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Empirical Copulas for CDO Pricing Using Minimum Cross Entropy

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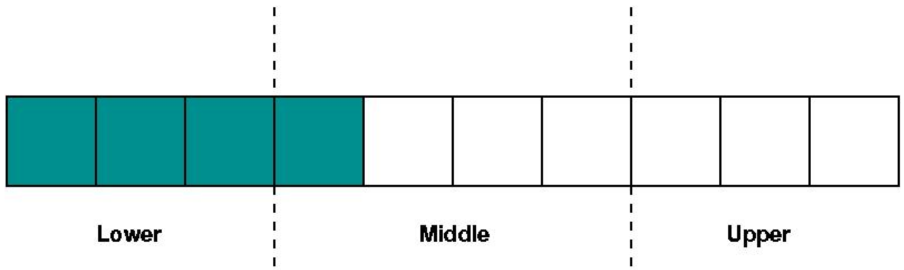
Outline

- **Introduction**
- **Principle of Minimum Cross Entropy**
- **Computational Results**
- **Conclusion**

Introduction

- Simplified example
  - 10 firms each issue $1 ZCB
  - 3 tranches on this CDO: lower, middle, and upper
  - Four firms default
  - Lower tranche pays $3, middle tranche pays $1, upper pays nothing



- Problem: How to price bespoke tranches consistently with liquid market quotes?
- Solution: We will use the minimum cross entropy copula between the market data and a specified prior copula

Introduction

- Early approaches – reduced-form with correlated intensities

- Default intensities are
  - Firm 1: $P\left(\tau_1 \in [t, t+dt) \middle| \lambda_1(t)\right) = \lambda_1(t)\,dt$
  - Firm 2: $P\left(\tau_2 \in [t, t+dt) \middle| \lambda_2(t)\right) = \lambda_2(t)\,dt$

  where $\boldsymbol{\lambda}_1$ and $\boldsymbol{\lambda}_2$ are correlated stochastic processes

- Problem: Empirical study suggests level of correlation between $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ achievable is not high enough
  Das *et al* (2005)

Introduction

- Current popular approach: copulas which directly introduce dependence between default times τ_1 and τ_2

- Idea: (e.g. Gaussian copula)

$$P(\tau_1 < T, \tau_2 < T) = \Phi(z_1, z_2; \rho)$$

where z_1 and z_2 are retrieved from single-name calibration and correlation ρ is determined from CDO tranche quotes

Introduction

- Choice of copula is exogenous and arbitrary – motivated by quality of fit to data
- Idea: imply the copula from market prices
- Problem is that only a finite number of market prices are available from which we are trying to retrieve a continuous distribution
- Solution is to introduce a regularizing function
- Several possibilities: e.g. goodness-of-fit, maximal smoothness Hull & White (2006) and entropy *cf* Avellaneda *et al* (1997)

Principle of Maximum Entropy

- Entropy is a measure of uncertainty
- We should choose a distribution which is consistent with the given data but otherwise has maximum uncertainty
- This approach is optimal in the sense that we do not assume anything about the distribution other than what is given by the data i.e. it is a distribution free method

Principle of Maximum Entropy

- Define entropy for a continuous multivariate density f as

$$H(f) := -\int f(x) \log f(x) dx$$

- Maximize H over f subject to
  - $\int f(x) dx = 1$
  - $\int a_i(x) f(x) dx = \bar{a}_i \qquad i = 1, \ldots, m$

where the a_i are discounted payoffs and the constants $\bar{a}_i$ are the m observed market data points

Principle of Maximum Entropy

- Easy to solve with calculus of variations via method of Lagrange multipliers with solution

$$\hat{f}(x) = \frac{1}{Z(\eta)} \exp\left\{ \sum_{i=1}^{m} \eta_i a_i(x) \right\}$$

where Z is the normalizing factor that makes $\hat{f}$ a density

- Must now solve for the η's numerically via first order conditions and Newton's method e.g. Agmon *et al* (1981) but computationally easier to solve the problem dual to the original problem i.e. minimize the primal value over the η's

Principle of Maximum Entropy

- Dual value function can be simplified to

$$L(\eta) = -\int \hat{f}(x)\log \hat{f}(x)dx + \sum_{i=1}^{m} \eta_i \left[\int a_i(x)\hat{f}(x)dx - \bar{a}_i \right]$$

$$= -\int \hat{f}(x)\left[-\log\left(Z(\eta)\right) + \sum_i \eta_i a_i(x) \right] dx + \sum_i \eta_i \left[\int a_i(x)\hat{f}(x)dx - \bar{a}_i \right]$$

$$= \log\left(Z(\eta)\right) - \sum_i \eta_i \bar{a}_i$$

Borwein & Lewis (2000)

- Minimize this unconstrained objective over η numerically

Principle of Maximum Entropy

- The dual function is strictly convex

$$L_{\eta_i \eta_j} = \frac{Z_{\eta_i \eta_j}}{Z} - \frac{Z_{\eta_i} Z_{\eta_j}}{Z^2}$$

$$\vdots$$

$$= \text{cov}\left(a_i(X), a_j(X) \right)$$

- So optimization is straightforward using a quasi-Newton method e.g. BFGS

Principle of Minimum Cross-Entropy

- Maximizing entropy means we are choosing a distribution closest to the uniform distribution while satisfying the data constraints
- But instead of the uniform distribution we could choose another distribution which is useful if we have prior knowledge or a view
- Use the concept of cross-entropy – a measure of distance from one probability distribution to another defined by

$$D(P \mid Q) := \int p(x) \log \frac{p(x)}{q(x)} dx$$

Principle of Minimum Cross-Entropy

- We wish to minimize cross-entropy – choose a distribution closest to our prior while satisfying the data constraints

- Solving this problem almost identical to MaxEnt with solution

$$\hat{p}(x) = \frac{1}{Z(\eta)} q(x) e^{\eta' a(x)}$$

- Presence of the factor $q(x)$ means it is possible to compute the required integrals via importance sampling

Application to CDO pricing

- To apply to CDO pricing we need an extra requirement – the marginals of the joint distribution are specified
- Hence we can attempt to infer the minimum cross-entropy copula density c from the data but this introduces extra constraints – the marginals must be uniformly distributed
- Difficult to satisfy this requirement completely but we can approximate it with

$$\int_0^p \int_{[0,1]^{n-1}} c(u)\,du_{-i}\,du_i = p \qquad i = 1,\ldots,125$$

where $p := 1 - e^{-hT}$ for some fixed maturity T and h is default intensity

Computational Results

- Preliminary results: Suppose 'market' prices are generated by some copula with prior the Gaussian copula and calibrate the minimum cross-entropy copula to the generated 'market' premia

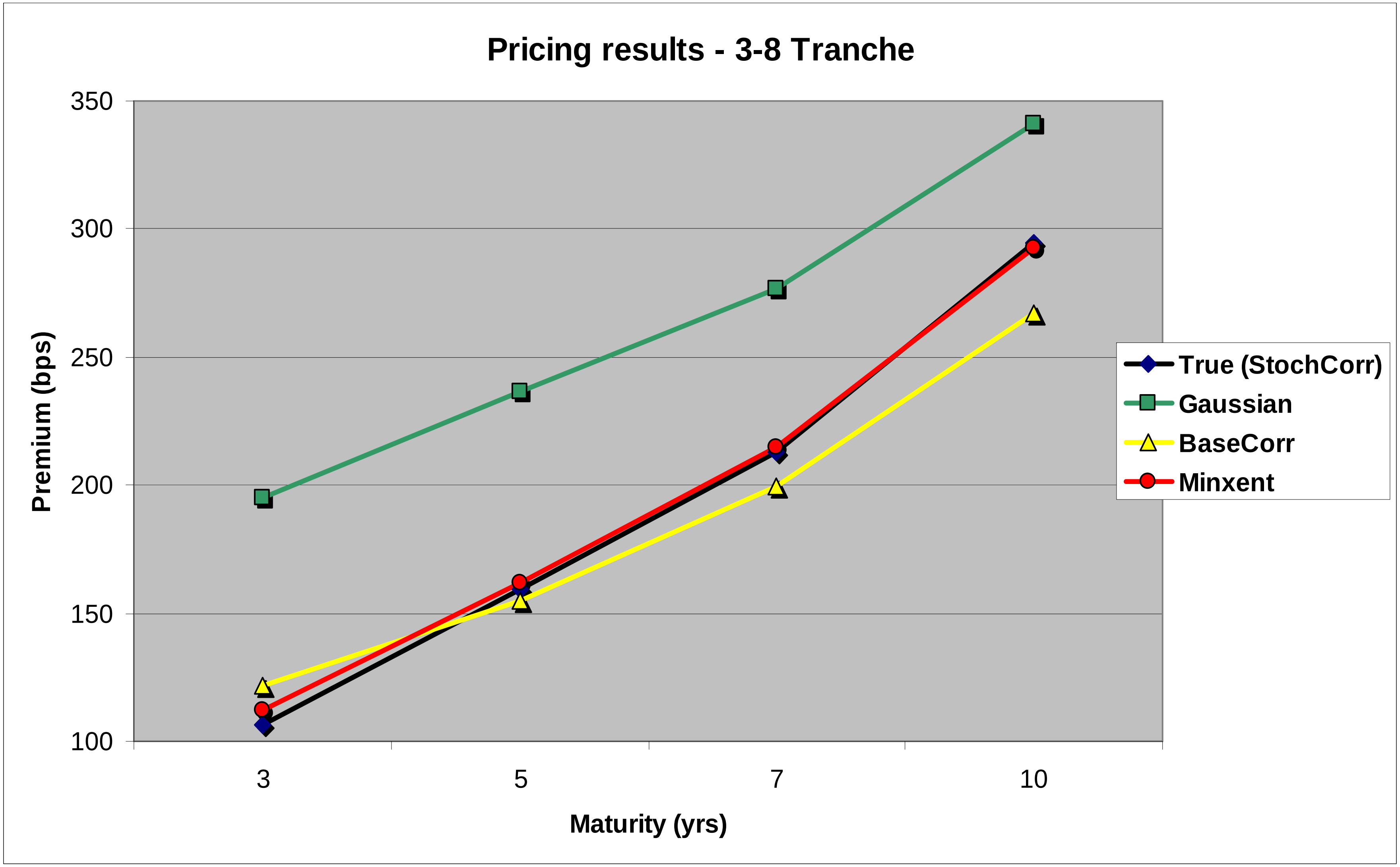
- The stochastic correlation Gaussian copula was chosen as the 'market' copula as it fits the market prices relatively closely (i.e. produces a correlation skew effect) and is very simple   Burtschell *et al* (2005)

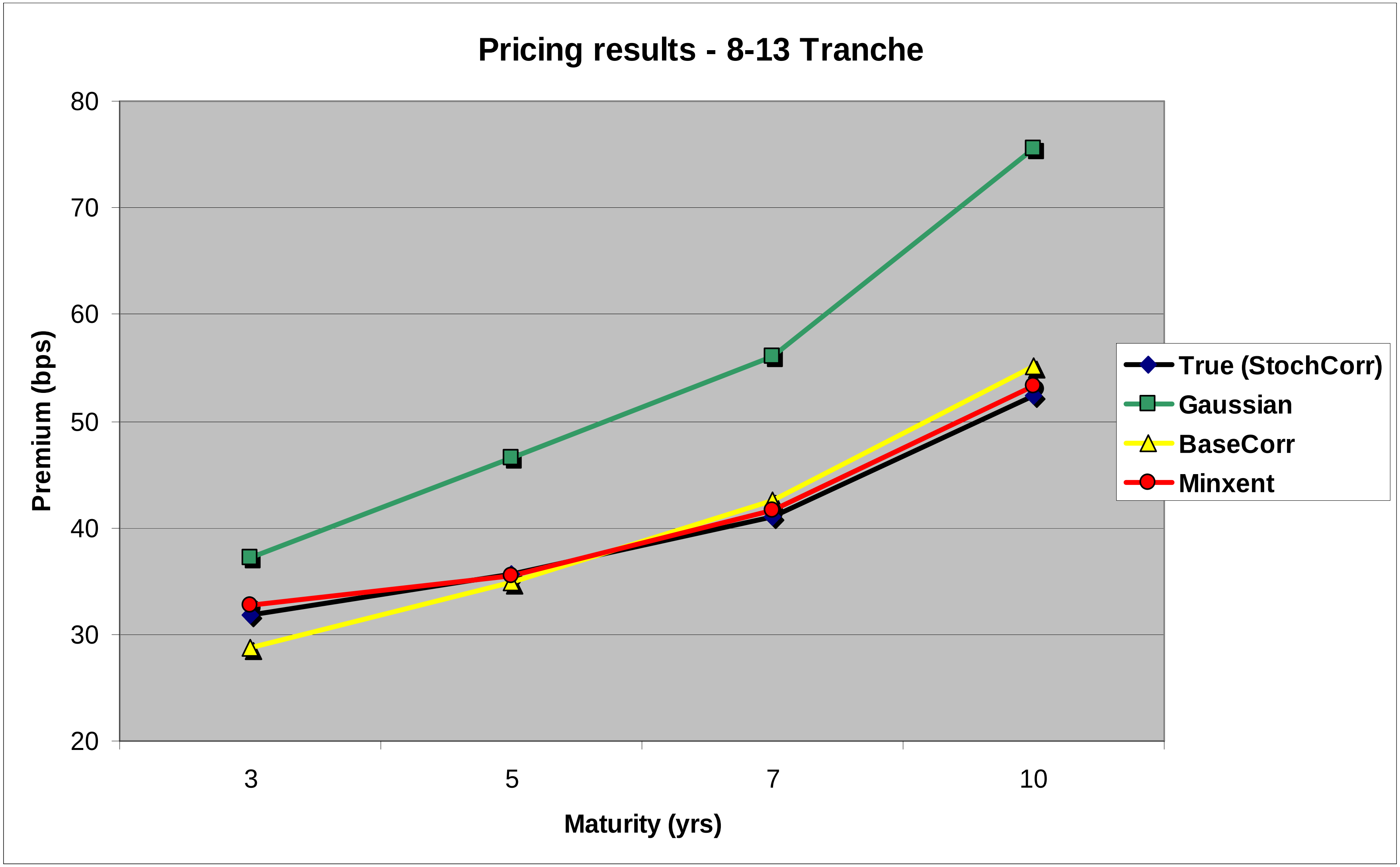
Thresholds (%)	Maturity	'Market' premium	Minxent premium	Absolute error (bps)
0-5	**3**	7.5%	7.4%	**0.1%**
5-10	**3**	51.3	50.5	**0.7**
10-15	**3**	25.2	25.8	**0.6**
15-30	**3**	12.3	11.9	**0.4**
0-5	**5**	13.2%	13.1%	**0.1**
5-10	**5**	68.9	68.9	**0.0**
10-15	**5**	28.3	27.9	**0.4**
15-30	**5**	13.6	13.8	**0.2**
0-5	**7**	18.7%	18.7%	**0.0**
5-10	**7**	91.6	92.7	**0.1**
10-15	**7**	30.7	31.6	**0.9**
15-30	**7**	14.5	15.2	**0.7**
0-5	**10**	26.0%	26.0%	**0.0**
5-10	**10**	131.0	131.8	**0.8**
10-15	**10**	35.2	36.6	**1.4**
15-30	**10**	15.4	15.8	**0.4**

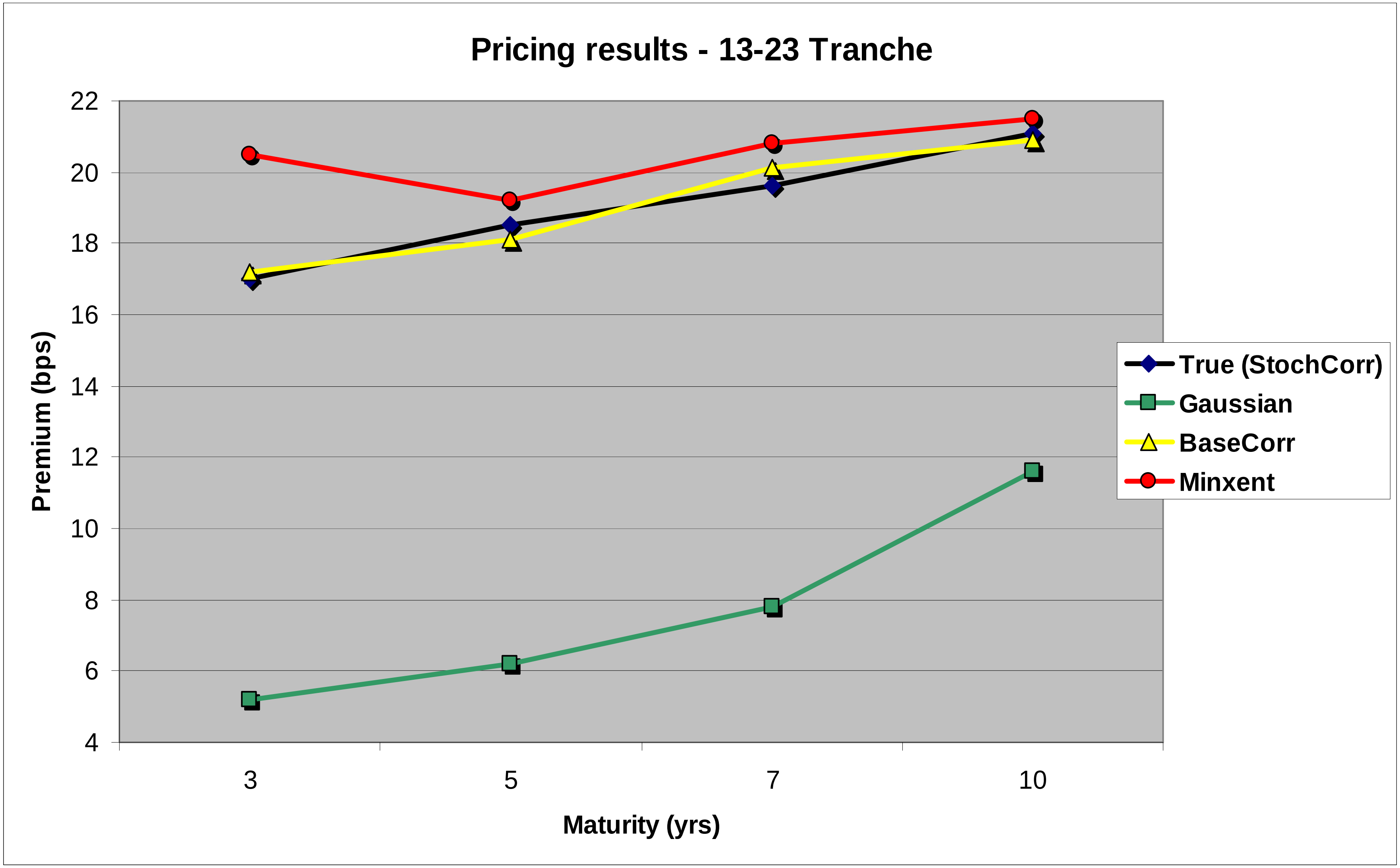
Computational Results

- Should be perfect calibration in theory – all errors are a result of inaccuracy in Monte Carlo integration
- Now use this minimum cross-entropy copula to price tranches with 'out-of-sample' threshold levels and compare to other models (Gaussian copula and base correlation)

Thresholds (%)	Maturity	'Market' premium bps	Gaussian bps	BaseCorr bps	Minxent bps
3-8	**3**	**106.3**	194.8	121.6	**112.2**
8-13	**3**	**31.8**	37.2	28.7	**32.7**
13-23	**3**	**17.0**	5.2	17.2	**20.5**
3-8	**5**	**159.2**	236.5	154.9	**162.0**
8-13	**5**	**35.6**	46.6	34.9	**35.5**
13-23	**5**	**18.5**	6.2	18.1	**19.2**
3-8	**7**	**213.2**	276.6	199.2	**215.0**
8-13	**7**	**41.0**	56.0	42.5	**41.6**
13-23	**7**	**19.6**	7.8	20.1	**20.8**
3-8	**10**	**294.8**	340.9	267.0	**292.4**
8-13	**10**	**52.4**	75.5	55.2	**53.3**
13-23	**10**	**21.1**	11.6	20.9	**21.5**
Total error	**-**	**-**	**-**	**-**	**21.2**


Pricing results - 3-8 Tranche
350
300
250
200
150
100
Premium (bps)
3
5
7
10
Maturity (yrs)
True (StochCorr)
Gaussian
BaseCorr
Minxent


Pricing results - 8-13 Tranche
80
70
60
50
40
30
20
Premium (bps)
3
5
7
10
Maturity (yrs)
True (StochCorr)
Gaussian
BaseCorr
Minxent


Pricing results - 13-23 Tranche
Premium (bps)
22
20
18
16
14
12
10
8
6
4
3
5
7
10
Maturity (yrs)
True (StochCorr)
Gaussian
BaseCorr
Minxent

Computational Results

- Next let us see if minimum cross-entropy can be fitted to real market data -- iTraxx and CDX across maturities

- Conducted calibration on two dates to see if the method is flexible enough to fit to different market conditions

- Used prior copula: Gaussian with $\rho = 0.25$

iTraxx 10 Apr 06

Thresholds (%)	Maturity	Market premium bps	Minxent premium bps	Absolute error bps
0-3	**5**	23.3%	23.4%	**0.1%**
3-6	**5**	68.0	67.8	**0.2**
6-9	**5**	19.0	19.9	**0.9**
9-12	**5**	9.5	10.1	**0.6**
12-22	**5**	4.5	5.3	**0.8**
0-3	**7**	43.1%	43.4%	**0.3%**
3-6	**7**	202.0	204.2	**2.2**
6-9	**7**	47.5	50.0	**2.5**
9-12	**7**	24.5	26.9	**2.4**
12-22	**7**	9.0	8.8	**0.2**
0-3	**10**	54.3%	53.9%	**0.4%**
3-6	**10**	580.0	580.4	**0.4**
6-9	**10**	117.0	117.2	**0.2**
9-12	**10**	51.5	52.0	**0.5**
12-22	**10**	20.0	20.6	**0.6**
Total error	-	-	-	**12.3**

iTraxx 21 Jun 05

Thresholds (%)	Maturity	Market premium bps	Minxent premium bps	Absolute error bps
0-3	**5**	30.2%	30.2%	**0.0%**
3-6	**5**	202.5	197.1	**5.4**
6-9	**5**	89.0	95.2	**6.2**
9-12	**5**	53.5	60.3	**6.8**
12-22	**5**	21.5	24.7	**3.2**
0-3	**10**	46.8%	48.0%	**1.2%**
3-6	**10**	399.0	394.7	**5.7**
6-9	**10**	206.0	199.3	**6.7**
9-12	**10**	137.0	128.8	**8.2**
12-22	**10**	62.5	58.3	**4.2**
Total error	-	-	-	**47.7**

CDX 4 Apr 06 Prior: Student t with $\rho = 0.25$ and $d = 2$

Thresholds (%)	Maturity	Market premium bps	Minxent premium bps	Absolute error bps
0-3	**5**	32.3%	32.7%	**0.6%**
3-7	**5**	92.0	88.4	**3.6**
7-10	**5**	17.5	12.6	**4.9**
10-15	**5**	9.0	7.7	**1.7**
15-30	**5**	4.5	6.1	**1.6**
0-3	**7**	50.0%	50.2%	**0.2%**
3-7	**7**	258.5	259.2	**0.7**
7-10	**7**	37.5	37.7	**0.2**
10-15	**7**	21.5	21.9	**0.4**
15-30	**7**	7.5	5.8	**1.7**
0-3	**10**	56.8%	57.1%	**0.3%**
3-7	**10**	617.5	616.5	**1.0**
7-10	**10**	105.0	106.3	**1.3**
10-15	**10**	51.5	49.9	**1.6**
15-30	**10**	14.5	12.0	**2.5**
Total error	-	-	-	**22.2**

CDX 4 Apr 06 5 year only Prior: Gaussian with $\rho = 0.25$

Thresholds (%)	**Maturity**	**Market premium bps**	**Minxent premium bps**	**Absolute error bps**
0-3	**5**	32.3%	32.6%	**0.3%**
3-7	**5**	92.0	94.2	**2.2**
7-10	**5**	17.5	17.2	**0.3**
10-15	**5**	9.0	9.9	**0.9**
15-30	**5**	4.5	5.5	**1.0**
Total error	**-**	-	-	**4.7**

Towards an exact solution to cross entropy CDO pricing

- Handle the uniformly distributed copula marginals by setting up the general marginal problem in Banach spaces over $T := [T_1, T_2]^n \subset \mathbb{R}^n$

$$\sup_{f \varepsilon L^1(T)} - \int_T \, f(x) \log \, f(x) dx$$

$$\text{s.t.} \qquad \int_T \, a_j(x) f(x) dx = \bar{a}_j \qquad j = 1, \ldots, m$$

$$\int_{T_1}^{p} \int_{T \setminus [T_1, T_2]} \, f(x) dx_{\sim i} dx_i = G_i(p) \quad p \varepsilon [T_1, T_2] \qquad i = 1, \ldots, n$$

- The constraint map carries $L^1(T)$ to $\mathbb{R}^m \times C(T)$ with dual $\mathbb{R}^m \times BV(T)$
- Then the convex dual function in terms of the Lagrange multipliers λ, Λ becomes

$$\mathcal{L}(\lambda, \Lambda; \hat{f}) = \log Z(\lambda, \Lambda) - \Sigma_{j=1}^{m} \lambda_j \bar{a}_j - \Sigma_{i=1}^{n} [G_i(T_2)\Lambda_i(T_2) - G_i(T_1)\Lambda_i(T_1)]$$
$$- \int_{T_1}^{T_2} \, \Lambda_i(p) dG_i(p)$$

- This can be globally minimized by the Banach space BFGS algorithm Edge & Powers (1976)

Example

- MaxEnt problem with Gaussian marginals

$$\max_{f \varepsilon C[-T,T]^2} -\int_{[-T,T]^2} f(x) \log f(x) dx$$

s.t.

$$\int_{[-T,T]^2} x_1 x_2 f(x) dx = \rho$$

$$\int_{-T}^{p} \int_{-T}^{T} f(x) dx_1 dx_2 = \Phi(p)$$

$$\int_{-T}^{T} \int_{-T}^{p} f(x) dx_1 dx_2 = \Phi(p)$$

where Φ is the standard normal distribution function

- Asymptotic solution in T is

$$\hat{f}(x) = \frac{1}{Z} \exp\{\lambda x_1 x_2 - \Lambda_1(x_1) - \Lambda_2(x_2)\} = \frac{1}{Z} \exp\{-\frac{1}{2(1-p^2)}(x_1^2 + x_2^2 - 2\rho x_1 x)\}$$

the bivariate standard normal density with normalizing constant Z

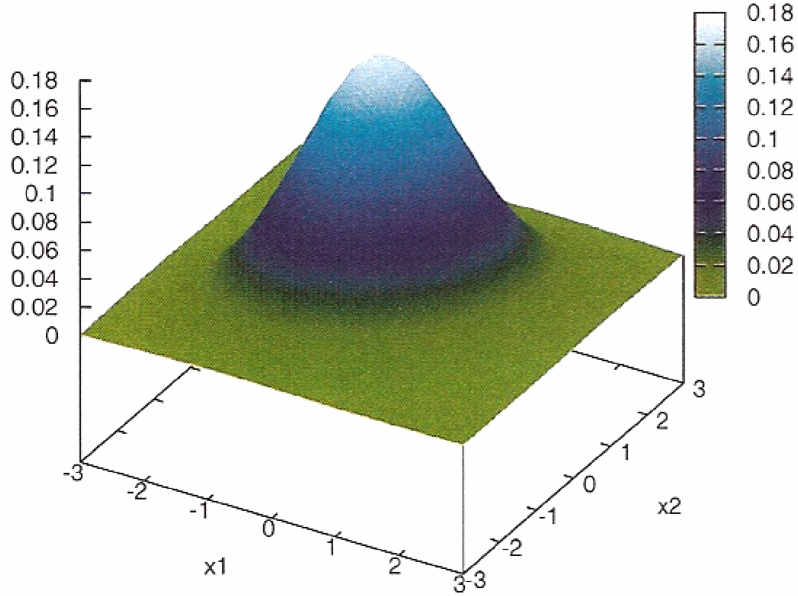


Figure 1: MaxEnt density, $\rho = 0.35$

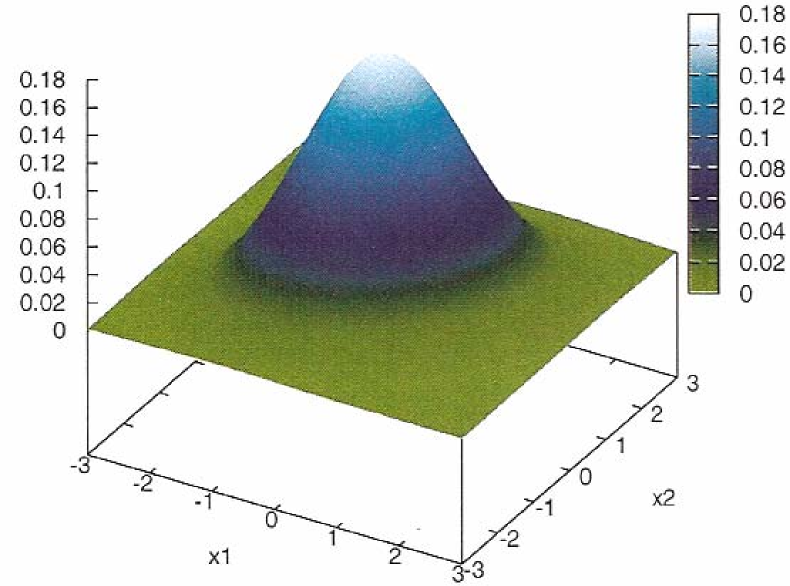


Figure 2: True bivariate normal density, $\rho = 0.35$

Conclusion

- Disadvantages and possible improvements
  - Computationally very slow since full Monte Carlo simulation is required but working on theory and speedups
  - Like typical copula approach the model is not dynamic but it does apply to multiple maturities

- Advantages of minimum cross-entropy copula approach
  - Provides motivating rationale for choice of copula
  - Near perfect calibration
  - Good out of sample performance
  - Heterogeneous portfolios possible
  - How does copula evolve from day to day and can its dynamics be simply specified?

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