Efficient Methods for managing Student's T Distribution

Equity, VaR and Credit Applications

William T. Shaw, June 2006

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Documentation

- Work available as working paper
- wp on my web site at:

  `www.mth.kcl.ac.uk/~shaww/web_page/papers/Tdistribution06.pdf`
- Final version to appear in the Journal of Computational Finance, Vol. 9 issue 4 (autumn 06), as “Sampling Student’s T Distribution - use of the inverse cumulative distribution function”
- There are further electronic resources on my web site as documented in the working paper/publication.

Terminology: The inverse cumulative distribution function has historically been known as the *Quantile function*.

Acknowledgements

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- I take no view on merits of copulas!

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- General power and asymptotic series
- Lots of technical detail in PDF/handout

Definition of T distribution

We shall begin by defining the Student's T-Distribution in a way that makes manifest one method of its simulation. We let $Z_0, Z_1, \ldots Z_n$ be standard normal random variables and set

$$\chi_n^2 = Z_1^2 + \cdots + Z_n^2$$

The density function of χ_n^2 is easily worked out, using moment generating functions, and is

$$q_n(z) = \frac{1}{2\Gamma(\frac{n}{2})} e^{-z/2} \left(\frac{z}{2}\right)^{\frac{n}{2}-1}$$

and gives a random variable with a mean of n and a variance of $2n$.

Normal variable + random variance

We now define a "normal variable with a randomized variance" in the form:

$$T = \frac{Z_0}{\sqrt{\chi_n^2/n}}$$

To obtain the density $f(t)$ of T we note that

$$f(t|\chi_n^2 = \nu) = \sqrt{\frac{\nu}{2\pi n}} e^{-\frac{t^2\nu}{2n}}$$

Then to get the *joint* density of T and χ_n^2 we need to multiply by $q_n(\nu)$.

The T PDF

Finally, to extract the univariate density for T, which we shall call $f_n(t)$, we integrate out ν. We observe that

$$\int_0^\infty f(t|\chi_n = \nu) q_n(\nu) d\nu$$

$$= \frac{1}{\sqrt{n\pi}} \frac{\Gamma[\frac{n+1}{2}]}{\Gamma[\frac{n}{2}]} \frac{1}{(1 + t^2/n)^{\frac{n+1}{2}}} = f_n(t)$$

A sample from this distribution can easily be obtained using $n + 1$ samples from the standard normal distribution. Known (to some), as is use of a normal variate divided by the square root of a scaled sample from χ^2, that being obtained by other methods.

Bailey's (1994) Method

Best known method to date for sampling without inverting from uniforms was published by R. Bailey as recently as 1994. He gave the polar method for the T analogous to the well-known method for the Normal: (you do not get two independent samples though)

1. Sample two uniform variates u and v from $[0, 1]$ and let $U = 2u - 1$, $V = 2v - 1$;
2. Let $W = U^2 + V^2$. If $W > 1$ return to step 1 and resample;
3. $T = U\sqrt{n(W^{-2/n} - 1)/W}$.

This wonderful algorithm also has the manifest limit that step 3 produces the result $T = U\sqrt{(-2\log W)/W}$ as $n \to \infty$, which is the well known polar formula for the Normal case.

The Direct Sampling Method

We want to get a grip on the use of the elementary result:

$$T = F_n^{-1}(U)$$

to define a sample from the T-distribution directly, where U is uniform and F_n is the cumulative density function for the T distribution with n degrees of freedom. We use the F^{-1} notation to denote the functional inverse (and not the arithmetical reciprocal!).

Why do this?

- We might be MUCH more efficient
- Exhibit the polynomial method, which is!
- In Quasi-Monte-Carlo want to sample e.g. basket size m using space-filling hypercube.
  Polar-Marsaglia/Box-Muller map uses hypercube dimensions $2m$.
- We have same motivation as Moro (RISK, 1995) to cut down to m by use of direct inverse.
- Inverse CDFs helpful in copula-based pricing via simulation

The exact forward CDF

$$F_n(x) = \int_{-\infty}^{x} f_n(t)dt$$

$$F_n(x) = \frac{1}{\sqrt{n\pi}} \frac{\Gamma[\frac{n+1}{2}]}{\Gamma[\frac{n}{2}]} \int_{-\infty}^{x} \frac{1}{(1+t^2/n)^{\frac{n+1}{2}}} dt$$

$$F_n[x] = \frac{1}{2} + \frac{x\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi}\Gamma\left(\frac{n}{2}\right)} {}_2F_1\left(\frac{1}{2}, \frac{n+1}{2}; \frac{3}{2}; -\frac{x^2}{n})\right)$$

$$F_n[x] = \frac{1}{2}\left(1 + \mathsf{sgn}(x)(1 - I_{(\frac{n}{x^2+n})}\left(\frac{n}{2}, \frac{1}{2}\right)\right)\right)$$

Uses the regularized beta function $I_x(a,b) = \frac{B_x(a,b)}{B(a,b)}$

Formal Inversion

$B(a,b)$ is the ordinary β-function and $B_x(a,b)$ is the incomplete form

$$B_x(a,b) = \int_0^x t^{(a-1)}(1-t)^{(b-1)}dt$$

This may be formally inverted to give

$$F_n^{-1}[u] = \sqrt{n\left(\frac{1}{I^{-1}_{\text{If}\left[u<\frac{1}{2},2u,2(1-u)\right]}\left(\frac{n}{2},\frac{1}{2}\right)}-1\right)}$$

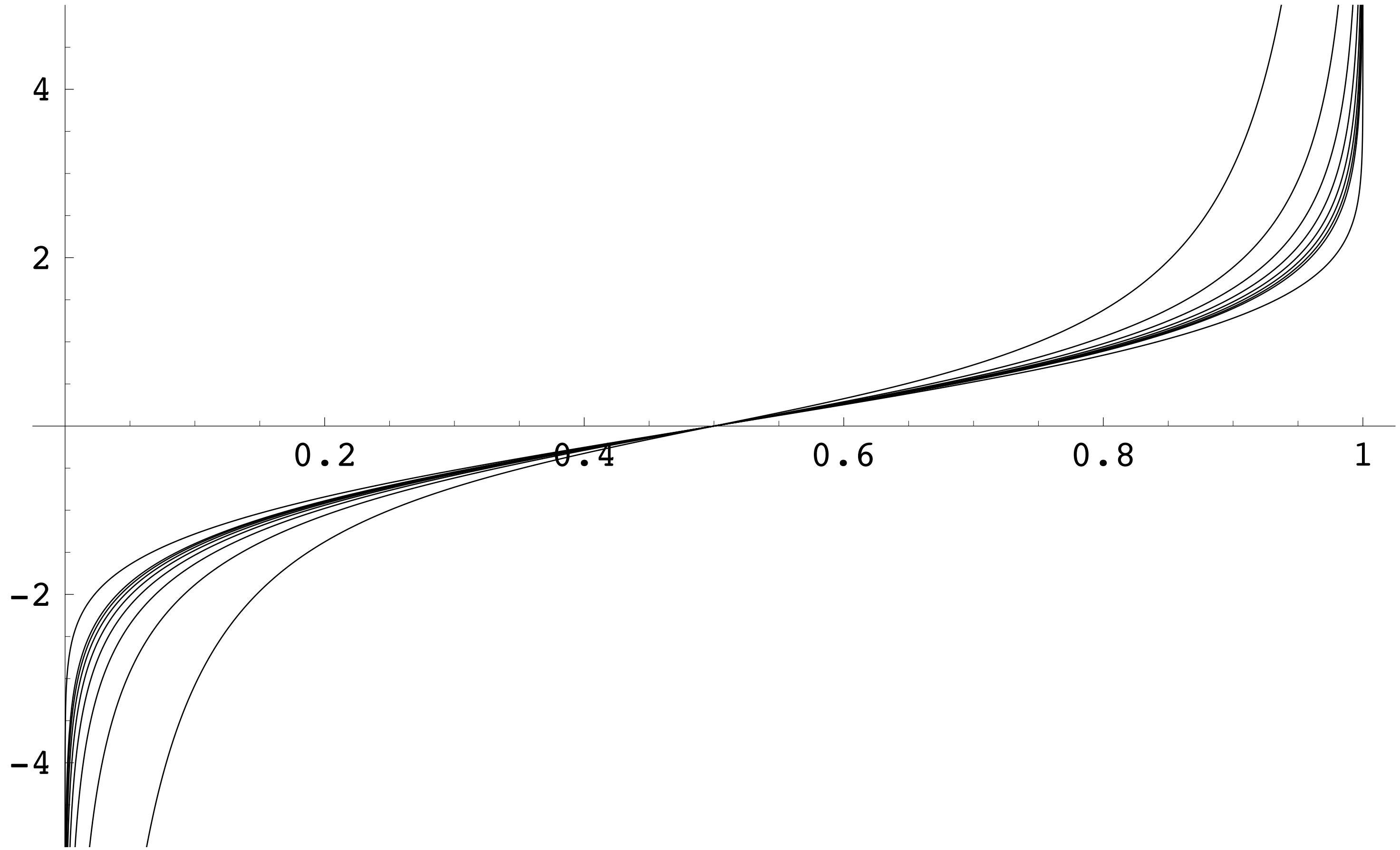
$$\times \text{sgn}\left(u-\frac{1}{2}\right)$$

Use of formal inverse

If one can access an accurate representation of the inverse β-function then one can work directly with the formal inverse. As an example, we can use a representation in *Mathematica* or other suitable symbolic system to visualize the inverse for various values of n. This is not computationally efficient any more than the inverse error function is an efficient way of sampling from a Normal, but a good check.

Also only OK for advanced mathematical computation languages. We shall see later how to write down series for these functions that are more directly useful in low-level languages such as C++. Such representations do not give any *insight* into what is happening.

Array of Inverse CDFs



Plot of inverse for the cases $n = 1, 2, 3, 4, 5, 6, 7, 8, \infty$

Getting some insight into inversion

Tabulate the first few:

n	$F_n(x)$
1	$\frac{1}{2} + \frac{1}{\pi}\tan^{-1}(x)$
2	$\frac{1}{2} + \frac{x}{2\sqrt{x^2+2}}$
3	$\frac{1}{2} + \frac{1}{\pi}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + \frac{\sqrt{3}x}{\pi(x^2+3)}$
4	$\frac{1}{2} + \frac{x\left(x^2+6\right)}{2(x^2+4)^{3/2}}$
5	$\frac{1}{2} + \frac{1}{\pi}\tan^{-1}\left(\frac{x}{\sqrt{5}}\right) + \frac{\sqrt{5}x\left(3x^2+25\right)}{3\pi(x^2+5)^2}$
6	$\frac{1}{2} + \frac{x\left(2x^4+30x^2+135\right)}{4(x^2+6)^{5/2}}$

Odd n a mix of algebraic and trigonometric functions (hard), even n always algebraic. More tractable. Defer non-integer n.

The beautiful algebra of even n

The case of even n can be massively simplified. We set

$$p = n + x^2$$

$$\alpha = 1 - 4(u - \frac{1}{2})^2$$

and manipulate the formulae arising from the family of equations

$$u = F_n(x)$$

We obtain an interesting sequence of purely polynomial equations, about half of whose terms vanish! We call these the “resolvent polynomials” of the T distribution.

The Resolvent Polynomials

$$n = 2 : \alpha p - 2 = 0$$

$$n = 4 : \alpha p^3 - 12p - 16 = 0$$

$$n = 6 : \alpha p^5 - 135p^2 - \frac{1215}{4}p - \frac{2187}{2} = 0$$

$$n = 8 : \alpha p^7 - 2240p^3 - 7168p^2 - 35840p - 204800 = 0$$

$$n = 10 : \alpha p^9 - \frac{196875p^4}{4} - \frac{1640625p^3}{8} - \frac{10546875p^2}{8} - \frac{615234375p}{64} - \frac{2392578125}{32} = 0$$

We now proceed to extract some solutions. The on-line supplement code to generate the resolvent polynomial equations for even $n \leq 20$ and exhibits them.

Problem is now Polynomial Solution

Exact solutions can be written down. We already knew $n = 1$ case from Cauchy distribution:

$$x = \tan(\pi(u - \frac{1}{2}))$$

The case $n = 2$ was known to Hill in 1970. This is now trivial as the resolvent polynomial is linear. After some simplification we obtain

$$x = \frac{2\sqrt{2}\left(u - \frac{1}{2}\right)}{\sqrt{1 - 4\left(u - \frac{1}{2}\right)^2}}$$

The case $n = 4$

The resolvent polynomial equation is now a cubic in reduced form (no quadratic term). A cubic in reduced form may be solved by exploiting the identity

$$(p-A-B)*(p-A\omega-B\omega^2)*((p-A\omega^2-B\omega) \equiv p^3-3ABp-A^3-B^3$$

where $\omega = e^{\frac{2\pi i}{3}}$ is the standard cube root of unity. We just have to solve some auxiliary equations for A and B (modern formulation of Tartaglia's solution!) Some work and simplification gives us:

$$p = \frac{4}{\sqrt{\alpha}} \cos\left(\frac{1}{3}\cos^{-1}\sqrt{\alpha}\right)$$

where, as before, $x = \operatorname{sign}(u - \frac{1}{2})\sqrt{p-4},\ \alpha = 1 - 4(u - \frac{1}{2})^2$.

Solution for higher n

Basically use your favourite polynomial solver. Paper suggests Newton-Raphson with a sensible seed value. Highly efficient due to nearly half the coefficients being zero. For example, $n = 6$ scheme is

$$p_{k+1} = \frac{2\left(8\alpha p_k^5 - 270 p_k^2 + 2187\right)}{5\left(4\alpha p_k^4 - 216 p_k - 243\right)}$$

For $n = 8$ we have

$$p_{k+1} = \frac{2}{7}\left(3p_k + \frac{640\left(p_k\left(p_k\left(p_k + 4\right) + 24\right) + 160\right)}{p_k\left(\alpha p_k^5 - 960 p_k - 2048\right) - 5120}\right)$$

You can blast out inverses very fast and hence do fast MC sampling by applying these maps to samples from a uniform distribution.

Comments on $n = 4$

There is more reason to consider this case than mere inversion “doability”. The case $n = 4$ has finite variance. As we decrease n from ∞ then $n = 4$ is the point at which the kurtosis becomes infinite. Therefore an interesting case from a risk management view, as it represents a good alternative base case to consider along with normal case. So perhaps VaR simulations might be tested in the log-Student-($n = 4$) case as well as log-normal case. Note that the variance also diverges as $n \to 2_+$.

Recent independent evidence supports $n = 4$ as an interesting case for purely financial reasons - Fergusson and Platen (AMF, Spring 2006) suggest that $n = 4$ T is a good representation of index returns in a global setting.

Managing general real n

These polynomial results are very pretty but do not extend to n odd. An approach to case $n = 3$ is discussed in the paper. It looks like we may have exhausted the simple and elegant methods and now need some more symbolic firepower. We are trying to solve

$$u - \frac{1}{2} = \frac{1}{\sqrt{n\pi}} \frac{\Gamma[\frac{n+1}{2}]}{\Gamma[\frac{n}{2}]} \int_0^x \frac{1}{(1 + s^2/n)^{\frac{n+1}{2}}} ds$$

So x is an odd function of $u - 1/2$. We work with the problem in the normalized power series form:

$$x = F_n^{-1}(u) = v + \sum_{k=1}^{\infty} c_k v^{2k+1}, \ v = (u - 1/2)\sqrt{n\pi} \frac{\Gamma[\frac{n}{2}]}{\Gamma[\frac{n+1}{2}]}$$

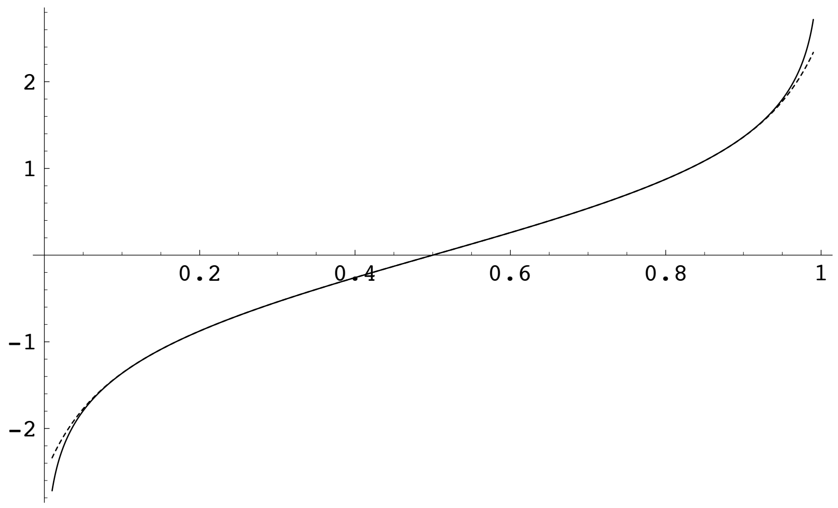
Symbolic Power Series Solution

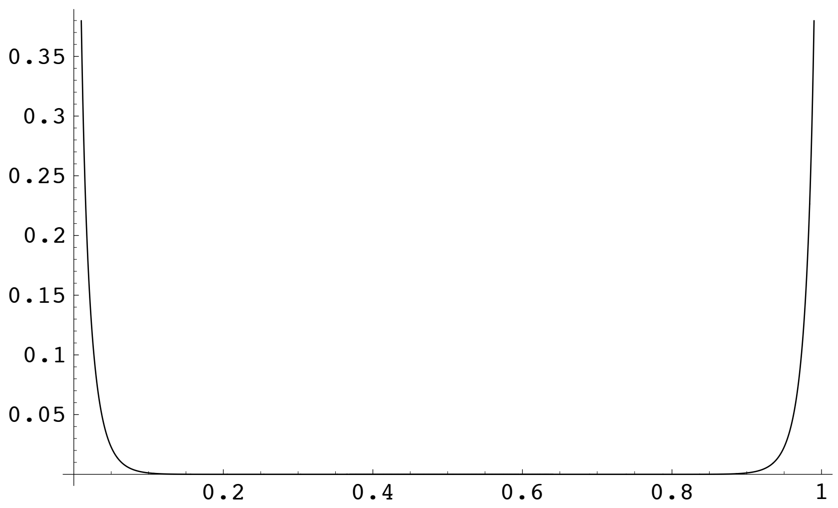
We use a non-linear iteration to obtain the coefficients as:

$$c_1 = \frac{1}{6} + \frac{1}{6n}$$
$$c_2 = \frac{7}{120} + \frac{1}{15n} + \frac{1}{120n^2}$$
$$c_3 = \frac{127}{5040} + \frac{3}{112n} + \frac{1}{560n^2} + \frac{1}{5040n^3}$$
$$c_4 = \frac{4369}{362880} + \frac{479}{45360n} - \frac{67}{60480n^2} + \frac{17}{45360n^3} + \frac{1}{362880n^4}$$

and so on. The written paper goes up to c_9; the on-line material has terms up to c_{30}.

Series vs exact, $n = 11$, terms to c_9





A series for the tail

We proceed as before, and a little experimentation tells us what series to seek. We solve for x as a function of w, where

$$(1-u)\sqrt{n\pi}\frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n+1}{2})} = w = \int_x^\infty \frac{1}{(1+\frac{s^2}{n})^{\frac{n+1}{2}}} ds$$

The series solution is written in the following form

$$x = \sqrt{n}\left(\sqrt{n}w\right)^{-1/n}\left(1 + \sum_{k=1}^{\infty}(\sqrt{n}w)^{\frac{2k}{n}} d(k)\right)$$

and giving the problem to a symbolic cruncher gives the results

Tail series coefficients

$$d_1 = -\frac{(n+1)}{2(n+2)}$$

$$d_2 = -\frac{n(n+1)(n+3)}{8(n+2)^2(n+4)}$$

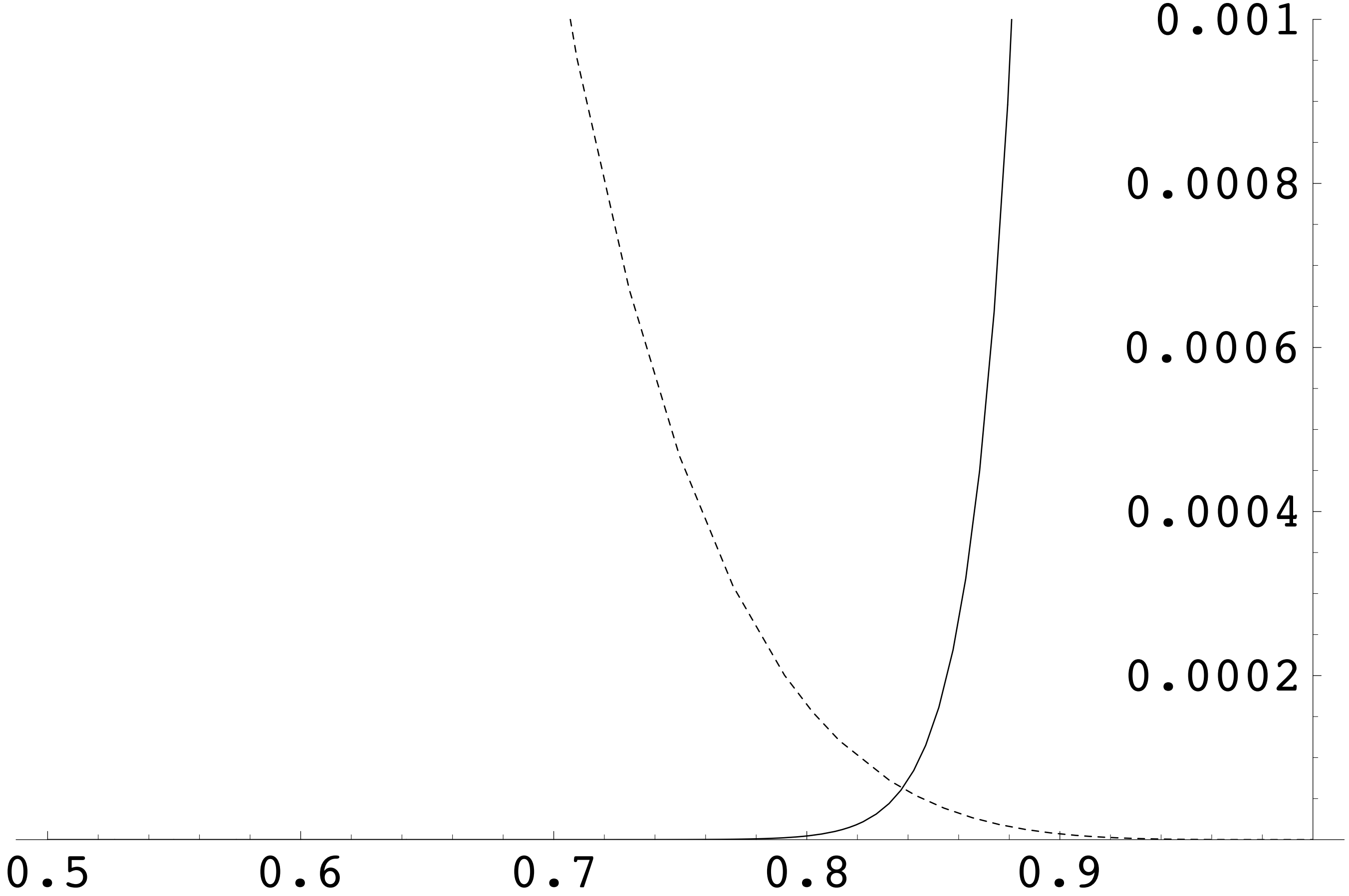
$$d_3 = -\frac{n(n+1)(n+5)\left(3n^2+7n-2\right)}{48(n+2)^3(n+4)(n+6)}$$

$$d_4 = -\frac{n(n+1)(n+7)\left(15n^5+154n^4+465n^3+286n^2-336n+64\right)}{384(n+2)^4(n+4)^2(n+6)(n+8)}$$

and so on, with more terms in the paper.

Combination for $n = 3$

Take 9-term power and 6 term tail (very short series!) and compare with exact solution. The graph shows the error.



Other material in paper

- Hazards of Cornish-Fisher expansions
- Making senses of the Excel function TINV
- Other benchmarks cases

Summary of Results on Simulation

- The iCDF in terms of inverse β-functions, suitable for benchmark computations;
- Exact solutions for the iCDF in terms of elementary functions for $n = 2, 4$, which are themselves of interest to "fat-tailed finance" applications;
- Fast iterative Newton-Raphson techniques the iCDF for even integer $n \leq 20$.
- A power series for the iCDF valid for general real n accurate except in the tails;
- A generalized power series for the tails that is good for low to modest n;
- A summary of known results on the Cornish-Fisher expansions, including tail problems.