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Dynamic Stochastic Programming for Asset Liability Management

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Outline

1. Introduction
2. Asset return and exchange rate dynamics
3. Scenario trees
4. Optimal dynamic asset allocation
5. Shaping portfolio NAV
6. System asset allocation backtests
7. Conclusions

1. Introduction

Experience with a variety of actual applications including:-

- **Long term asset allocation**
- **Asset liability management**
- **Derivative portfolio pricing and hedging strategies**
- **Risk management**
- **Capital allocation**
- **Real options evaluation**
- **Financially hedged logistics operations**

Financial modelling for pension plan management

The Problem:

- State pension schemes are currently under severe stress
- In the future new retirees will face a substantial **"pension gap"**
- Are private fund management companies in a position to fill it?

The pension gap problem

There are different ways to solve this problem:

A. **Individual** asset liability management

B. **Structured funds** with a guaranteed return on investment

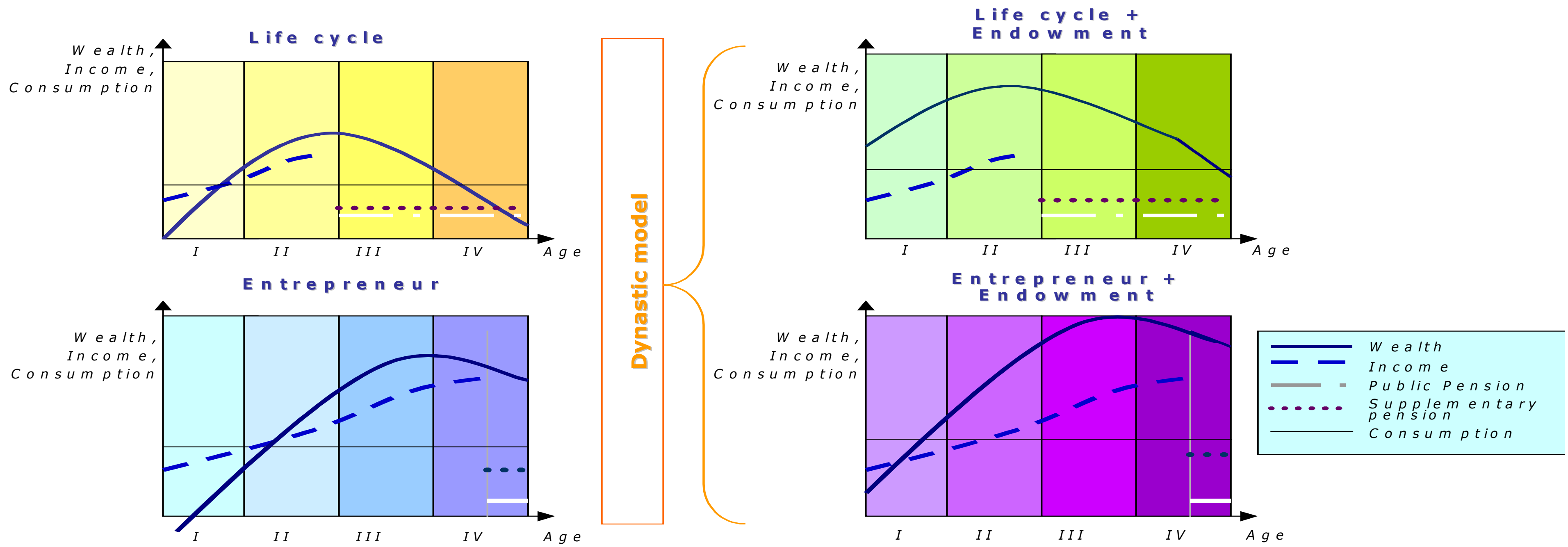
C. **New products** for retirement

Modelling individual liabilities and net cash flows is **different** from modelling at the pension fund or insurance company level – higher degrees of uncertainty than at pension/insurance level are involved!

Individual asset liability management

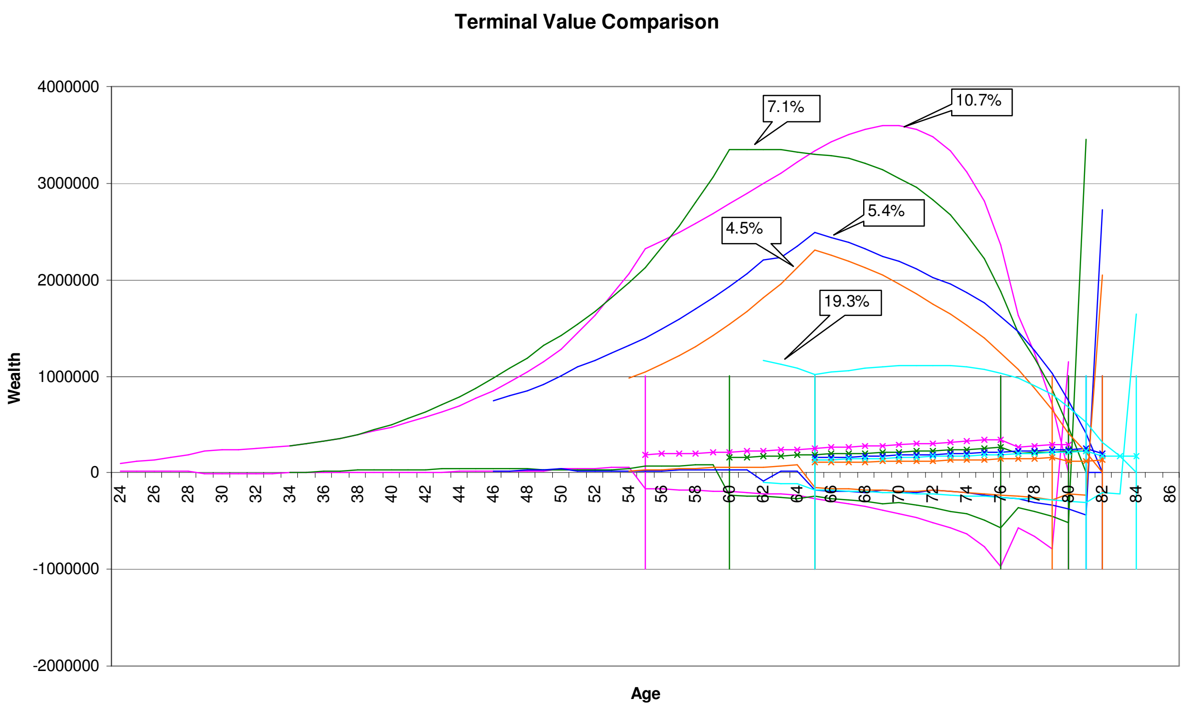
Classic **life cycle** models **cannot explain** the individual behaviour of **all investors**

Wealth-age and income-age profile: life-cycle and entrepreneur models + endowment



Individual asset liability management

- Modelling **individual liabilities** and **net cash flows** involves data gathering on individual households of different age groups and wealth with given income, liabilities and certain goals over life



Structured funds with a guaranteed return on investment

- This approach involves the design of a fund (fund of funds) with guarantees which are carefully matched to the risk profiles of the specified class of individual investors – the guarantees are sufficient to cover the aggregated investors' liabilities and their consumption

New products for retirement

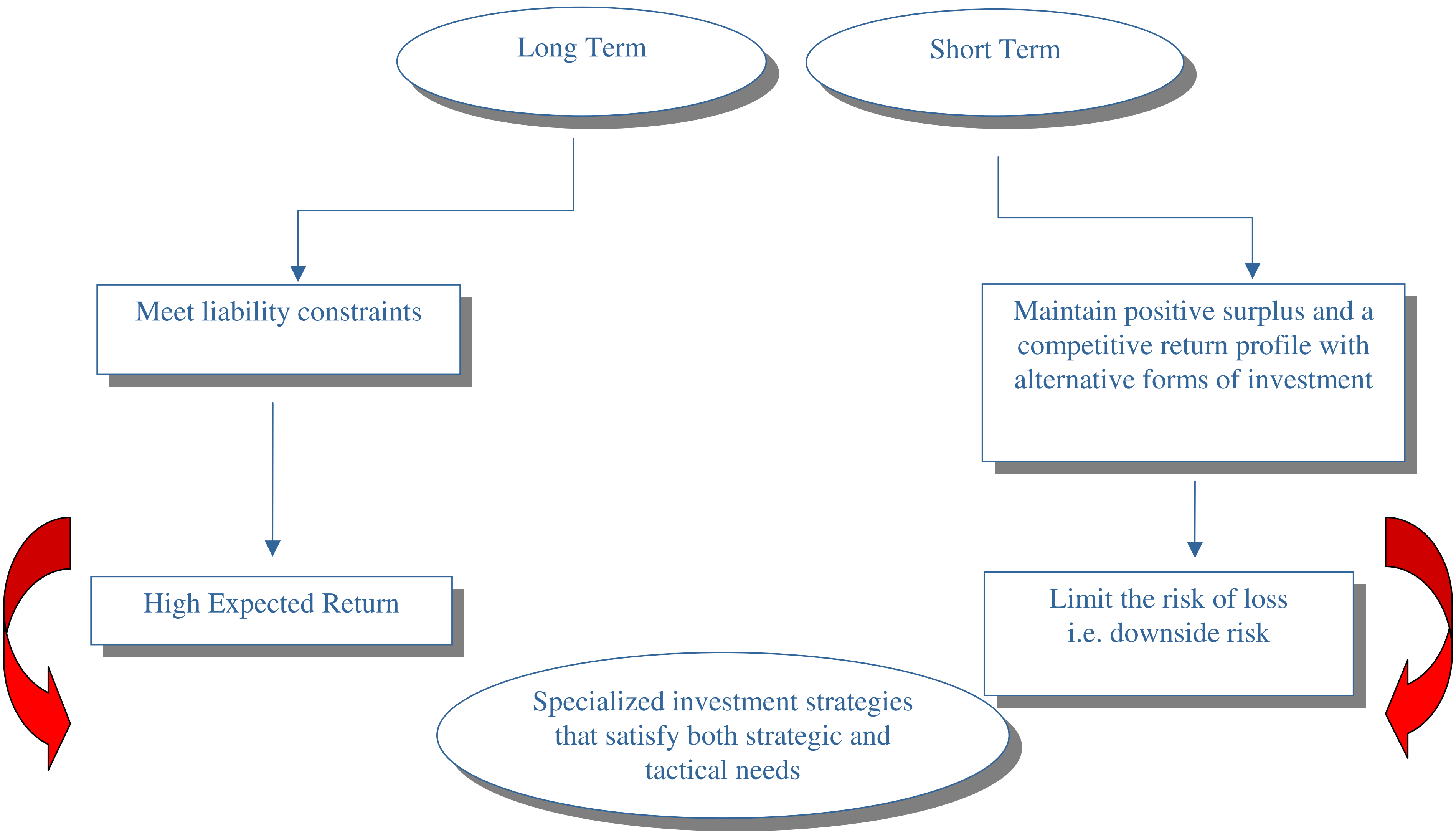
- More research is needed to understand the stochastic liabilities
- Guaranteed fund products – a first step towards the construction of a range of retirement products Dempster et al (2003)
- Alternatives requiring similar analysis are guaranteed annuity options Wilkie et al (2003) Boyle & Hardy (2003)

Financial modelling for pension plan management

What is needed for fund design supporting new products?

- Capacity to perform long-term asset allocation
- Ability to guarantee returns over long time horizons, in the face of uncertainty about:
  - changing economic and market conditions
    Need for active risk management
  - changing demographic and actuarial conditions
    Need for unified asset liability management

The pension plan "dilemma"



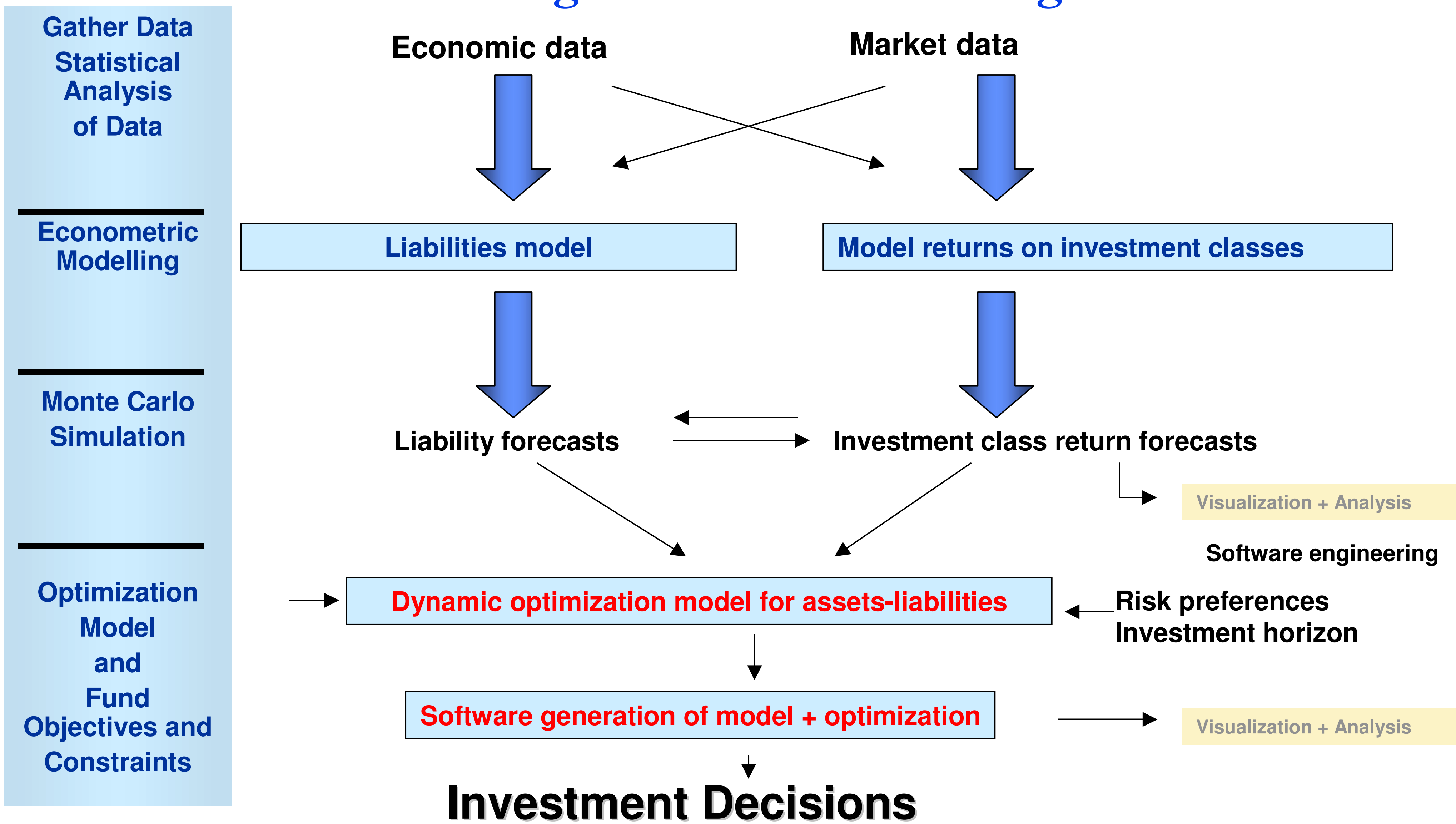
Strategic ALM problem definition

Given a set of uncertain assets and liabilities, a fixed planning horizon and set of rebalance dates, find the trading strategy that maximizes expected risk adjusted net return subject to the constraints

Dynamic stochastic programming (DSP) solution

- Discrete set of trading times
- Uncertainty represented by a finite set of scenarios
- Assets are stocks, bonds and cash denominated in different currencies
- Fund operates from the point of view of one currency called the home currency
- Fund begins with an initial wealth in the home currency
- Fund may face market frictions such as proportional transaction costs and portfolio restrictions such as position limits

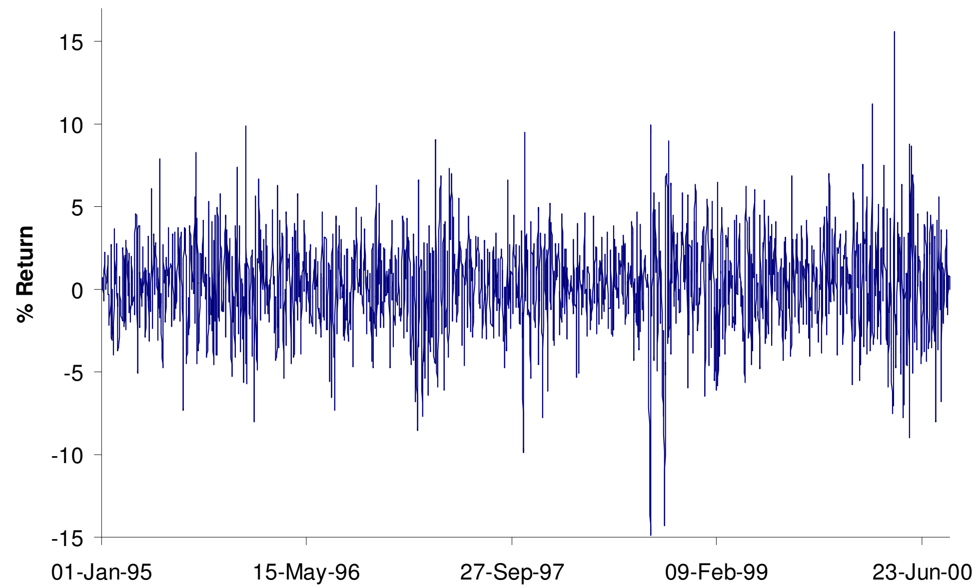
Strategic Financial Planning



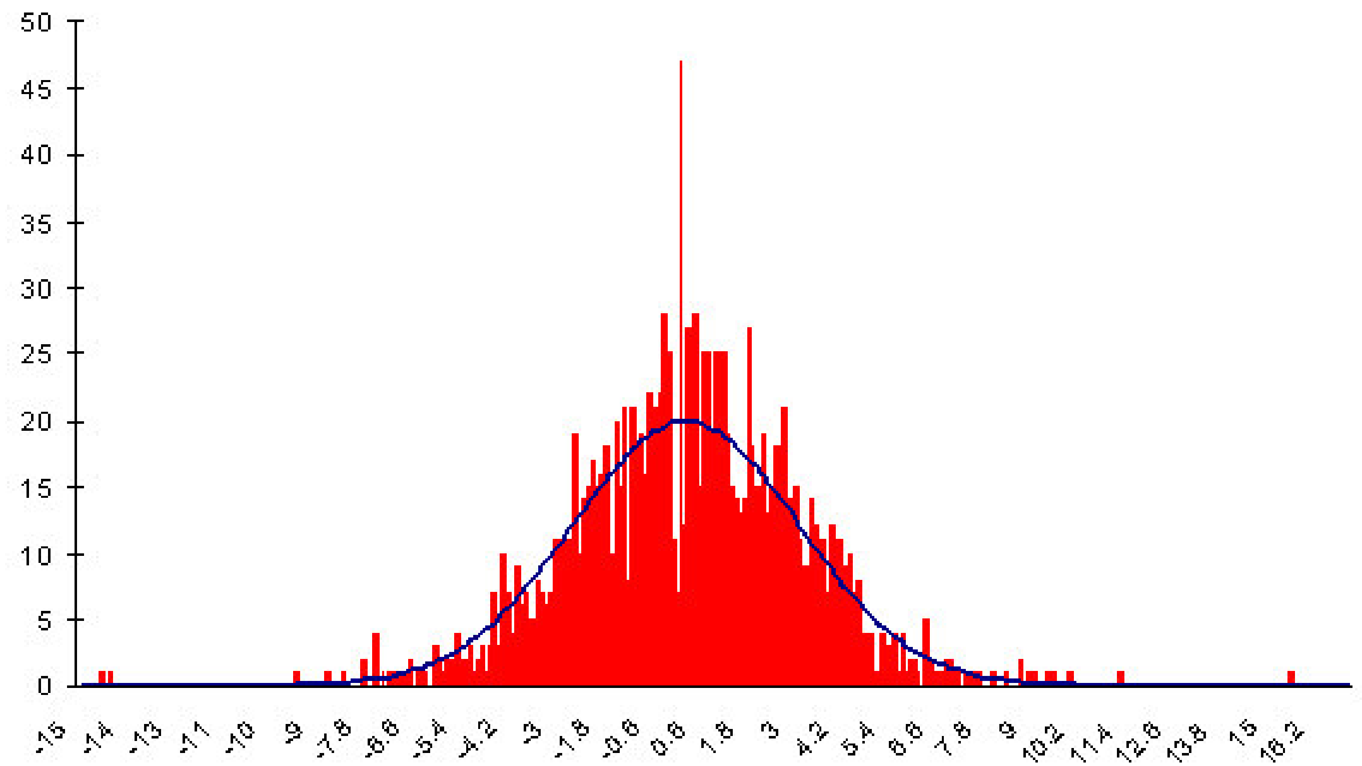
2. Asset Return and Exchange Rate Dynamics

Histograms from financial time series

Cisco closing stock prices and daily returns - *Jan 1995 - Sept 2000*

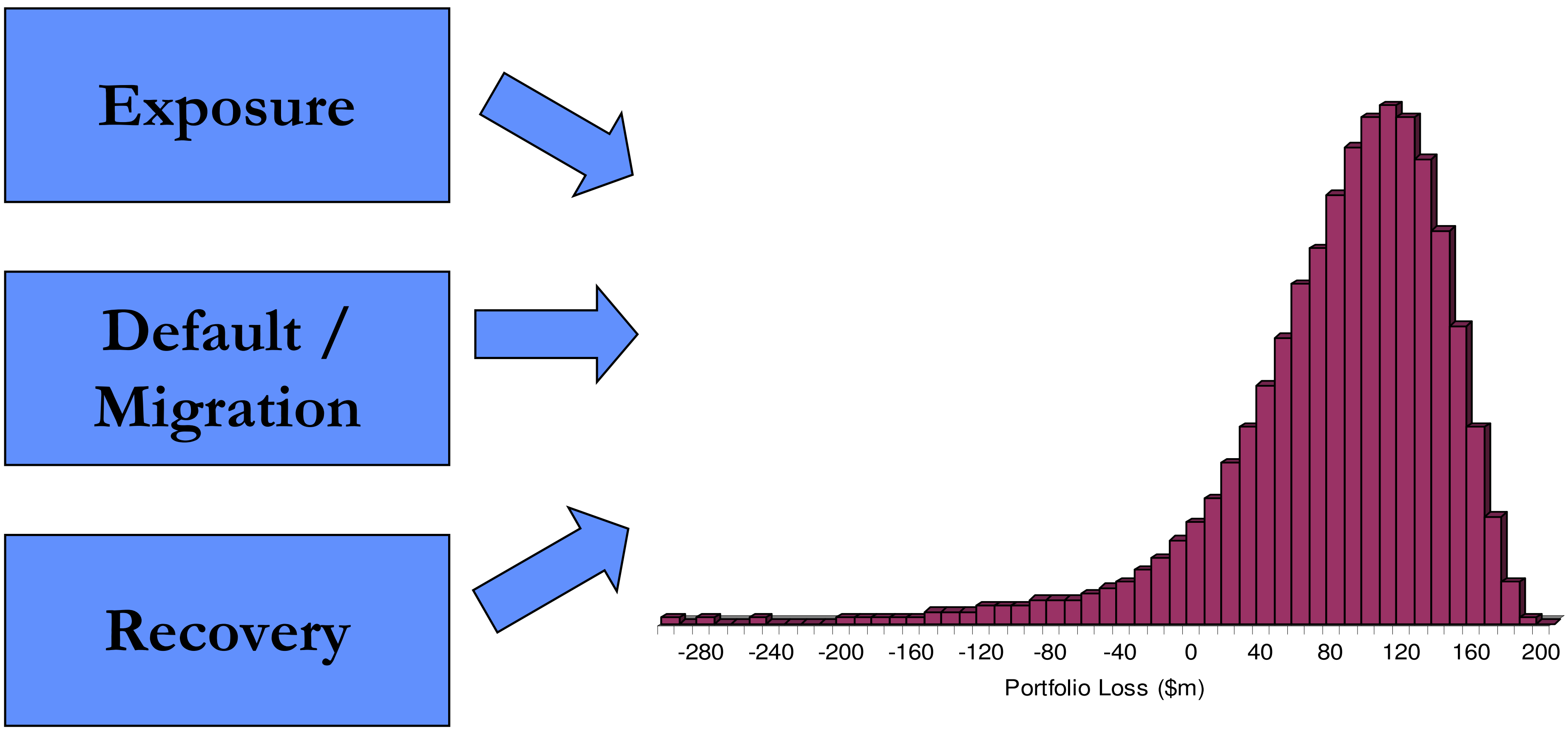


Gaussian two moment fit to the symmetric Cisco return histogram with long (fat) tails



Source: J-P Bouchaud & M Potters, *Theory of Financial Risks,* CUP (2000)

Skewed credit return distribution (skew measured by the 3rd moment)



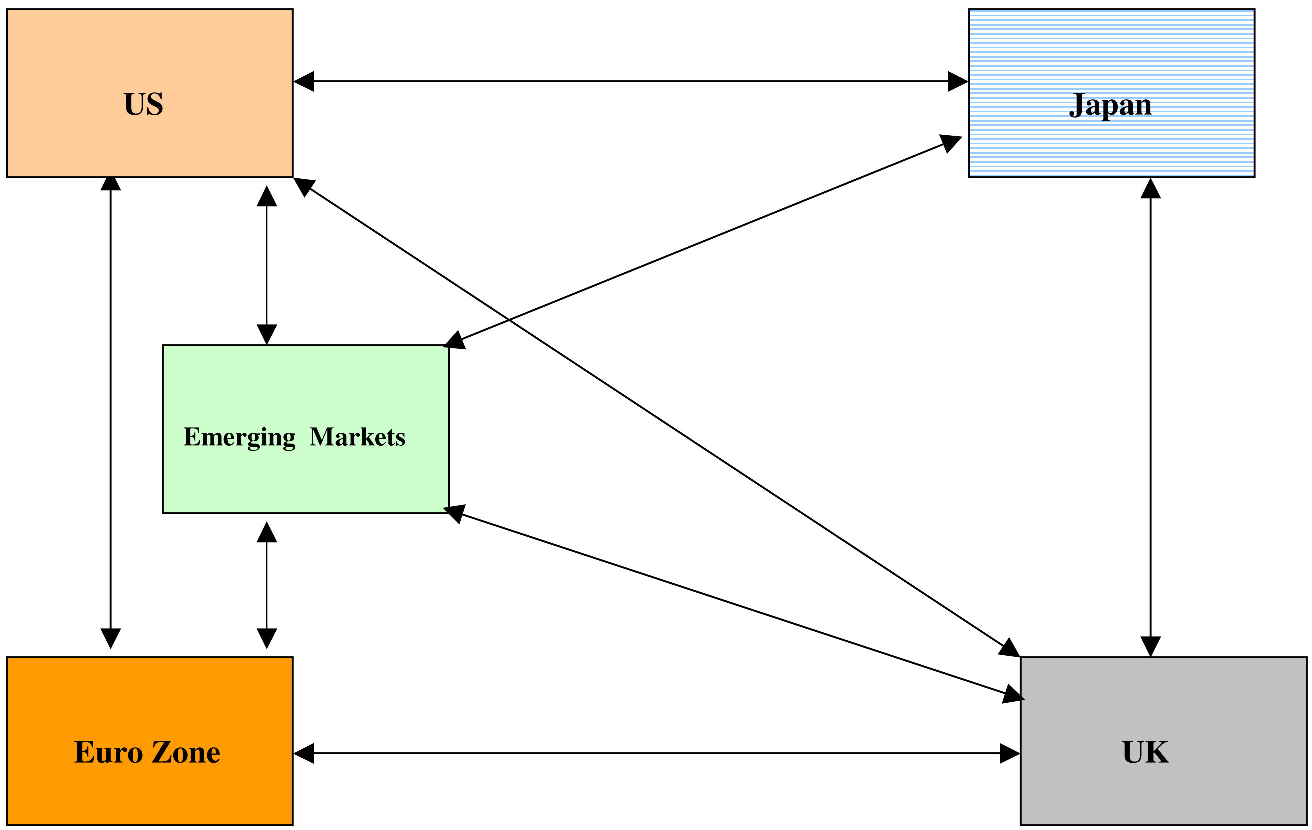
Asset returns and exchange rates

- Scenarios represent uncertain future asset returns and exchange rates
- Scenarios are generated by simulating from an underlying dynamic model of the assets and exchange rates
- 3 types of dynamic models considered
  - **Nonlinear** model (BMSIM)
  - **Vector autoregressive** (VAR) models (VARSIM and USMACRO)
  - **Historical bootstrap** model (HSIM)

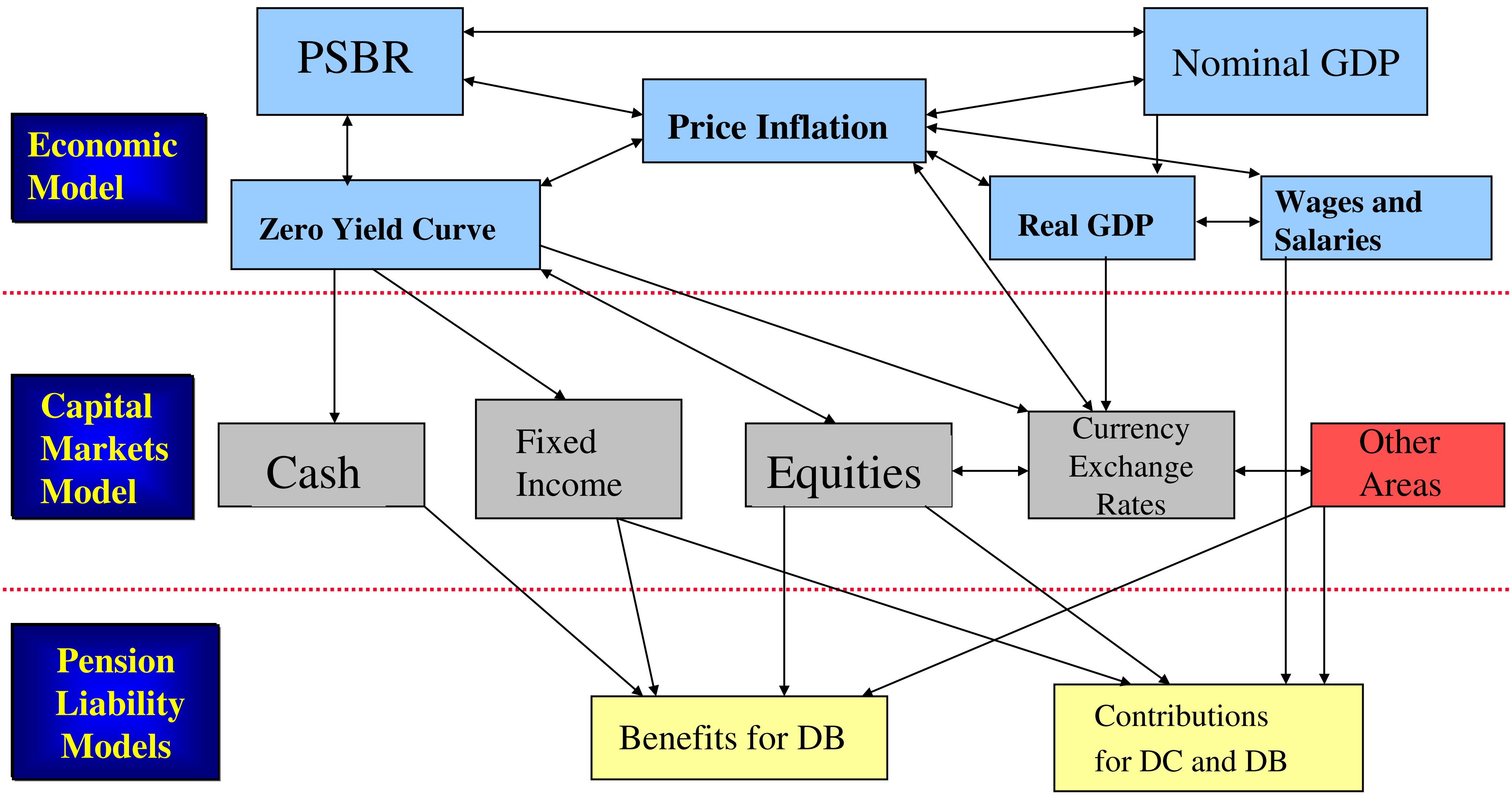
Nonlinear (BMSIM)

- Home currency is USD
- 2nd order model for
  - US stocks, long rate, short rate, GDP, inflation, wages and public sector borrowing
  - UK stocks, long rate, short rate and UK/US exchange rate
  - EU stocks, long rate, short rate and EU/US exchange rate
  - Japanese stocks, long rate, short rate and JP/US exchange rate
  - Emerging markets stocks and bonds
- Collection of country models linked by correlated innovation terms and exchange rate equations

Asset return model: Global structure



Asset return model: Currency area structure



VaR model (VARSIM)

- Home currency is EUR
- 3rd order model for
  - EU stocks, bonds and cash
  - US stocks and US/EU exchange rate
  - Japanese stocks and JP/EU exchange rate
- Can be expressed as

$$y_t = \mu + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \phi_3 y_{t-3} + \eta_t$$

where y_t are the variable net returns at time t, μ is a vector of constants, ϕ_i is the lag i coefficient matrix and the η_t are vectors distributed $N(0,\Sigma)$ and are uncorrelated across time

- Variables are only allowed to depend on other variables in its country (for EU variables this includes the US/EU exchange rate)

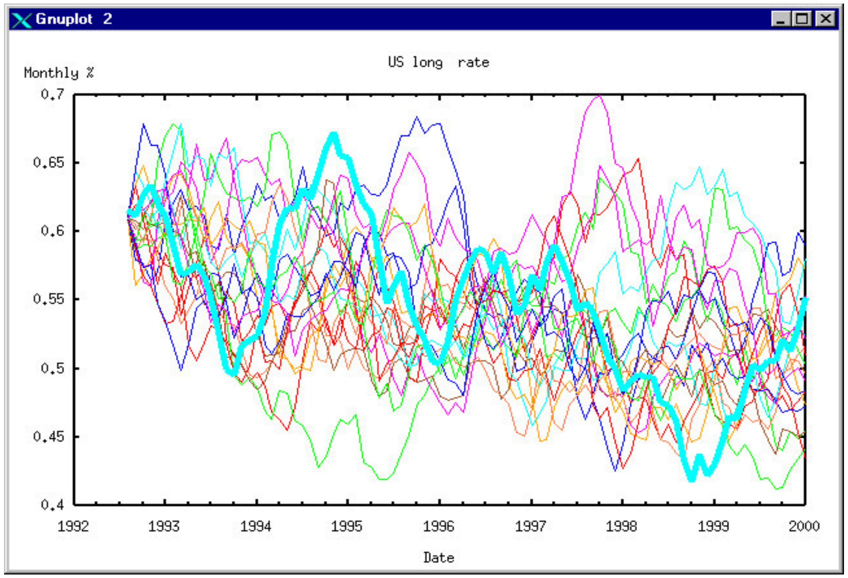
Estimation and simulation

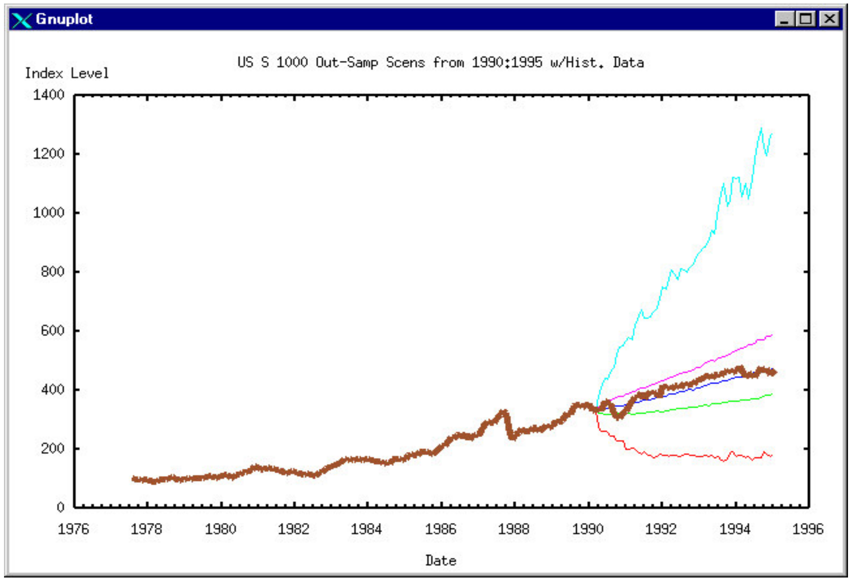
- Estimated using monthly data (July 1977 – February 2002)
- Estimated using regression/maximum likelihood
- Simulation using Monte Carlo methods with Gaussian and t innovations

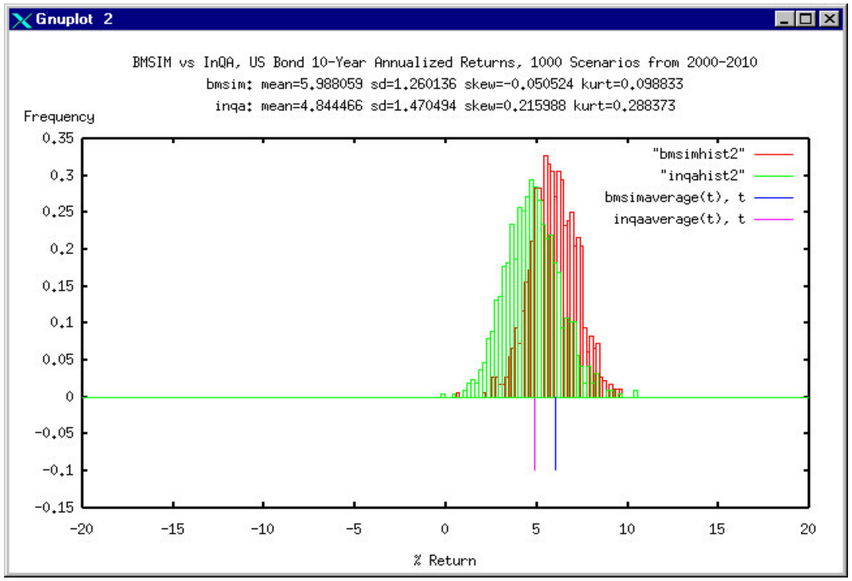
Historical bootstrap model (HSIM)

- Home currency is USD
- Bootstrap model for
  - US stocks, long rate and short rate
  - UK stocks, long rate, short rate and UK/US exchange rate
  - EU stocks, long rate, short rate and EU/US exchange rate
  - Japanese stocks, long rate, short rate and JP/US exchange rate
- Given a historical time series of returns simulation draws returns randomly from the corresponding histogram
- Monthly returns from July 1977 – February 2002 used

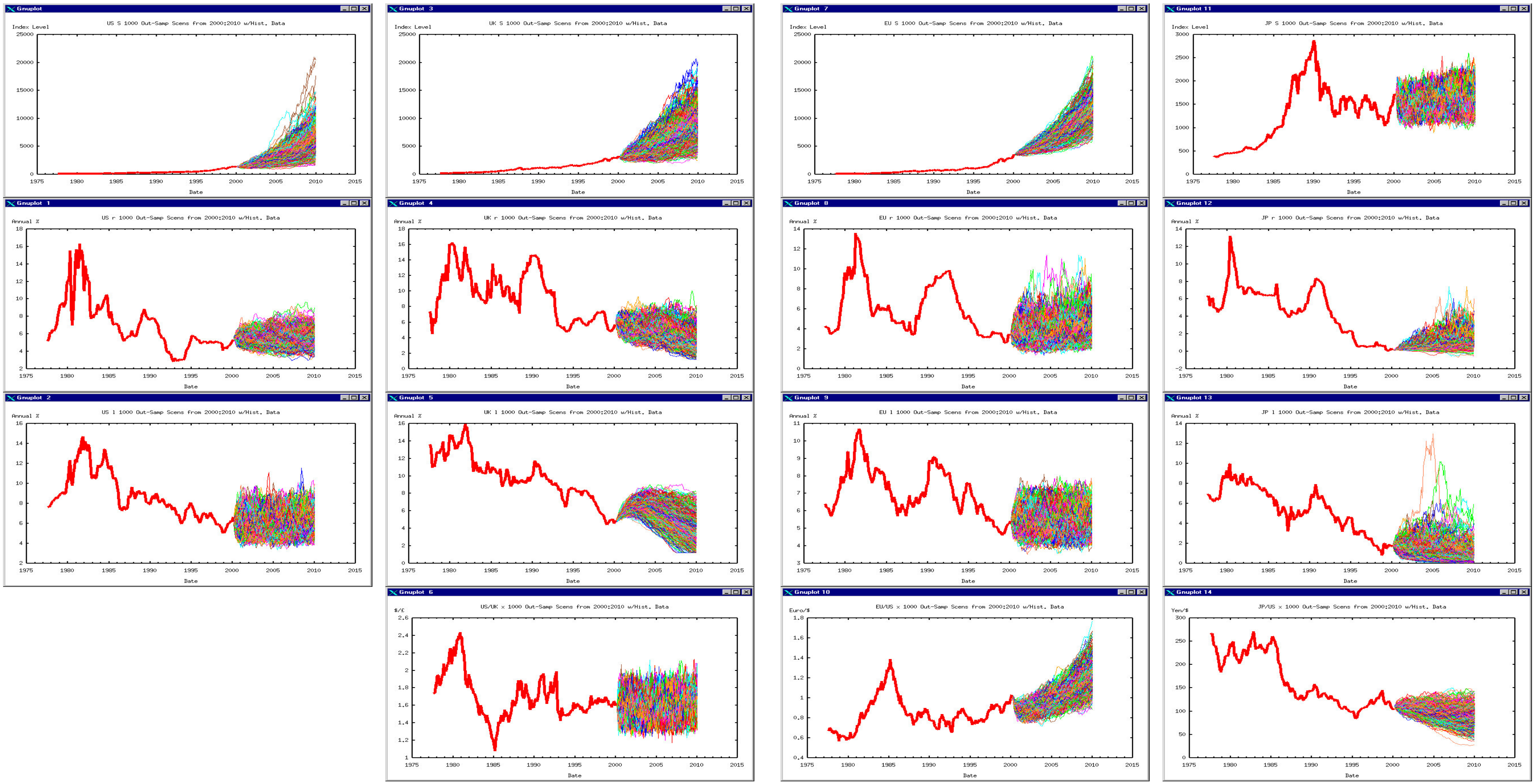
Visualization of scenarios



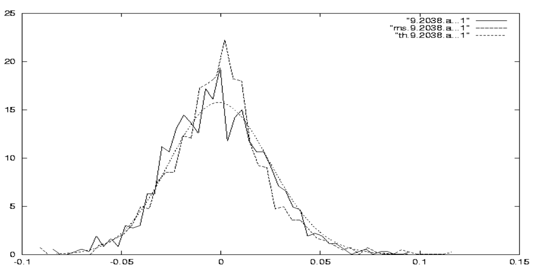




Ten year out-of-sample scenario forecasts   1977-2000-2010

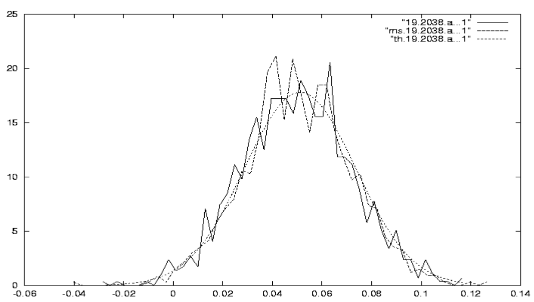


Simulated Processes



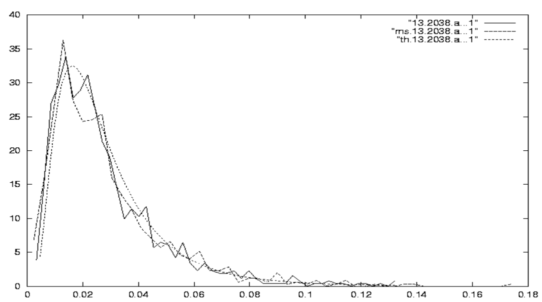
TIPS

GBM



TIPS Coupon

OU

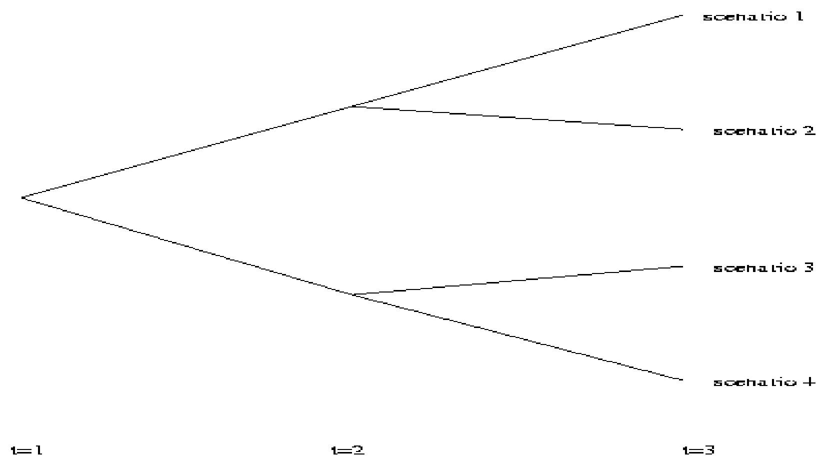


International
Equity Dividends

Geometric OU

3. Scenario Trees

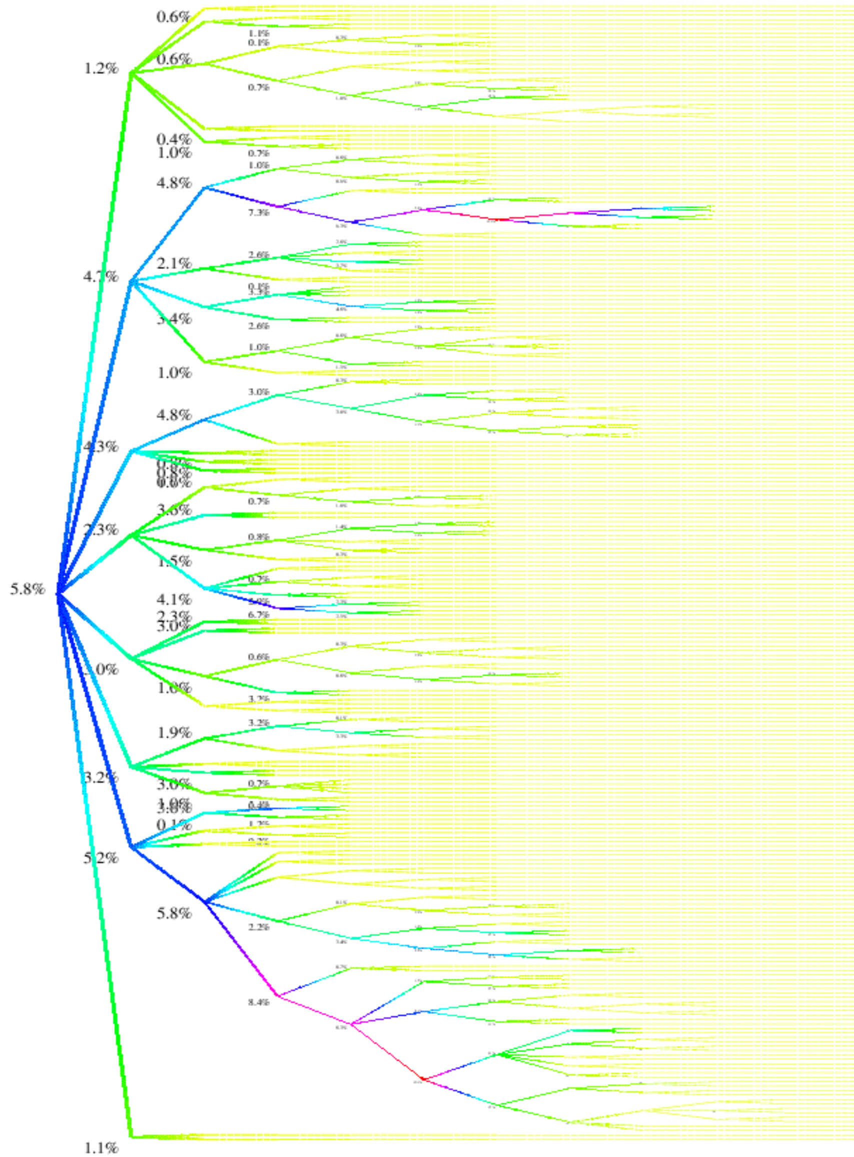
- Scenarios must be represented in the form of a scenario tree
- Example "2 2" tree



- Each path through the tree corresponds to a scenario and each node in the tree corresponds to a time along one or more scenarios

Dynamic stochastic optimization:

Representing future portfolio decisions in the face of uncertainty



Scenario tree generation

- Generation of the scenario tree is a crucial step in the problem solving process
- No optimal method of tree generation but goal should be the best representation of the uncertainty
- Methods can be assessed on stability of the first stage portfolio and the expected utility distribution (with respect to seed)
- The most important factor for both these issues is the branching factor – the number of branches at each node
- Different scenario tree generation methods
  - **Random sampling** (Bradley & Crane, 1974)
  - **Binary lattices** (Zenios, 1991)
  - **Adjusted random sampling** (Carino *et al*., 1994)
  - **Optimization** based methods (Hoyland & Wallace, 2001)
- Arbitrage elimination

Comparison of methods

- Comparison of the following methods
  - **Random sampling**
  - **Mean matching**
  - **Mean-covariance matching**
- Experiment
  - **1 stage problem**
  - Downside-Quadratic utility function with transaction costs and portfolio restrictions
  - VARSIM underlying dynamic model
  - For each method and a given branching factor generate 100 scenario trees using different seeds and solve
  - Find **lowest branching factor** needed for the **problem** to be **stable**
    - Problem considered stable if the standard deviation of each asset weight is less than 0.1 and the standard deviation of the expected terminal wealth is less than 10% of its mean
  - **Compare** the **mean and standard deviation of** the **expected utility distribution** at this branching factor **to** the **true solution**
    - True solution **obtained** by solving problems **with very large branching factors** until convergence

Results

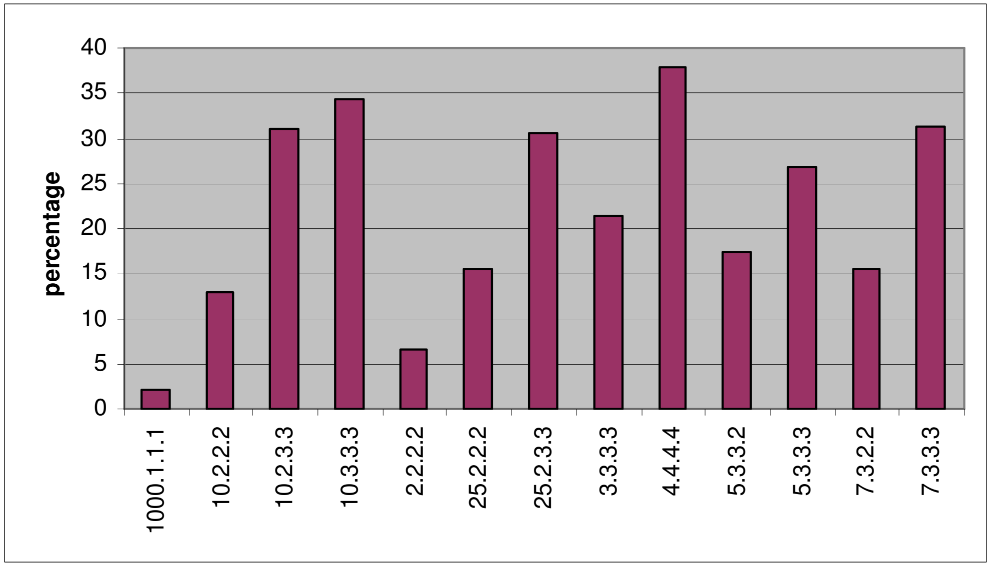
- The following table shows the lowest branching factor which makes each method stable

Method	Branching Factor
Random Sampling	50
Mean Matching	40
Mean-Covariance Matching	20

- The moment matching methods also approximated the true solution better than random sampling

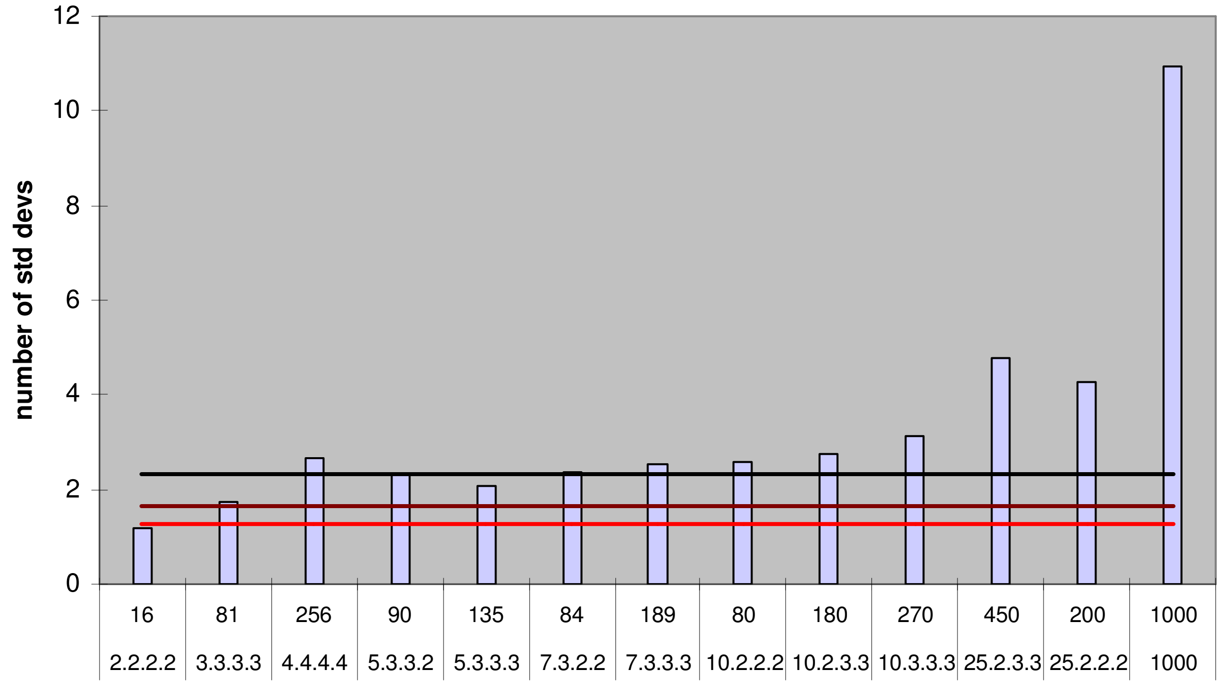
Scenario Clipping Events

Average for each tree of the percentage failure of K-S test at 5% level for 100 seeds



Rare Events

Test for the difference of mean number of occurrences of events from zero



4. Optimal Dynamic Asset Allocation

Problem formulation

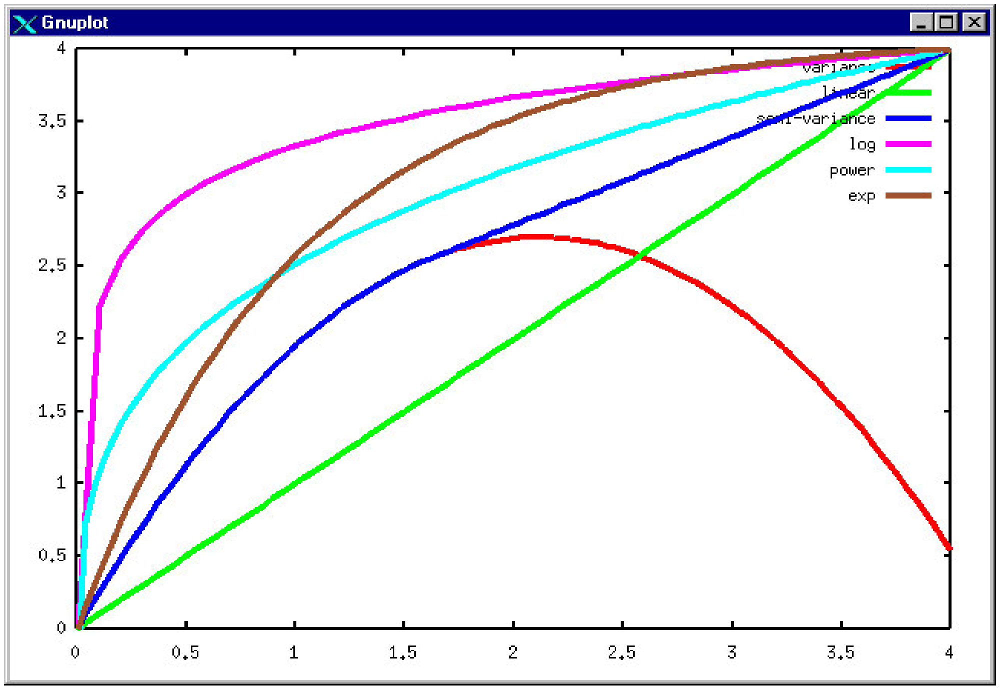
- Fixed planning horizon: 3, 5, 10, …, 40 years
- Portfolio rebalance dates: quarterly, semi-annually, annually, …
- Dynamic investment strategy maximizes risk adjusted fund wealth subject to constraints such as position and loss limits

$$\begin{aligned} &\textit{maximize} \quad E[u(\mathbf{w}(x))] \\ &\textit{subject to} \quad Ax \le b \end{aligned}$$

- Here u is a utility function and x is a decision process representing the portfolio composition at each rebalance date in each scenario subject to the data (A,b) representing the constraints

Utility functions

Utility functions are normally increasing to capture the investor's preference for a higher terminal wealth and concave to capture the investor's risk averse attitude – the greater the curvature the greater the aversion to risk



Exponential	$u(w) = -e^{-aw}$
Power	$u(w) = \frac{1}{a} w^{a}$
Log	$u(w) = \log(w)$
Downside quadratic	$u(w) = (1-a)w - a(w - tw)_{-}^{2}$
Variance	$u(w) = (1-a)w - a(w - tw)^{2}$
Linear	$u(w) = w$

Accounting constraints

- **Cash balance**

$$\sum_{i \in I} p_{it}(\omega)(g x_{it}^{-}(\omega) - f x_{it}^{+}(\omega)) = 0$$

- **Inventory balance**

$$x_{it}(\omega) = x_{it-1}(\omega)(1 + v_{it}(\omega)) + x_{it}^{+}(\omega) - x_{it}^{-}(\omega)$$

- **Wealth definition**

$$w_t(\omega) = \sum_{i \in I} p_{it}(\omega) x_{it}(\omega)$$

- **Non-negativity**

$$x_{it}^{+}(\omega), x_{it}^{-}(\omega) \geq 0$$

Portfolio constraints

- Short/borrowing limits

$$p_{it}(\omega)x_{it}(\omega) \geq \underline{x_i}$$

- Position limits

$$p_{it}(\omega)x_{it}(\omega) \leq \phi_i w_t(\omega)$$

- Turnover constraints

$$| p_{it}(\omega)x_{it}(\omega) - p_{it-1}(\omega)x_{it-1}(\omega) | \leq \alpha_i w_t(\omega)$$

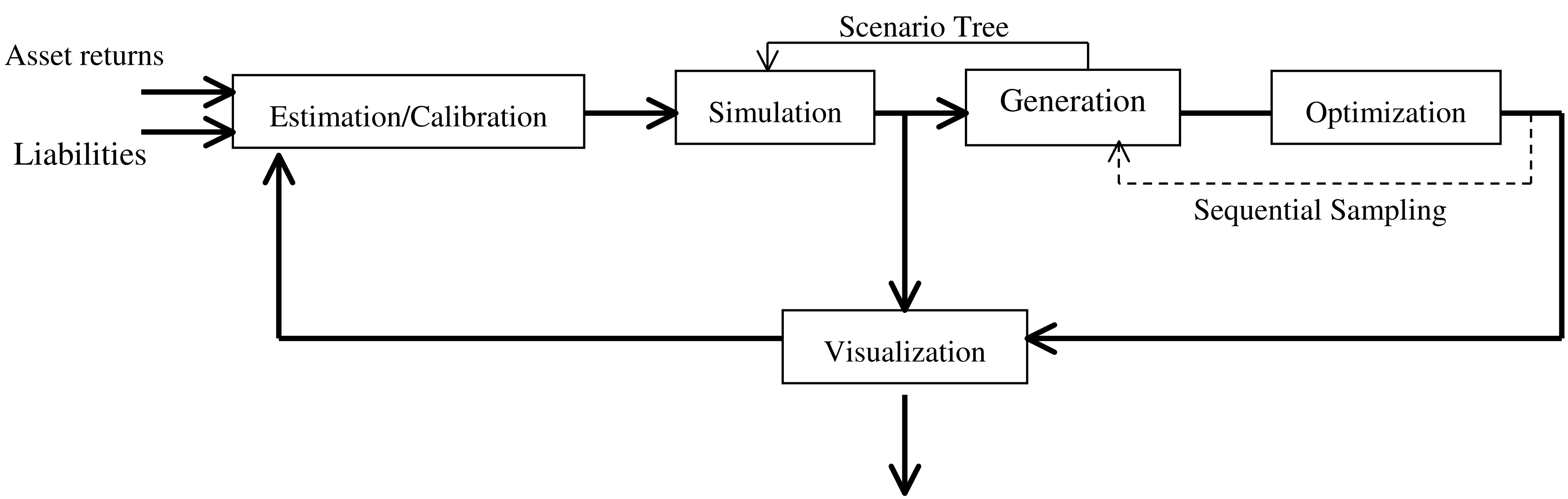
- Solvency constraints

$$w_t(\omega) \geq 0$$

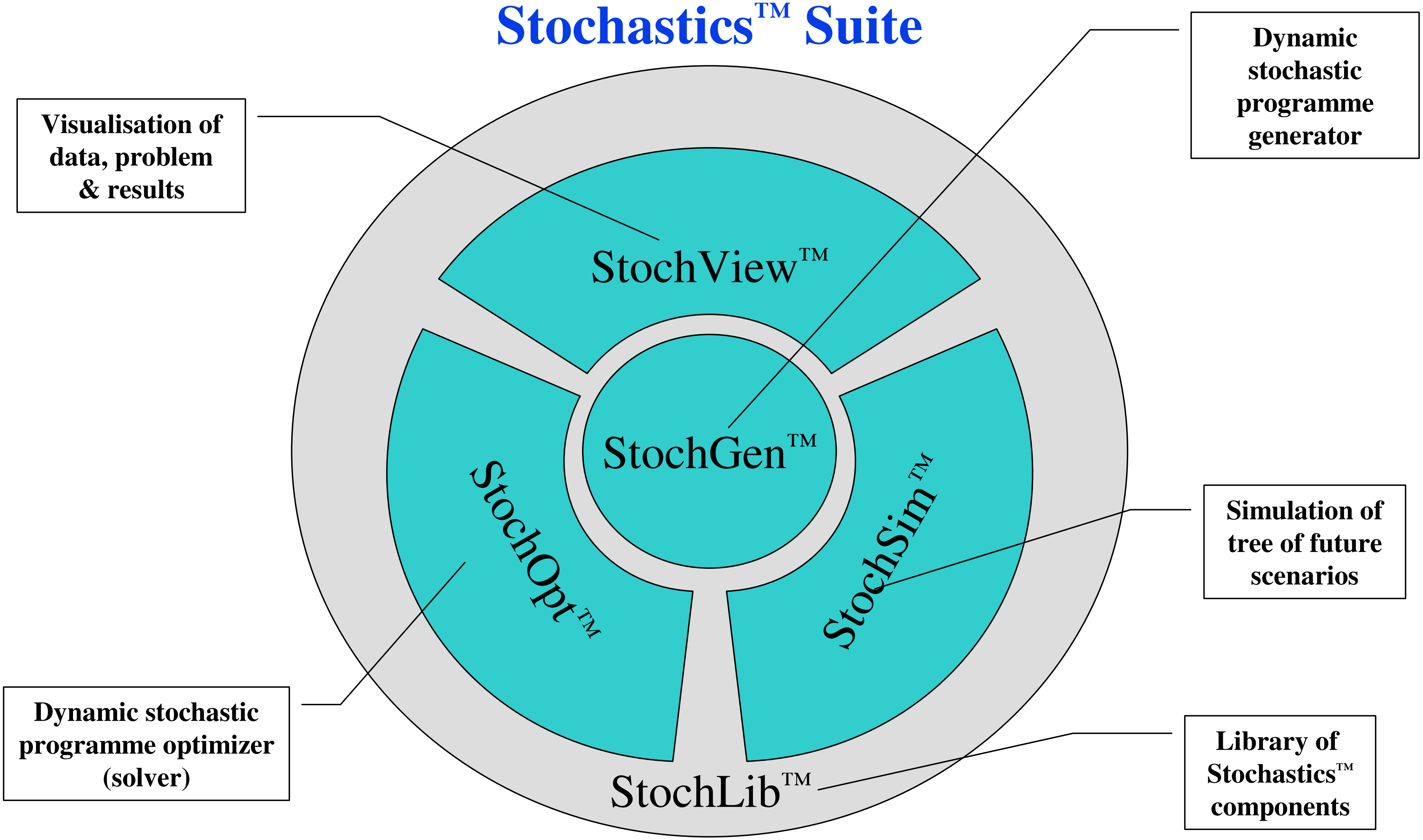
Problem generation and solution methods

- Resulting stochastic program (SP) is convex possibly nonlinear
- Problems are generated in a standard linear (MPS) or stochastic (SMPS) programming format using the STOCHASTICS™ system
- Solution method depends on utility functions
  - Downside-quadratic: **nested Benders** or **interior point**
  - Exponential, Power/Log: nested Benders
  - Linear: nested Benders, interior point or **simplex**

Fundamentals of the *Stochastics*™ System

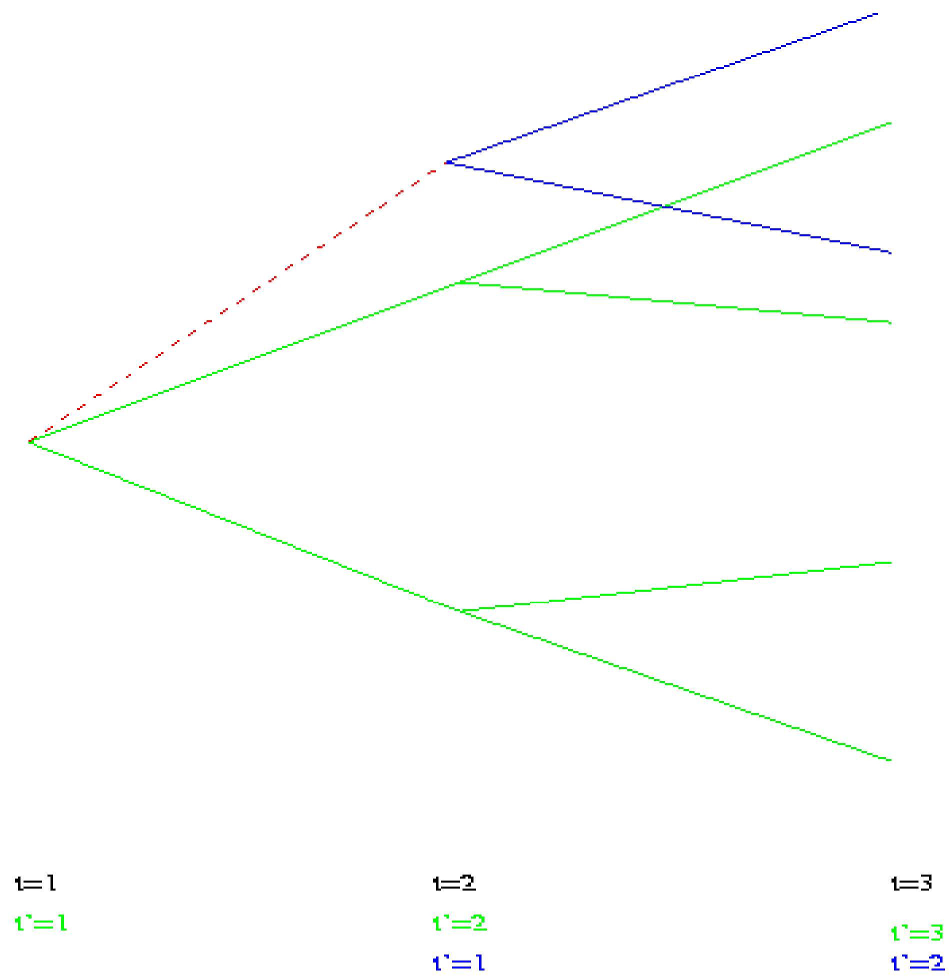


Stochastics™ Suite



Implementation

- In practice a separate SP is solved for each trading time and only the first stage solution is implemented since
  - Realized values of variables are unlikely to coincide with simulated values
  - Opportunity to update underlying dynamic statistical model at each time



5. Shaping Portfolio NAV

- Formulation and solution of ALM problems with risk-averse utility functions
- Formulation and solution of ALM problems which use a "fixed mix" strategy to construct a benchmark portfolio
- Formulation and solution of ALM problems with probabilistic VaR and capital guarantee constraints

Constraints

- **Regulatory constraints**
  - Borrowing and position limits

$$\underline{X}_{it} \leq p_{it} x_{it} \leq \overline{X}_{it} \qquad \underline{Z}_{kt} \leq p_{kt}(z_{kt}^{+} - z_{kt}^{-}) \leq \overline{Z}_{kt} \quad t = 1, \ldots, T \quad \forall i$$

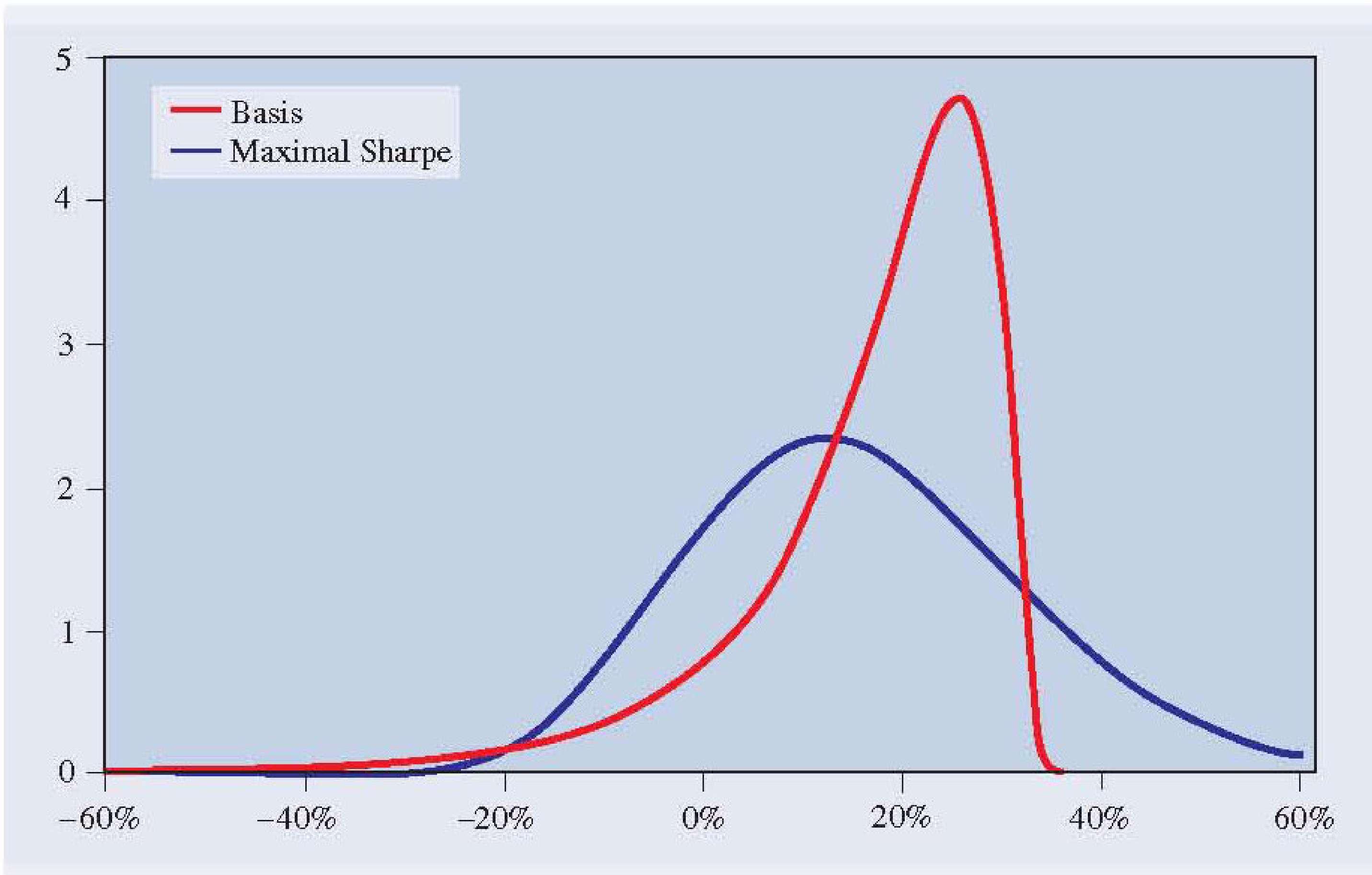
- **Fixed mix constraints**
  - The amount invested in each asset is rebalanced to a benchmark proportion of total fund wealth at each trading date

$$p_{it} x_{it} = \lambda_i \left(\sum_{j=1}^{I} p_{jt} x_{jt} \right) \quad t = 1, \ldots, T \quad \forall i$$

- **Performance constraints**
  - Guaranteed return

$$\frac{w}{w_1} \geq R$$

Wealth distribution of a Sharpe ratio maximising portfolio
Goetzmann *et al.* (2002)



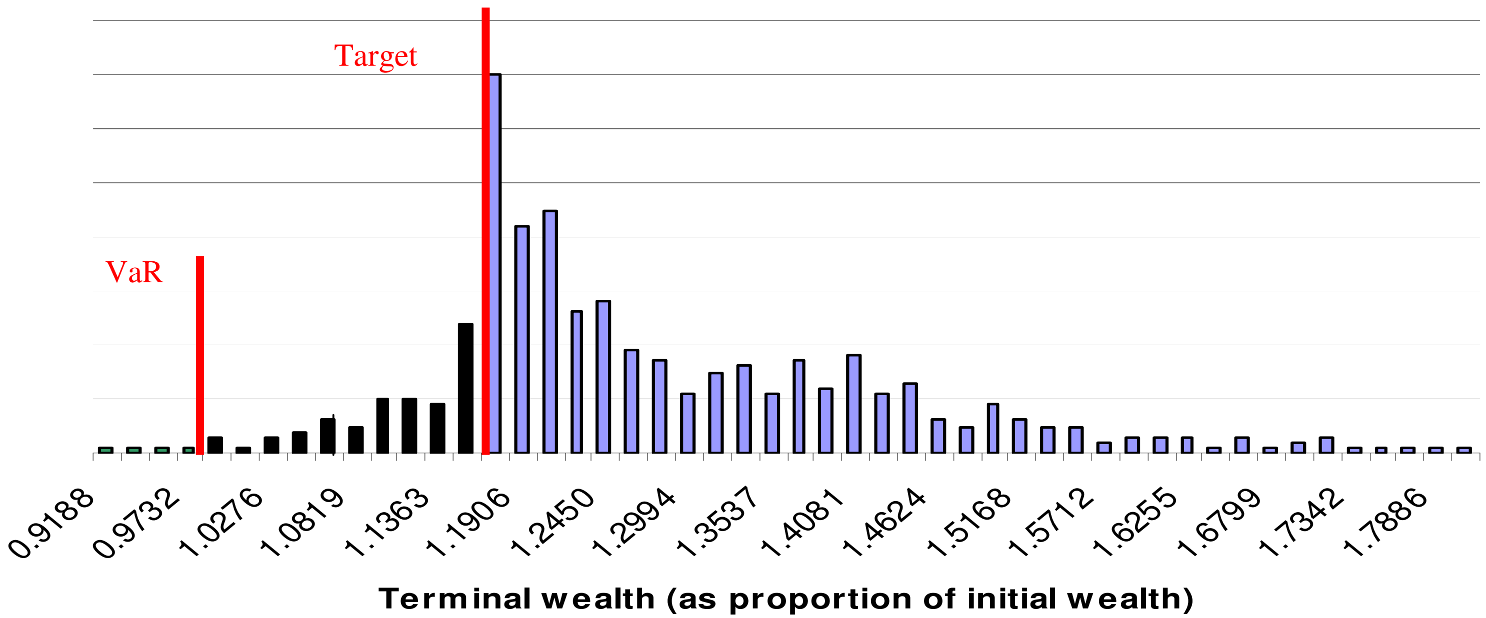
Source: H Till & J Eagleeye, *Quantitative Finance* **3** (2003) C42-48

Risk management and capital guarantees

$$\max \mathrm{E}[\beta \times \text{return} - (1-\beta) \times \text{risk}]$$

- Terminal wealth distribution (from scenario tree)

  Portfolios are penalized for each scenario in which they under-perform relative to the target



Alternative to Probability Constraints

- **Shortfall**:

$$h_t(\omega) = \max\left(0, L_t(\omega) - W_t(\omega)\right) \qquad \forall \omega \in \Omega \quad \forall t \in T^{\text{total}}$$

where **L** represents the fund's liability and **W** its wealth

- **Maximum Shortfall** for each scenario

$$H(\omega) = \max_{t \in T^{\text{total}}} h_t(\omega) \qquad \forall \omega \in \Omega$$

- **Probability Constraint**:

$$P\left(\max_{t \in T^{\text{total}}} h_t(\boldsymbol{\omega}) > 0\right) \leq \alpha$$

- Alternatively, shortfall can be penalized in objective function

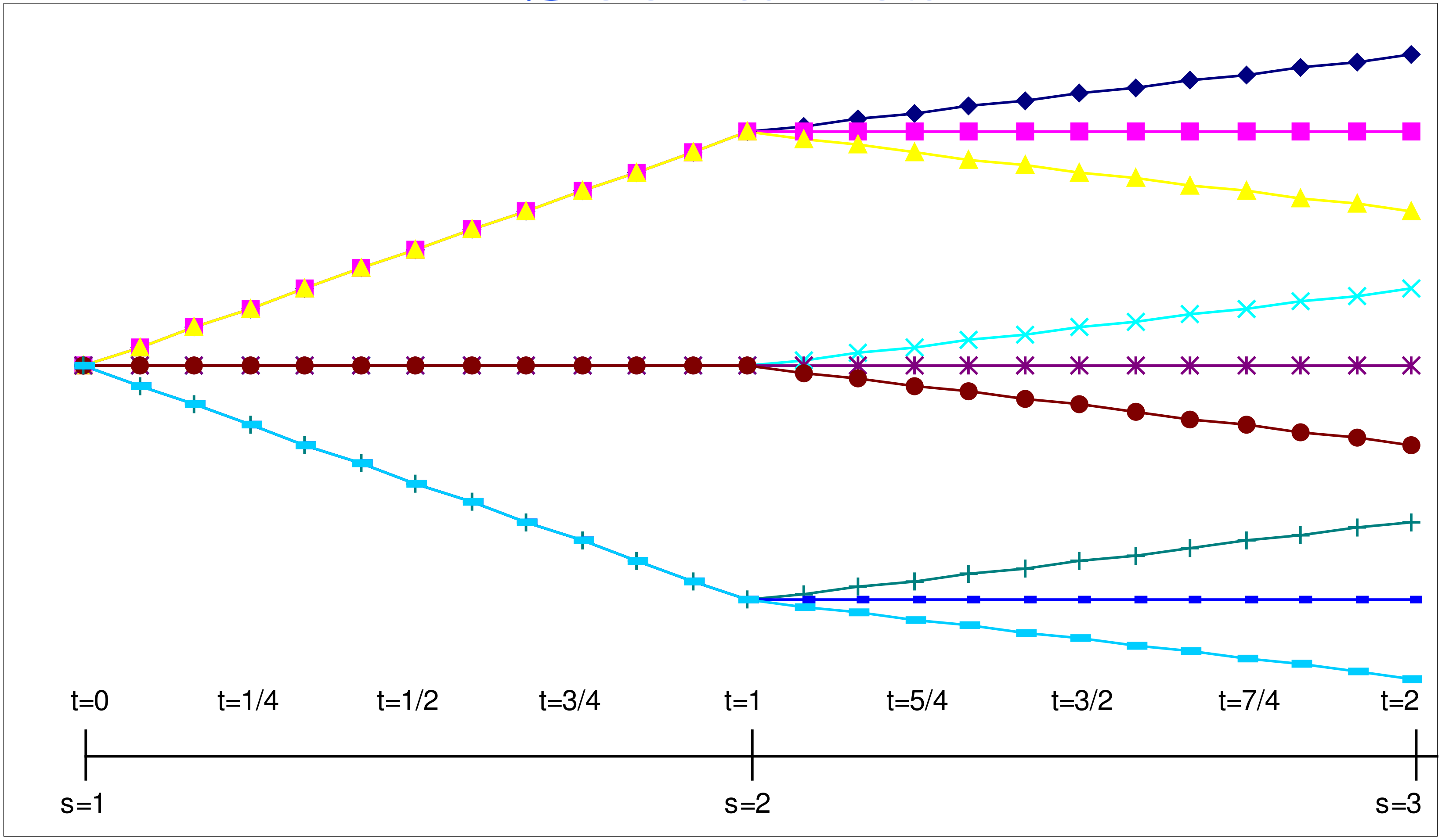
Alternative to Probability Constraints

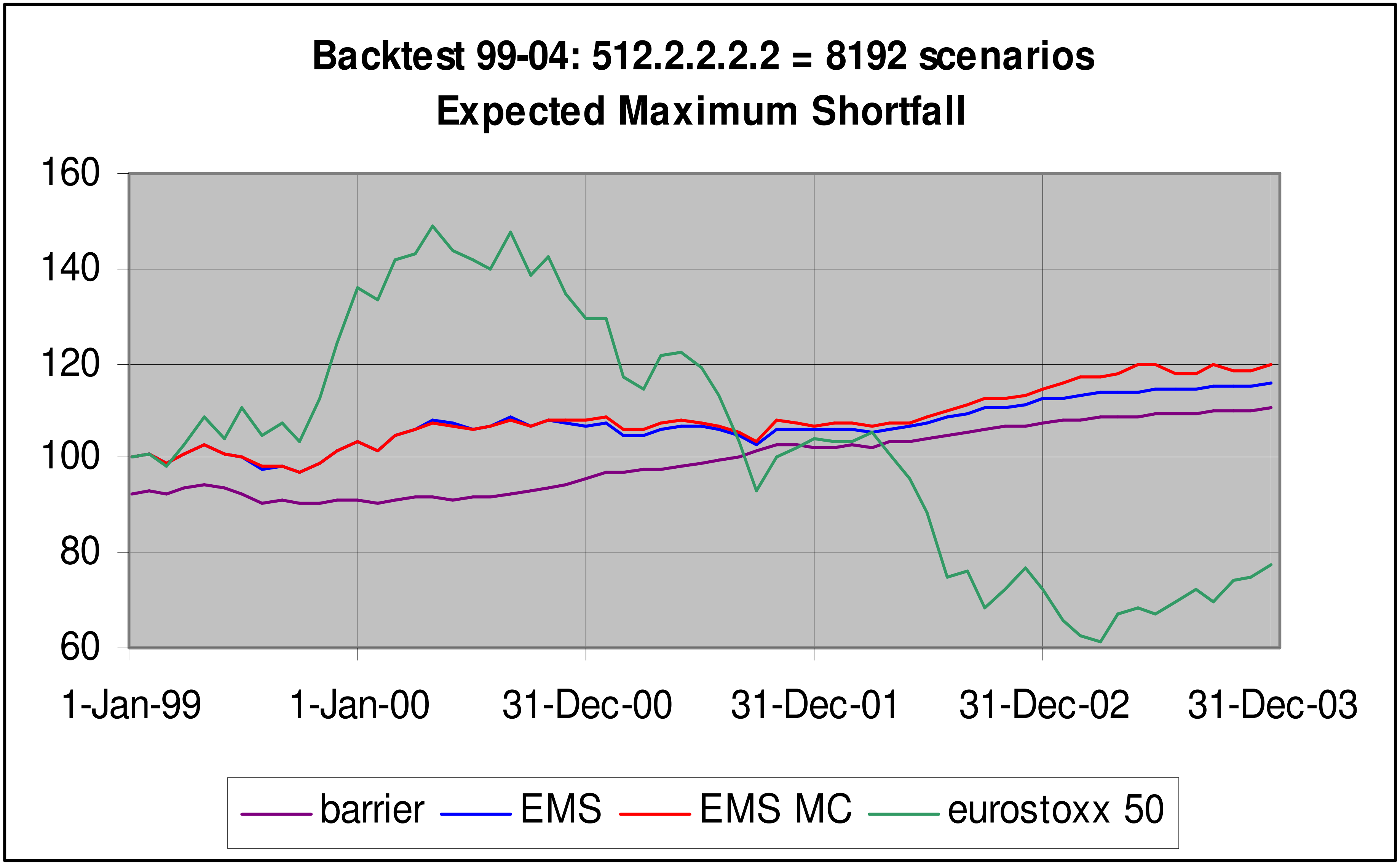
- **Objective Function**:

$$\max_{\substack{\{x_{t,a}(\omega),x^+_{t,a}(\omega),x^+_{t,a}(\omega): \\ a\in A,\omega\in\Omega,t\in T^d\cup\{T\}\}}} \left\{(1-\beta)\left(\sum_{\omega\in\Omega} p(\omega)\sum_{t\in T^d\cup\{T\}} W_t(\omega)\right)-\beta\left(\sum_{\omega\in\Omega} p(\omega)H(\omega)\right)\right\}$$

- Trade-off between fund wealth and expected maximum shortfall
- β is risk aversion measure
- Shortfall is measured on a monthly basis even though rebalancing is only allowed once a year

Graphical Representation of Scenarios





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Portfolio Allocations

	1y	2y	3y	4y	5y	10y	30y	Stock
Jan 99	0	0	0	0	**0.48**	**0.34**	0	**0.18**
Jan 00	0	0	0	0	0	**0.72**	0	**0.28**
Jan 01	0	0	0	0	**0.44**	**0.25**	0	**0.31**
Jan 02	0	0	0	**0.28**	**0.68**	0	0	**0.04**
Jan 03	0	0	0	**0.10**	**0.88**	0	0	**0.02**

6. System Asset Allocation Backtests

- Viewpoint of US dollar investor
- All portfolio rebalances subject to 2% transaction value tax (Eire)
- Monthly data available 1977-2002
- Three historical out-of-sample periods

Period	Length	S&P500 Return	Asset Return Model	Rebalance Frequency
1990-95	5 years	7.41% p.a.	3 areas (ex Japan)	Annual
1996-2000	5 years	14.28% p.a.	4 areas	Annual
1999-2001	2 years	0% p.a.	4 areas 4 areas + emerging markets	Semi-annual
	2.5 years	-2.30% p.a.	4 areas 4 areas + emerging markets	Semi-annual

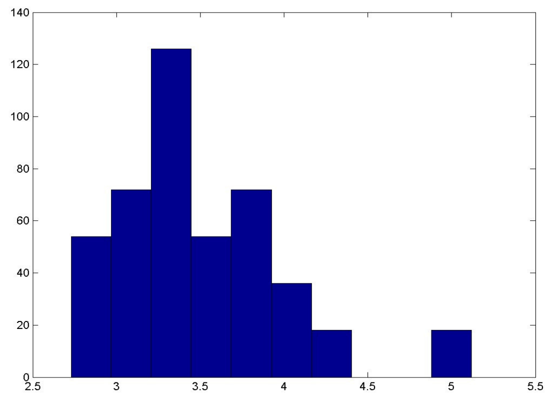
- Various fund objectives and attitudes to downside risk
- In all historical backtests **system outperformed S&P500 by up to 10% p.a.**
- **All system returns were positive** – even through the recent high tech crash!

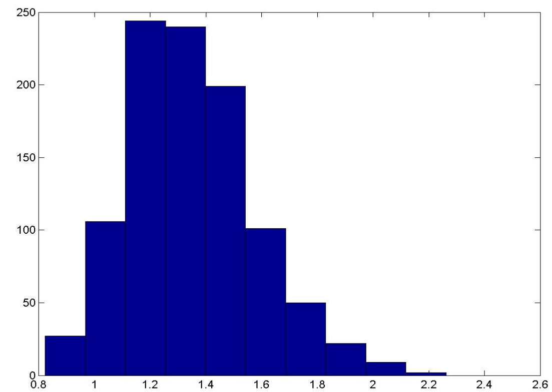
Initial Estimation Period	Out-of-sample Period	Length	Asset Return Model	Simulator	Number of Scenarios k	Rebalance Frequency	Risk Management Criterion	Horizon	Constraint Annualised Return % (see Section 5.4)			S&P 500 Benchmark Annualised Return %
									T1	T2	T3	
1972-1990	1990-1995	5 years	3 areas (ex Japan)	BMSIM	4	annual	terminal	telescoping	10.33	9.34	-	7.41
1992-1996	1996-2001	5 years	4 areas	BMSIM	4	annual	terminal	telescoping	13.36	7.13	-	14.12
1992-1996	1996-2001	5 years	4 areas	VARSIM	4	annual	terminal	telescoping	1.51	8.30	-	14.12
1992-1999	1999-2001	2.5 years	4 areas	BMSIM	8.2	semi-annual	terminal	telescoping	27.89	6.48	2.69	-2.30
1992-1999	1999-2001	2.5 years	above + emerging markets	BMSIM	8.2	semi-annual	terminal	telescoping	16.98	5.72	3.38	-2.30
1992-1999	1999-2001	2.5 years	above + US economy	BMSIM	8.2	semi-annual	terminal	telescoping	19.16	4.64	-0.38	-2.30
1992-1999	1999-2001	2.5 years	4 areas	VARSIM	8.2	semi-annual	terminal	telescoping	-6.40	-	-3.92	-2.30
1990-1996	1996-2001	5 years	4 areas	BMSIM	8.2	annual	all periods	telescoping	8.54	-	8.37	14.12
1990-1996	1996-2001	5 years	4 areas	VARSIM	8.2	annual	all periods	telescoping	5.78	9.99	**9.37**	14.12
1990-1996	1996-2001	5 years	4 areas	HSIM	8.2	annual	all periods	telescoping	4.95	-	6.04	14.12
1972-1991	1991-2001	10 years	4 areas	VARSIM	8.2	annual	all periods	5-year rolling	3.56	-	9.98	12.72

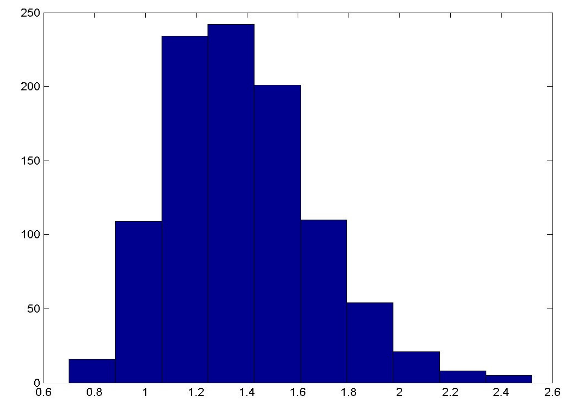
T1 – no shorting/borrowing
T2 – no shorting/borrowing and position limits
T3 – no shorting/borrowing with position limits and turnover constraints

A Cautionary Tale

Portfolio Wealth Distributions







In Tree Distribution

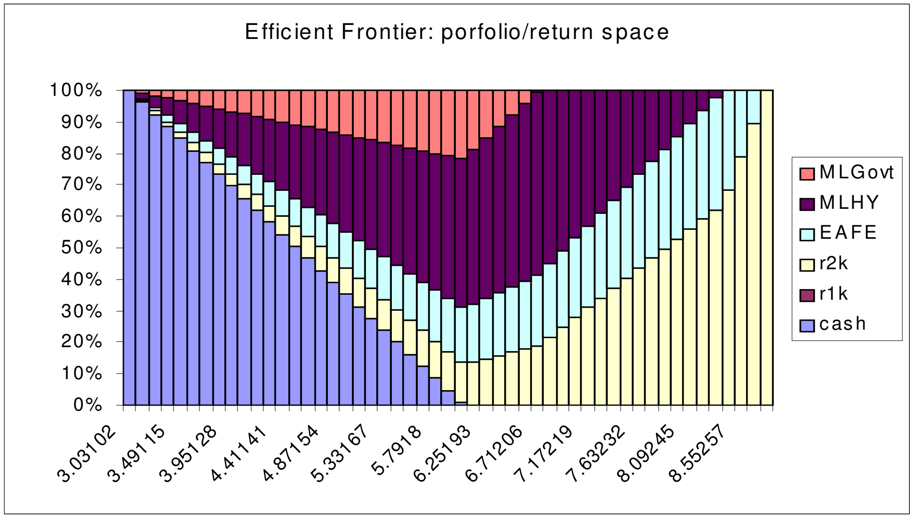
Root Node Portfolio

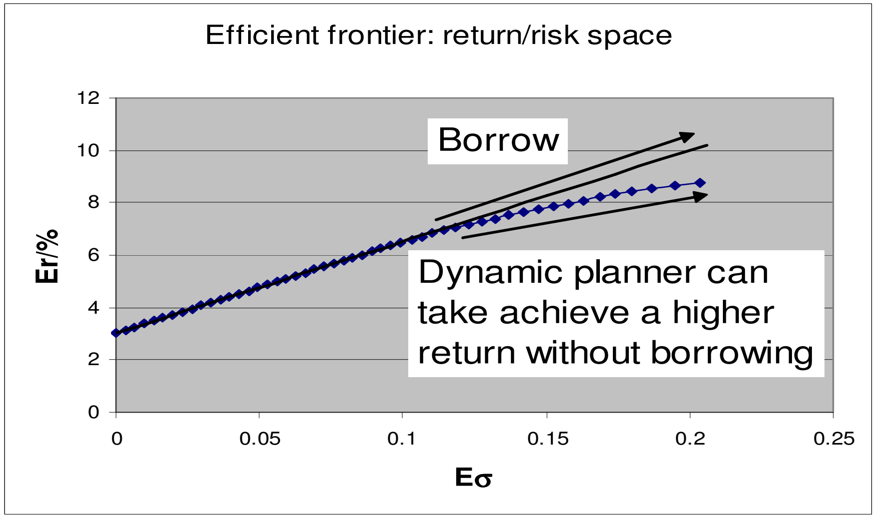
Flat Scenarios

Average Portfolios

Flat Scenarios

The Markowitz Investor





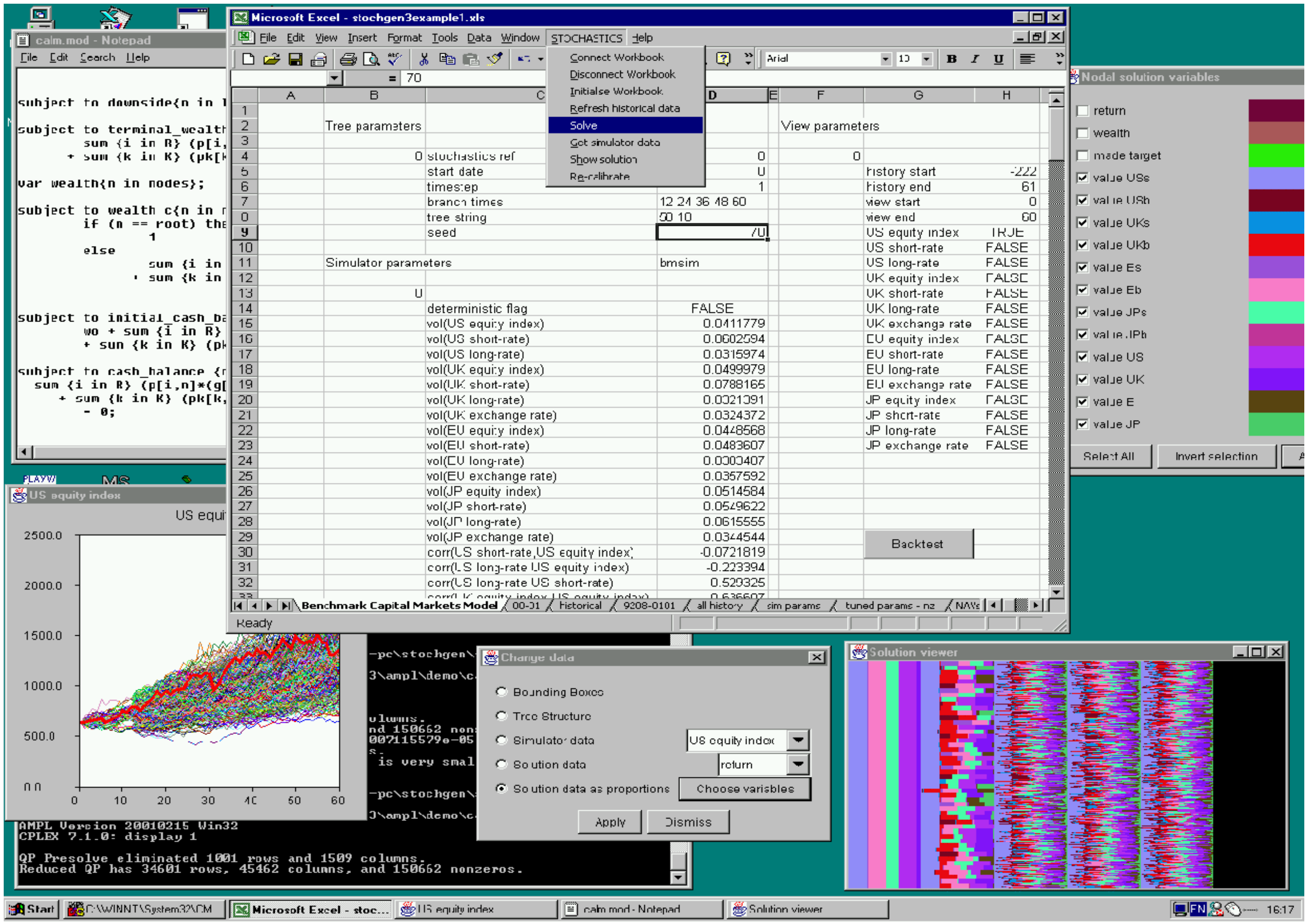
- We calculate a market portfolio based on 5 risky assets and cash
- This portfolio is implemented at each rebalance point
- Between rebalances we allow deviation from this portfolio within a narrow band

7. Conclusions

- Strategic ALM and tactical risk systems for funds and individuals are a reality today
- Multiperiod model yields multiple advantages
  - Robust portfolios in the face of dynamic uncertainty
  - Significantly outperforms single period buy-and-hold models
  - Best, worst and VaR limited what-if portfolios views available
  - Forewarned is forearmed!
- All business structures and regulatory constraints can be accurately modelled
- Models involving millions of equations and variables can be solved in minutes on PCs
- Flexibility and visualization are the keys to effective decision support for strategic fund management

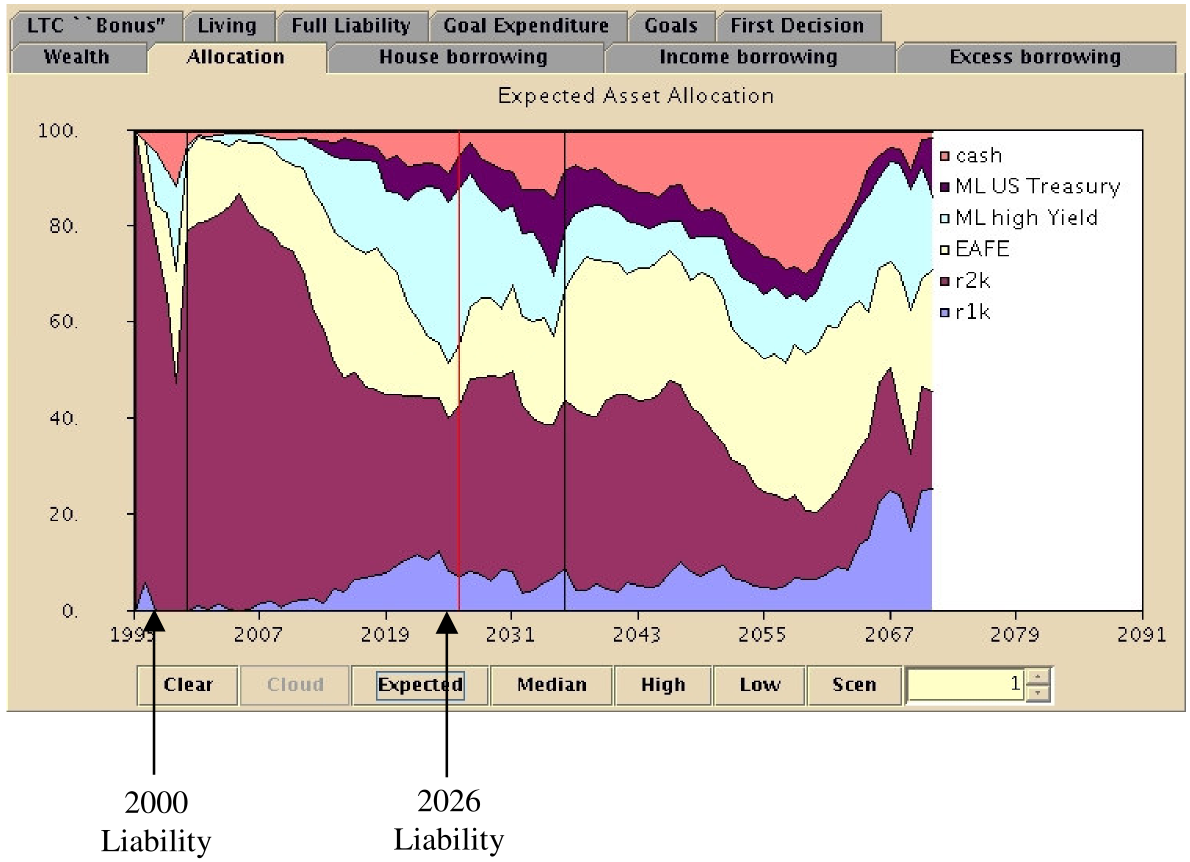
Prototype user interface for the fund manager

STOCHASTICS™ System Stochgen 3.0

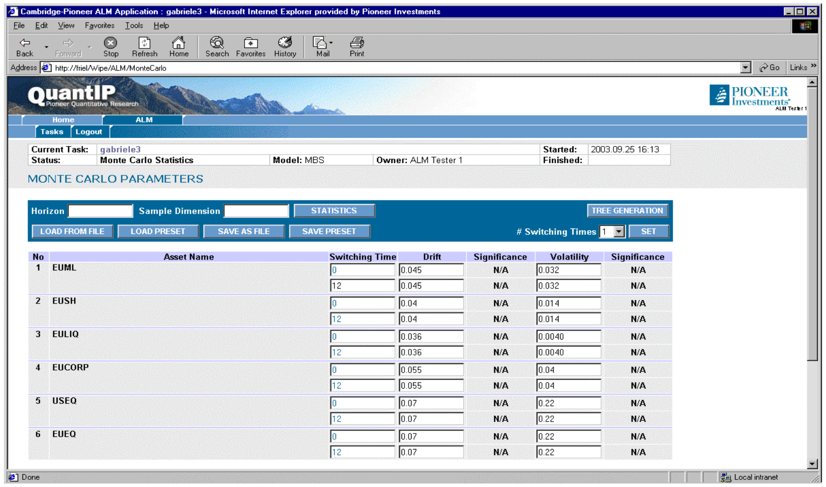


Dynamic Asset Allocation

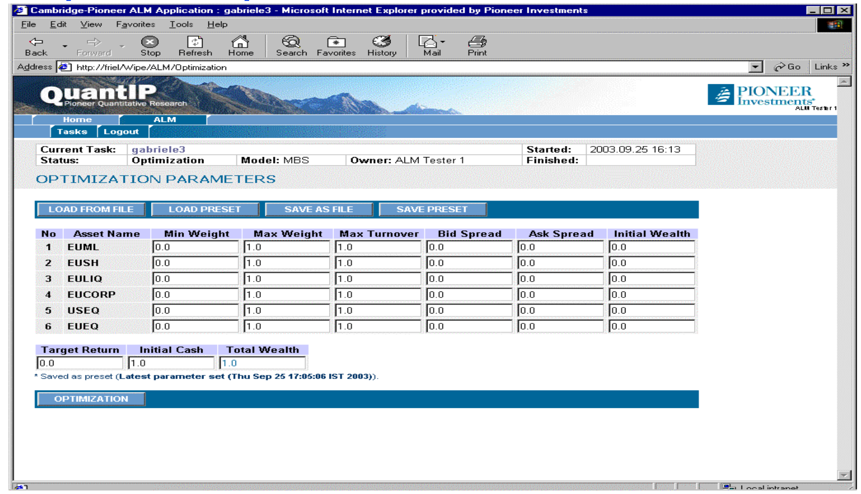
- **Asset allocation should switch into less risky assets before a major liability to ensure that the investor has enough to pay the liability in adverse market conditions**
- **In this model investment assets cannot be sold down between rebalance points to meet liabilities so that cash must be kept to cover any liabilities occurring before the next rebalance point**



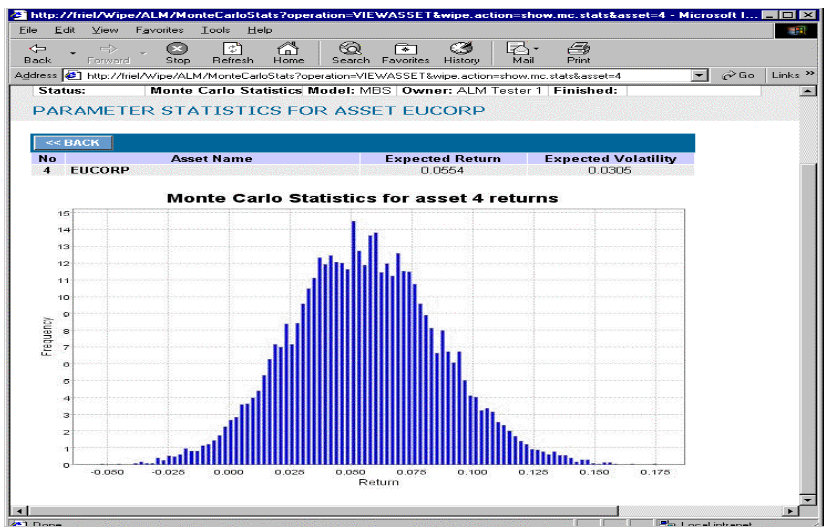
Monte Carlo parameters



Optimization parameters



Monte Carlo expected returns



Results