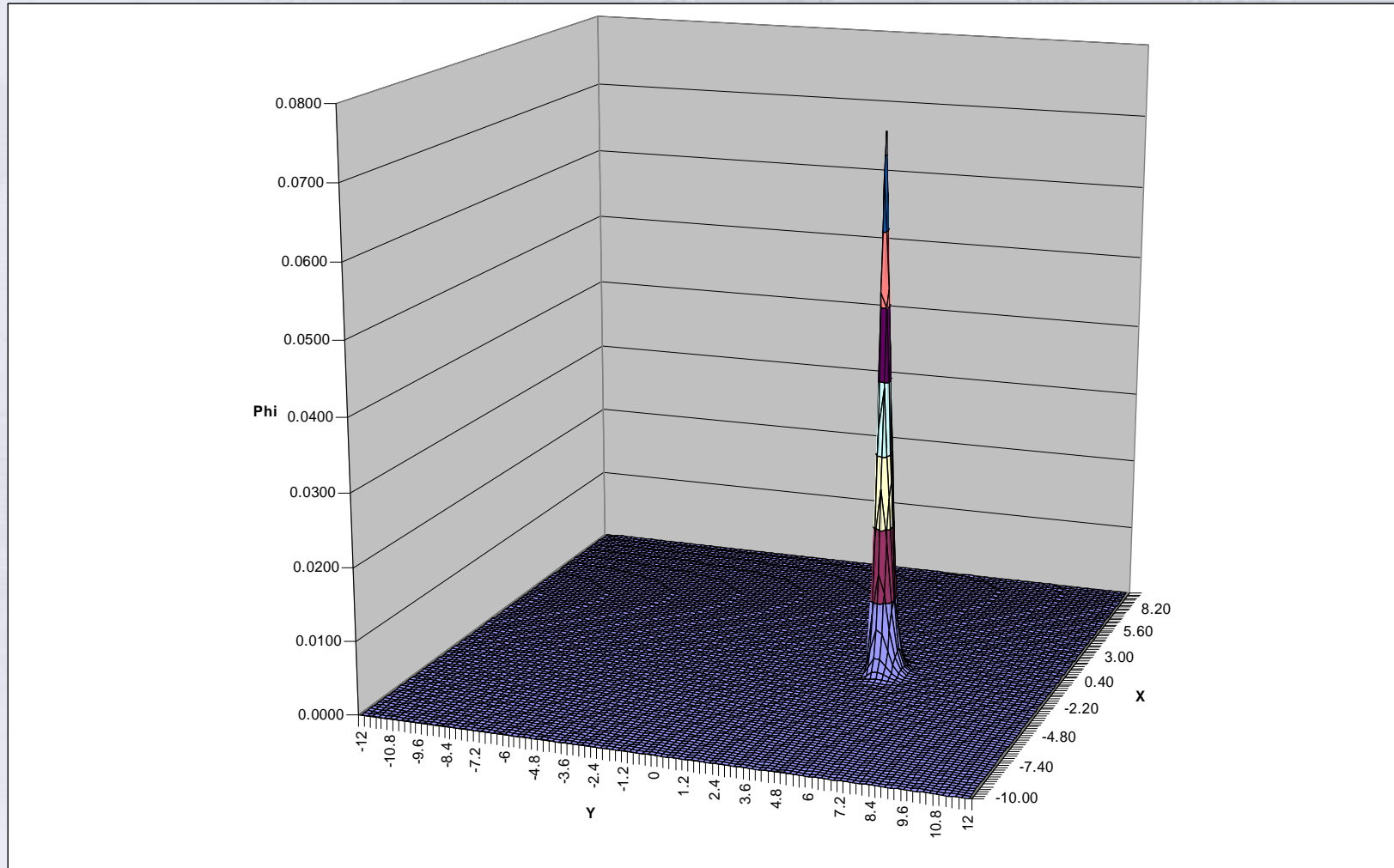
- The unconditional probability is defined can be found by a simple integration.

$$\Psi(T, Y) = \int_{-\infty}^\infty \Phi(T, X, Y) dX$$

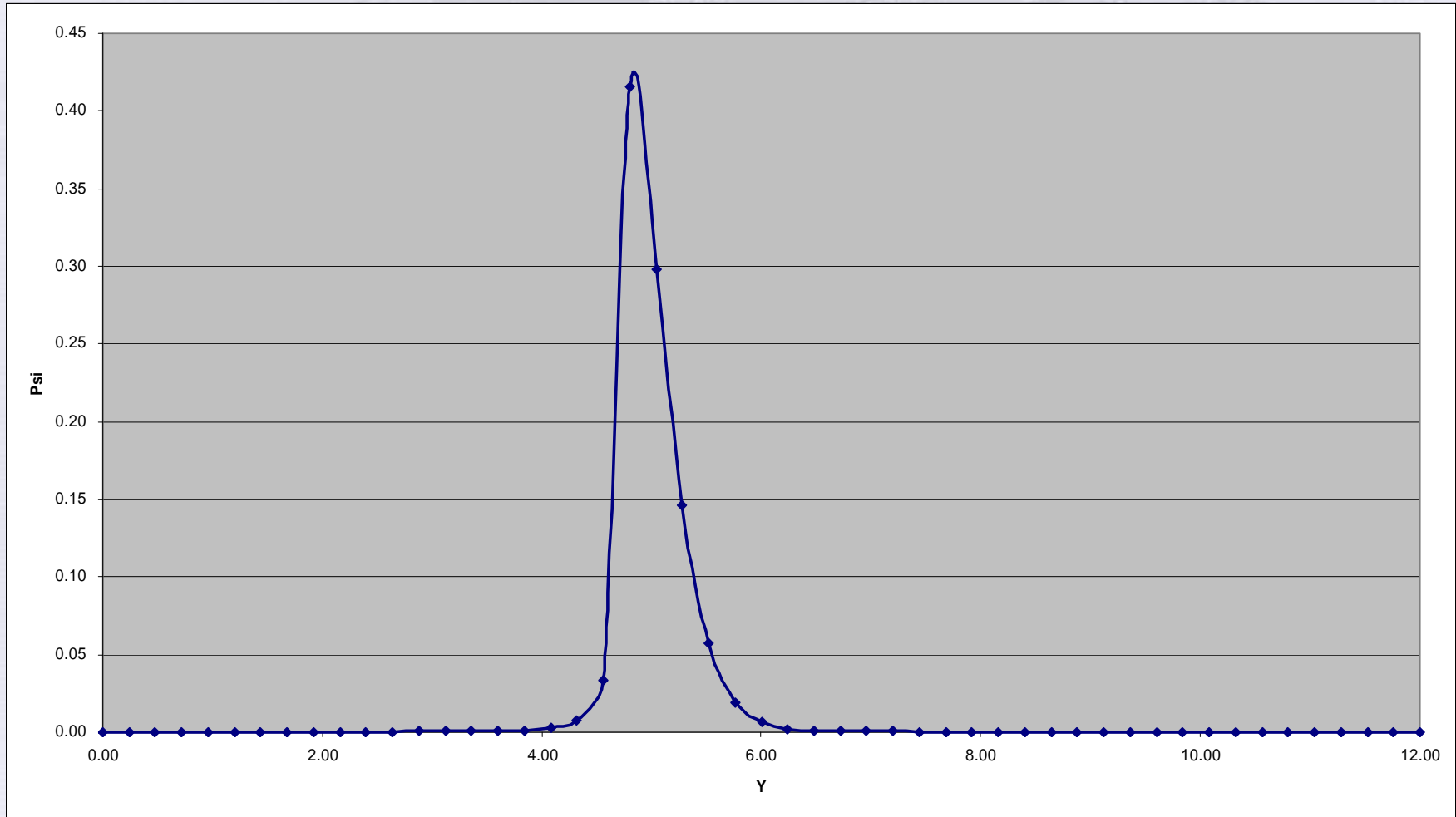
- Once it is known, all the quantities we are interested in can be computed by taking averages.



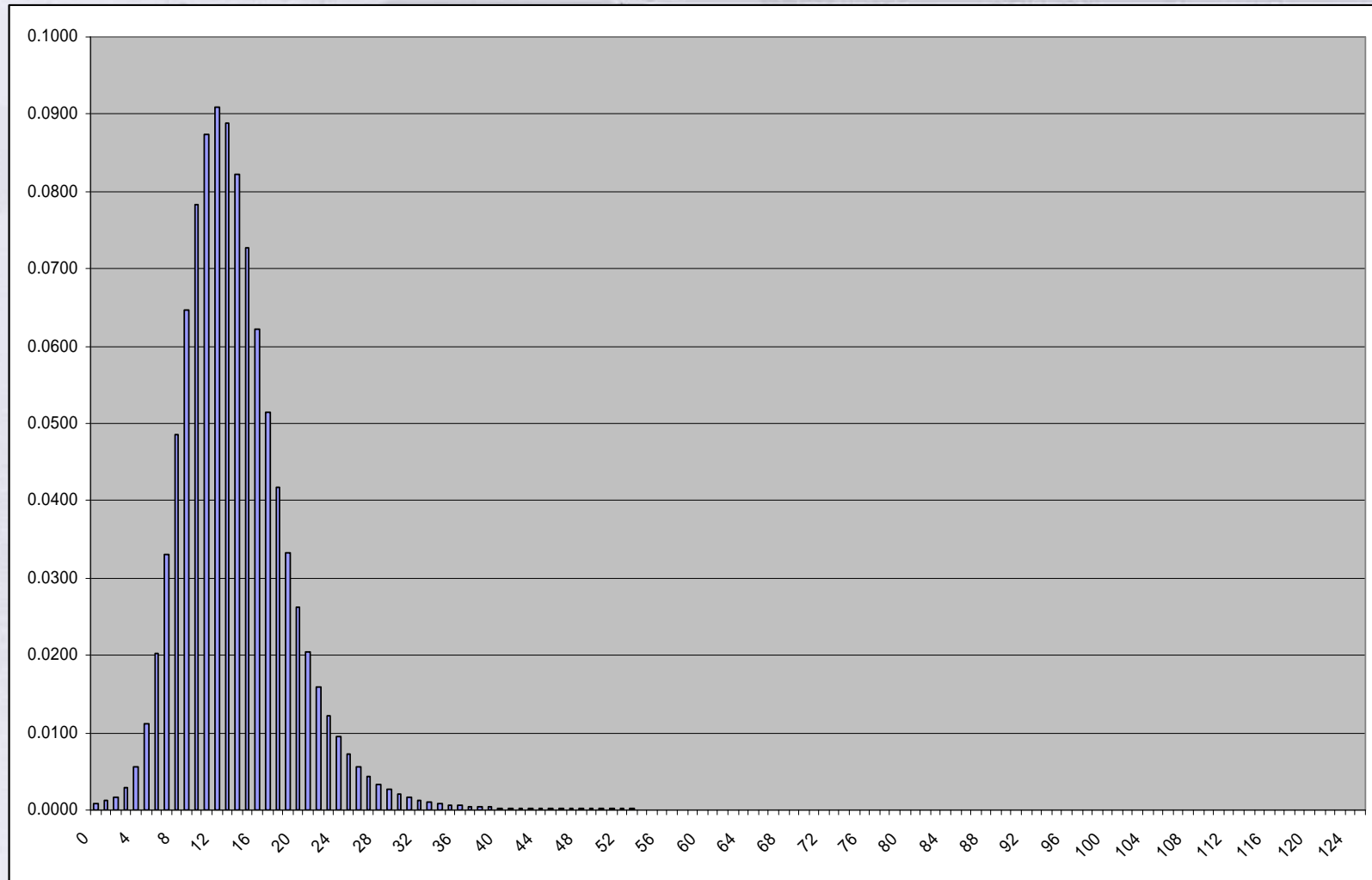
Our model



Our model



Our model



A simplified model

- To make the above arguments more transparent, we introduce a simplified version of the model and apply it to CDX7 on Jan 12, 2007. On that date, the market data has the form

Year	0-3%	3-7%	7-10%	10-15%	15-30%	0-100%
3	2.300%	0.0600%	0.0013%	0.0006%	0.0002%	0.172%
5	23.500%	0.730%	0.135%	0.055%	0.033%	0.340%
7	42.875%	2.018%	0.425%	0.180%	0.074%	0.470%
10	53.063%	4.760%	1.055%	0.490%	0.149%	0.590%

- We can extract expected tranche losses by balancing default and payment legs in the usual manner.

A simplified model

$$DL_m^{(k)} = \sum_{i=1}^{n_m} DF\left(\frac{t_i + t_{i-1}}{2}\right) (L_i^{(k)} - L_{i-1}^{(k)})$$

$$PL_m^{(k)} = uf_m^{(k)} N^{(k)} + s_m^{(k)} \sum_{i=1}^{n_m} \delta_i DF(t_i) (N^{(k)} - \phi L_i^{(k)})$$

$$L_i^{(k)} = \frac{(t_i - T_{n_{j-1}}) \bar{L}_j^{(k)} + (T_{n_j} - t_i) \bar{L}_{j-1}^{(k)}}{(T_{n_j} - T_{n_{j-1}})}, \quad n_{j-1} + 1 \leq i \leq n_j$$

$$PL_m^{(k)} = D_m^{(k)} - s_m^{(k)} \sum_{j=1}^m (B_j \bar{L}_j^{(k)} + C_j \bar{L}_{j-1}^{(k)})$$

$$DL_m^{(k)} = \sum_{j=1}^m A_j (\bar{L}_j^{(k)} - \bar{L}_{j-1}^{(k)})$$

$$\sum_{j=1}^m \left\{ (A_j + s_m^{(k)} B_j) \bar{L}_j^{(k)} + (-A_j + s_m^{(k)} C_j) \bar{L}_{j-1}^{(k)} \right\} = D_m^{(k)}$$

A simplified model

- By solving the above system of lower-triangular equations we find expected tranche losses as follows

	0-3%	3-7%	7-10%	10-15%	15-30%	0-100%
2.93	0.1583	0.0017	0.0000	0.0000	0.0000	0.0050
4.93	0.4830	0.0380	0.0071	0.0029	0.0017	0.0171
6.93	0.7805	0.1490	0.0324	0.0138	0.0056	0.0336
9.93	0.9568	0.4827	0.1175	0.0556	0.0167	0.0604

- In the spirit of our previous consideration, we introduce a market factor with four distinct values $y_0 = 0, y_1(T), y_2(T), y_3(T) = \infty$ occurring with probabilities $\pi_0(T), \pi_1(T), \pi_2(T), \pi_3(T), \sum \pi_k = 1$

- We assume that survival probabilities of individual names are given by the familiar expression

$$Q^{(i)}(T | y_k) = \frac{1}{1 + e^{y_k + \Theta^{(i)}(T)}}$$

- and calibrate model parameters to match the market.



A simplified model

- We achieve almost perfect calibration to the input expected tranche losses and obtain the following set of breakeven coupons and upfront payments (for clarity differences are shown)

0.50%	0.00%	0.00%	0.00%	0.00%	0.00%
0.11%	0.00%	0.00%	0.00%	0.00%	0.00%
0.11%	0.00%	0.00%	0.00%	0.00%	0.00%
0.11%	0.00%	0.00%	0.00%	0.00%	0.00%

- The corresponding parameters have the form

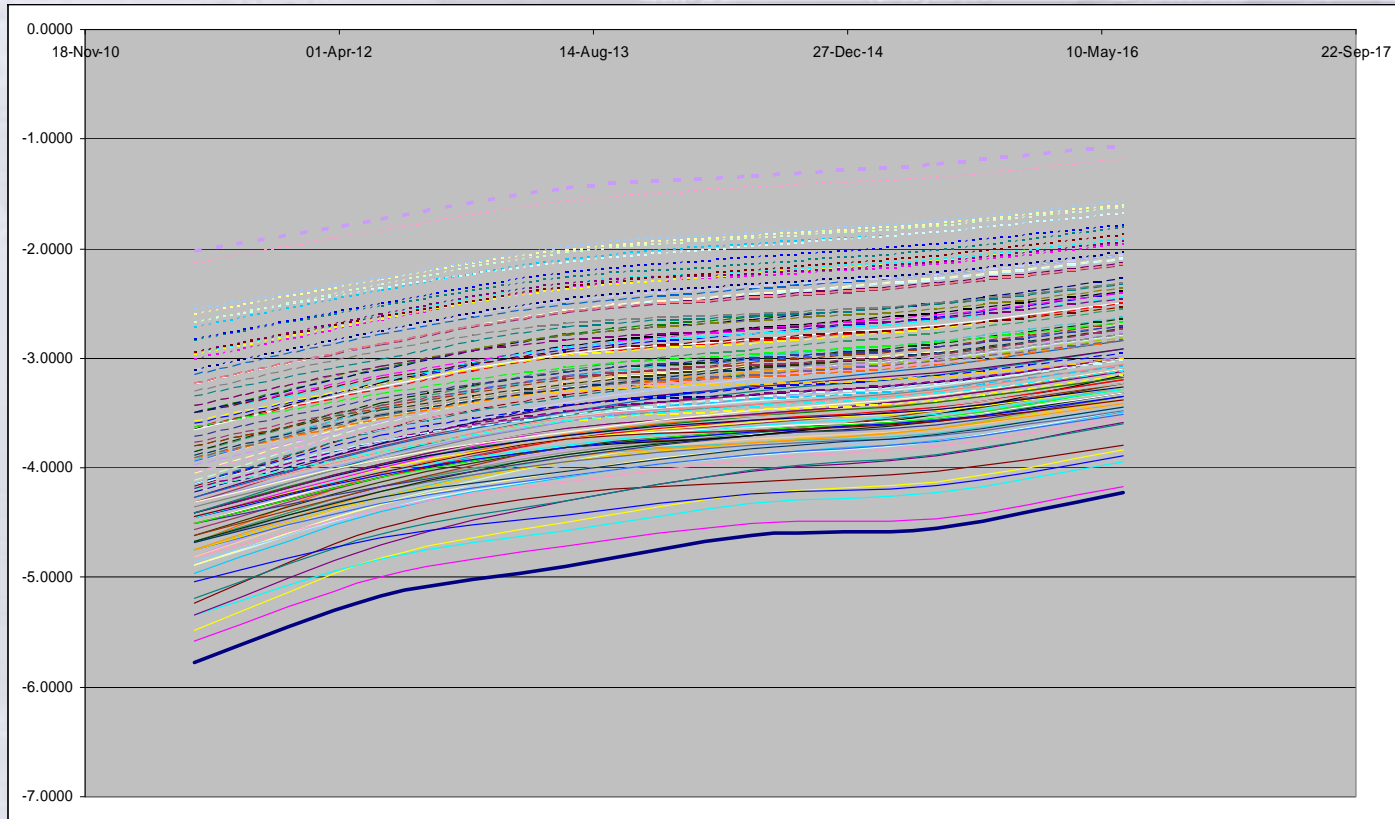
0.8833	0.1119	0.0048	0.0000	1.3007	2.5505
0.9030	0.0891	0.0062	0.0017	1.2746	2.4306
0.8382	0.1342	0.0219	0.0057	0.9768	1.9647
0.5475	0.3943	0.0464	0.0118	0.7428	2.0239



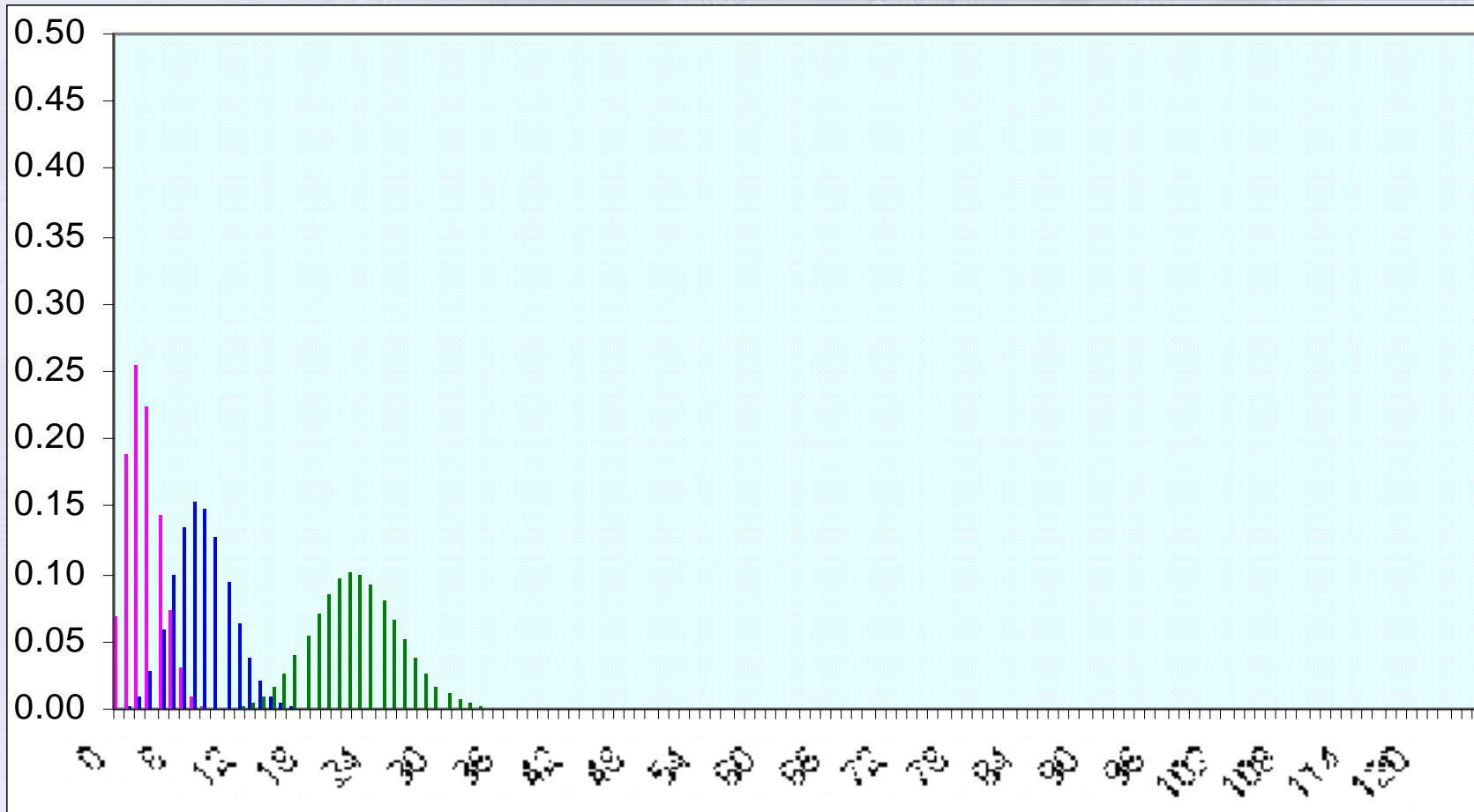
A simplified model

- It is very instructive to look at the corresponding thetas and loss distributions which are given in Figures 11-16.
- In principle, we can calibrate two indices at once and hence build a framework for pricing bespoke portfolios.

A simplified model

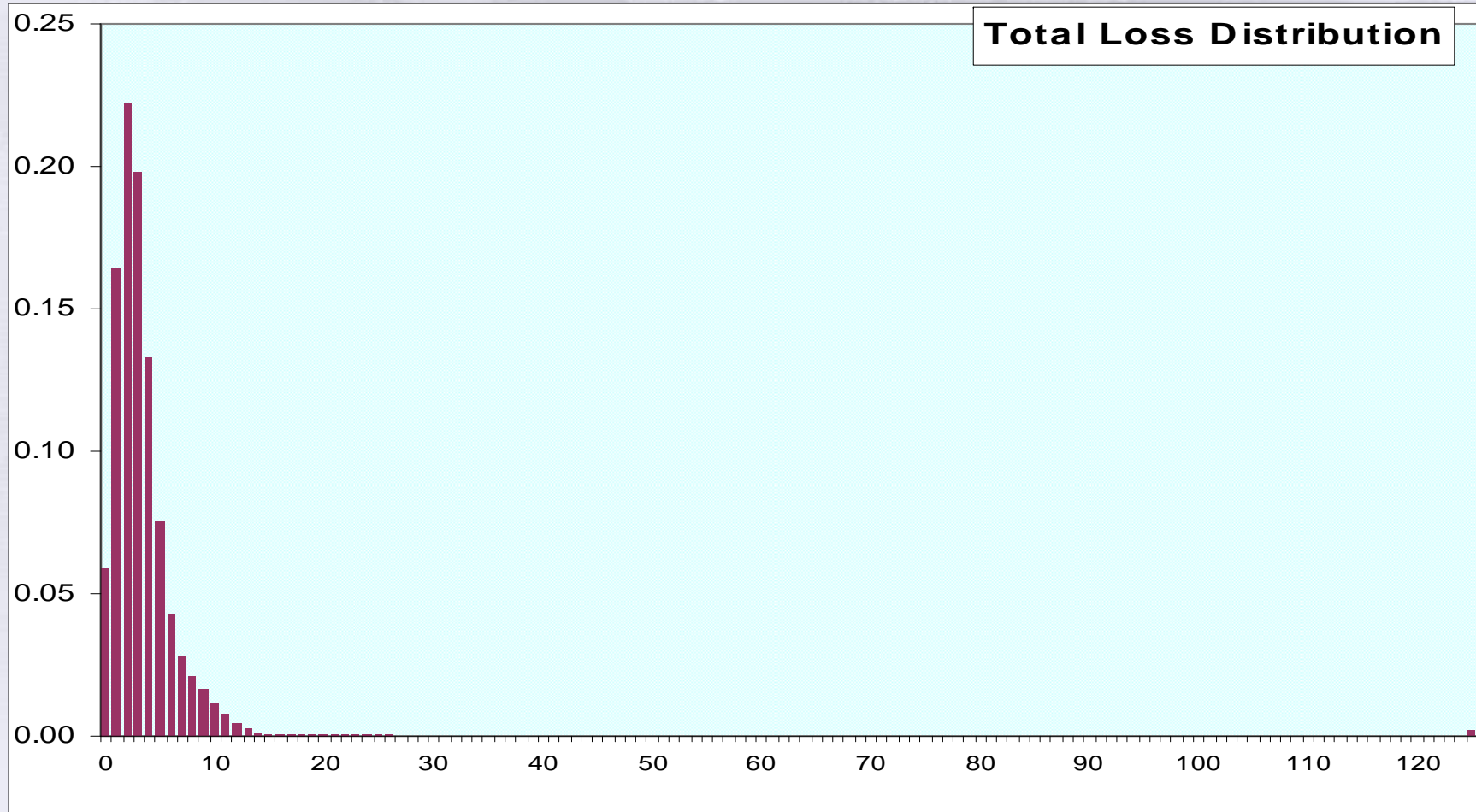


A simplified model



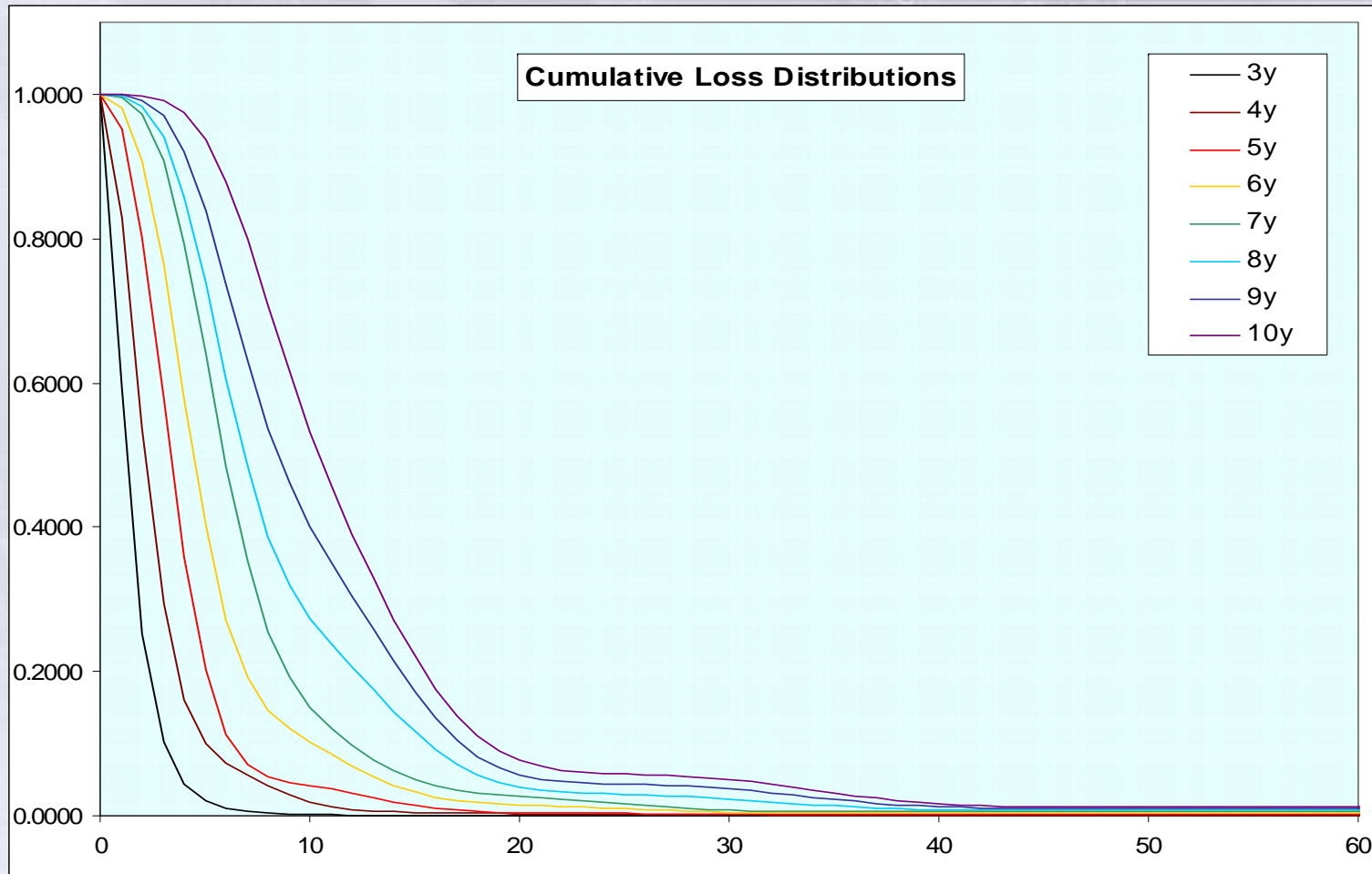
A simplified model

The loss distribution is built as weighted average of loss distributions corresponding to individual factors



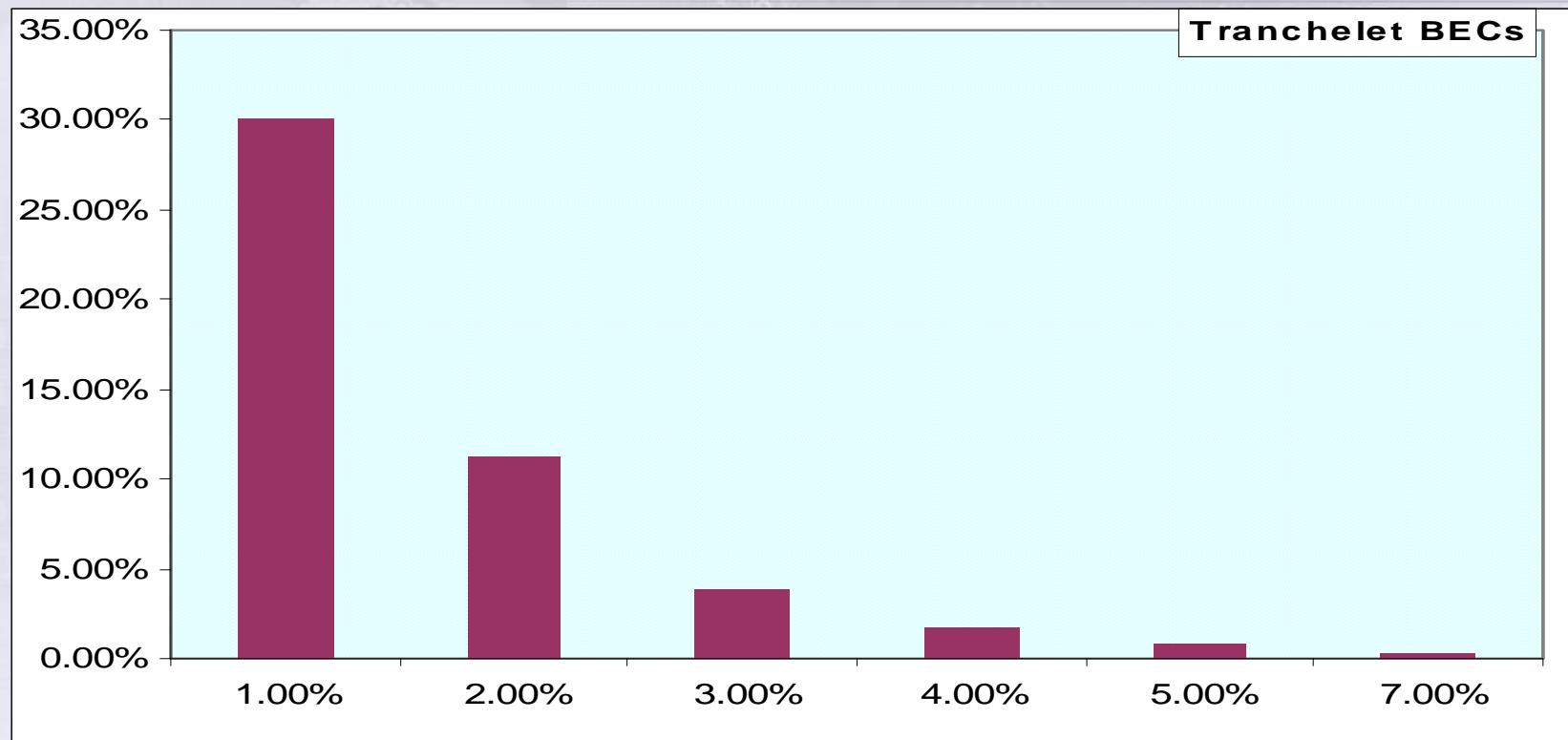
A simplified model

- Time-dependent loss distribution cannot be arbitrated.



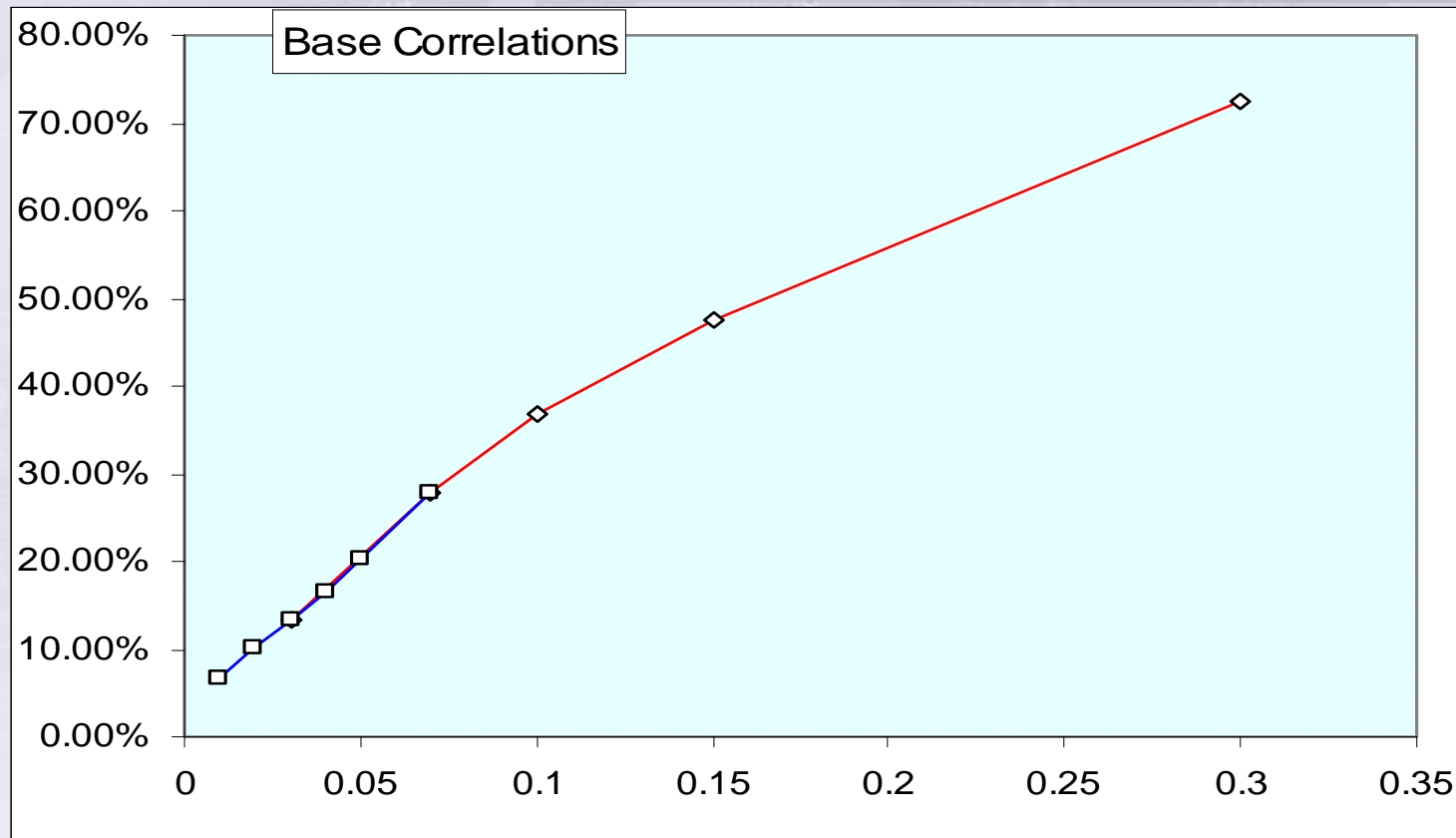
A simplified model

- Since we know loss distributions for all maturities, we can calculate break-even coupons for all 1% tranchelets, see Figure 14.



A simplified model

- The corresponding base correlation has the form



A simplified model

- In case the model does not calibrate to the market exactly, we can apply the usual relative entropy minimization to do so. The set of equations we need to solve is by now standard:

$$p_m(l) \geq 0,$$

$$\sum_{l=0}^{125} p_m(l) = 1$$

$$\sum_{l=0}^{125} p_m(l) E^{(k)}(l) = \bar{L}_m^{(k)}, \quad k = 1, \dots, 6$$

$$\sum_{l=0}^{125} p_m(l) \ln \left(\frac{p_m(l)}{q_m(l)} \right) \rightarrow \min$$

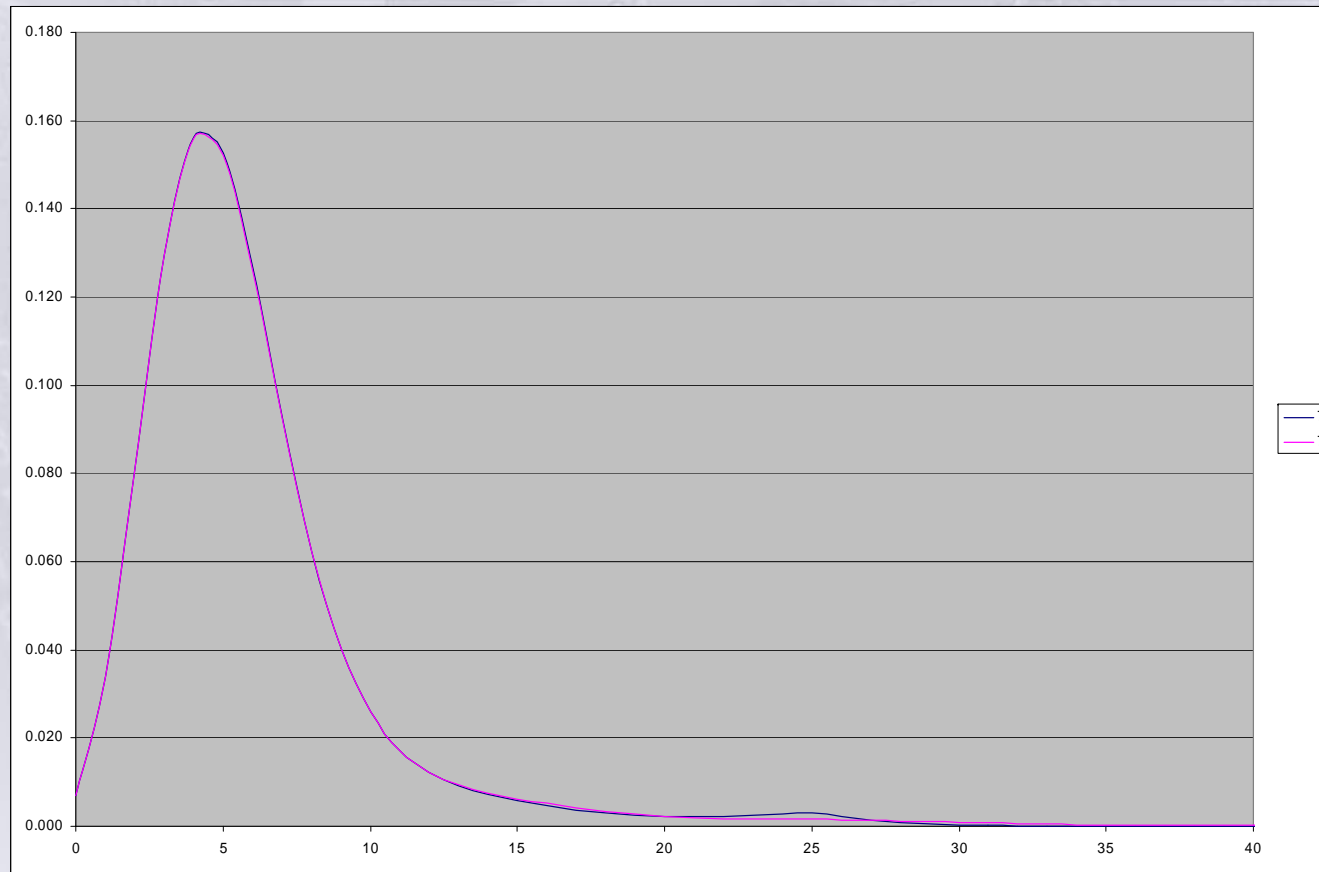
$$p_m(l) = \exp(\bar{\lambda}_0 + \lambda_1 E^{(1)}(l) + \dots + \lambda_6 E^{(6)}(l)) q_m(l)$$

$$\exp(-\bar{\lambda}_0) = \sum_{l=0}^{125} \exp(\lambda_1 E^{(1)}(l) + \dots + \lambda_6 E^{(6)}(l)) q_m(l)$$

$$\ln \left\{ \sum_{l=0}^{125} \exp(\lambda_1 E^{(1)}(l) + \dots + \lambda_6 E^{(6)}(l)) q_m(l) \right\} - (\lambda_1 \bar{L}_m^{(1)} + \dots + \lambda_6 \bar{L}_m^{(6)}) \rightarrow \min$$

A simplified model

- Results of applying minimum cross-entropy are shown in Figure 17.



A simplified model

- Calibration parameters for iTRAXX are similar to the one for CDX.
- Hence we can use the same parameters to price bespoke baskets (within reason).
- More complicated instruments (such as LSS deals and the like can be priced in the framework of the complete model).

Conclusions

- We demonstrated how to construct a one-factor dynamic factor model for credit baskets.
- Via a special ansatz we solved the calibration problem for individual names without using analytically tractable (but not necessarily financially motivated) dynamics.
- We showed that even the simplified version of the model can reproduce the market exactly.
- We use the calibrated simplified model to analyze the structure of loss distributions implied by market quotes.
- As a result we achieved a financially meaningful “completion” of the break-even coupon surface.
- In its full form, the model can be used to price a variety of exotic credit products, build dynamic hedges for tranches and tranchelets, etc.

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