A Two-Person Game for Pricing Convertible Bonds

Mihai Sîrbu
Department of Mathematics
Columbia University

Steven E. Shreve
Department of Mathematical Sciences
Carnegie Mellon University

What is a convertible bond?

It is a bond issued by a firm with the following provisions:

- Pays coupons.
  In our model, coupons are paid continuously at rate $c > 0$.
- Can be called by the firm.
  In our model, the firm may call at any time the firm value exceeds K by offering to pay K. $K > 0$ is the *call price*.
- Can be converted to stock by the bondholder.
  In our model, the bondholder may convert the bond to stock worth a fraction $\gamma \in (0, 1)$ of the value of the firm. γ is the *conversion factor*.
- Rules of the call.
  If the firm calls, the bondholder may *surrender* the bond in exchange for payment K or may *convert* it to stock worth a fraction γ of the value of the firm.

Two-person zero-sum game. The firm seeks a call strategy that *minimizes* the value of the bond. The bondholder seeks a conversion strategy that *maximizes* its value.

Model assumptions.

- Volatility – Value of firm has constant volatility $\sigma > 0$.
- Interest rate – Constant rate of interest $r > 0$.
- Dividends – Firm continuously pays a fixed fraction $\delta \in (0, 1)$ of its equity value as dividends.

Standing Assumption. $\delta < r$.

Notation.

- Value of firm – $X(t)$
- Value of convertible bond – $D(t)$
- Equity value of firm – $S(t)$
- Miller-Modigliani – $X(t) = D(t) + S(t)$

Dynamics prior to call, conversion and bankruptcy

$$dX(t) = rX(t)\,dt + \sigma X(t)\,dW(t) - c\,dt - \delta S(t)\,dt$$

Literature.

M. Brennan and E. Schwartz, *Convertible bonds: valuation and optimal strategies for call and conversion*, J. Finance 32 (1977), 1699–1715.

- Values convertible bond as a contingent claim on the firm value.
- Dividends and coupons are paid at discrete dates.
- Conversion and/or call occurs only immediately prior to dividend payments. Between dividend payments, firm value evolves as a geometric Brownian motion.
- Optimal call and conversion policies are found by a backward recursion over payment dates.

J. E. Ingersoll, *A contingent-claims valuation of convertible securities*, J. Financial Econom. 4 (1977), 289–322.

- Values convertible bond as a contingent claim on the firm value.
- Does not consider the possibility of default, except at maturity.
- Argues that when no dividends are paid, optimal conversion does not occur until maturity and call should occur the first time $\gamma X(t) = K$.
- Some qualitative results are stated when dividends are positive and a function of the firm value.
- Companion paper observes that firms seem to delay call and presents some possible reasons for this.

Dynamics prior to call, conversion and bankruptcy.

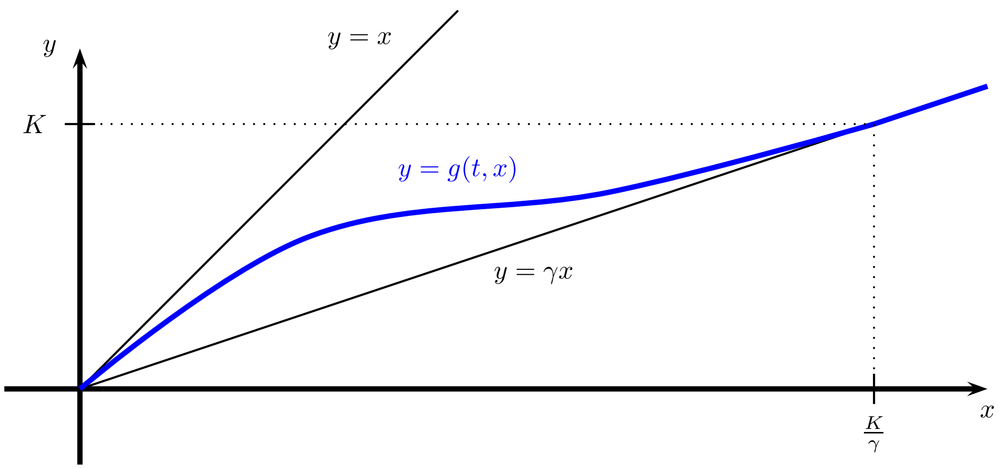
$$dX(t) = rX(t)\,dt + \sigma X(t)\,dW(t) - c\,dt - \delta\big(X(t) - D(t)\big)\,dt.$$

We seek a function $g(t,x)$ such that $D(t) = g\big(t, X(t)\big)$.

Properties of g.

1. $g(t,x) \geq \gamma x$
2. $g(t,x) = \gamma x$ for $x \geq \frac{K}{\gamma}$
3. $g(t,x) \leq K$ for $x \leq \frac{K}{\gamma}$
4. $0 \leq g(t,x) - g(t,y) \leq y - x$ for $y \geq x$.

Let $\mathcal{G}$ be the set of continuous functions on $[0,T] \times [0,\infty)$ satisfying (1)–(4).



Given $g \in \mathcal{G}$, $s \in [0,T]$ and $x \geq 0$, define $X^{s,x}$ by $X^{s,x}(s) = x$ and

$$dX^{s,x}(t) = rX^{s,x}(t)\,dt + \sigma X^{s,x}(t)\,dW(t) - c\,dt - \delta\Big[X^{s,x}(t) - g\big(t, X^{s,x}(t)\big)\Big]\,dt.$$

Time of bankruptcy:

$$\tau_0^{s,x} \triangleq \inf\{t \in [s,T] : X^{s,x}(t) = 0\}.$$

Time of conversion:

$$\tau \in \mathcal{S}^{s,x} \triangleq \Big\{\text{Stopping times } \theta \in [s, T \wedge \tau_0^{s,x}] \cup \{\infty\}\Big\}.$$

Time of call:

$$\rho \in \mathcal{S}_K^{s,x} \triangleq \Big\{\theta \in \mathcal{S}^{s,x} \text{ such that } X^{s,x}(\theta) \geq K \text{ if } \theta < \tau_0^{s,x}\Big\}.$$

Risk-neutral expected payoff of the game:

$$J_g(s,x,\rho,\tau) \triangleq \mathbb{E}\Bigg[\int_s^{\rho\wedge\tau\wedge T} e^{-r(u-s)} c\,du + e^{-r(\rho\wedge\tau\wedge T - s)}\Big(\mathbb{I}_{\{\tau \leq \rho\wedge T\}} \gamma X^{s,x}(\tau) + \mathbb{I}_{\{\rho<\tau\}} K + \mathbb{I}_{\{\rho=\infty,\tau=\infty\}}\big(L \wedge X^{s,x}(T)\big)\Big)\Bigg].$$

Par value: $L \leq K$

Lower value of the game:

$$\underline{v}_g(s,x) \triangleq \sup_{\tau\in\mathcal{S}^{s,x}} \inf_{\rho\in\mathcal{S}_K^{s,x}} J_g(s,x,\rho,\tau).$$

Upper value of the game:

$$\overline{v}_g(s,x) \triangleq \inf_{\rho\in\mathcal{S}_K^{s,x}} \sup_{\tau\in\mathcal{S}^{s,x}} J_g(s,x,\rho,\tau).$$

Theorem 1 (Value of the game). The game corresponding to g has a value, i.e.,

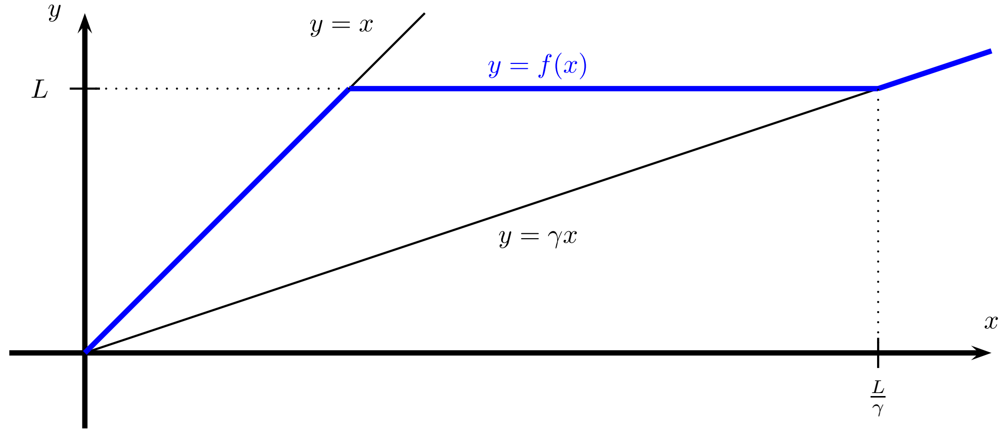
$$\underline{v}_g = \overline{v}_g.$$

We define v_g to be this common value.

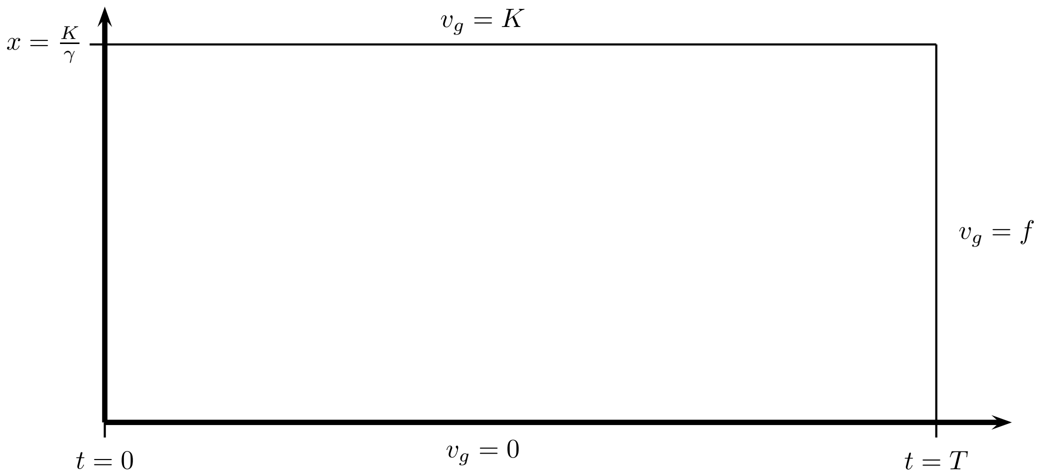
Value of v_g at maturity.

Define

$$f(x) \triangleq \begin{cases} x & \text{if } 0 \le x \le L, \\ L & \text{if } L \le x \le \frac{L}{\gamma}, \\ \gamma x & \text{if } x \ge \frac{L}{\gamma}. \end{cases}$$



Value of v_g on boundaries.



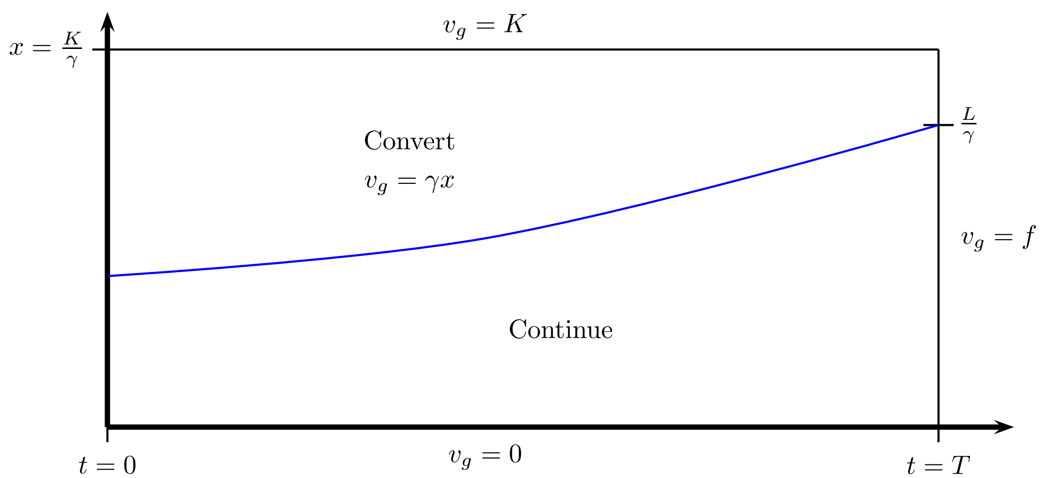
Characterization of v_g.

Case I: $r \geq \frac{c}{K}$. Conversion precedes call and v_g is the unique continuous viscosity solution of

$$\min\left\{ -v_t + rv - (rx-c)v_x + \delta(x-g)v_x - \frac{1}{2}\sigma^2 x^2 v_{xx} - c,\ v - \gamma x \right\} = 0 \tag{1}$$

satisfying the boundary conditions

$$v(t,0)=0,\ v\left(t, \frac{K}{\gamma}\right) = K \text{ for } 0 \leq t \leq T,\ v(T,x) = f(x) \text{ for } 0 \leq x \leq \frac{K}{\gamma}. \tag{2}$$



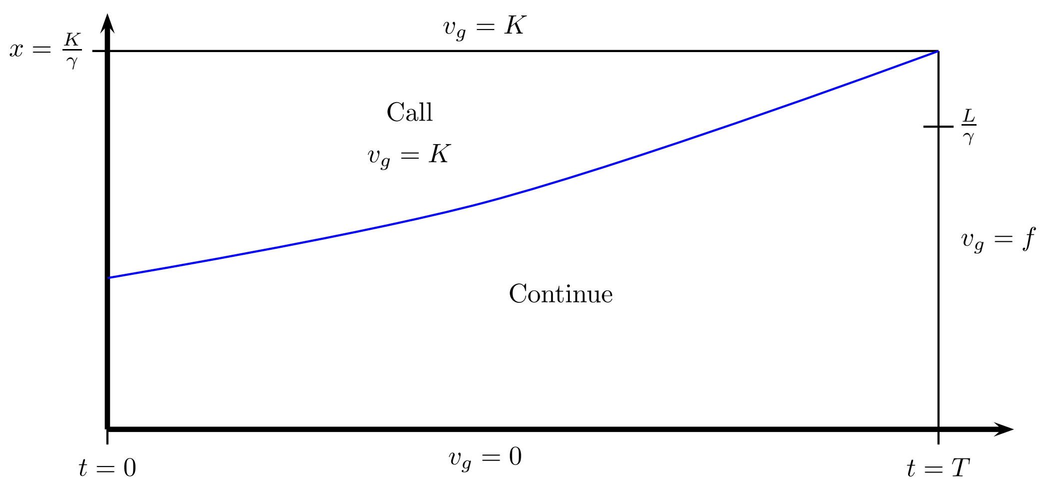
Idea of proof. $h(t,x) = K$ is a supersolution of (1), so h dominates the solution v of this equation.

Characterization of v_g (continued).

Case II: $\delta \le \frac{c}{K}$. Call precedes conversion and v_g is the unique continuous viscosity solution of

$$\max\left\{ -v_t + rv - (rx - c)v_x + \delta(x-g)v_x - \frac{1}{2}\sigma^2 x^2 v_{xx} - c,\ v - K \right\} = 0 \tag{3}$$

satisfying the boundary conditions (2).



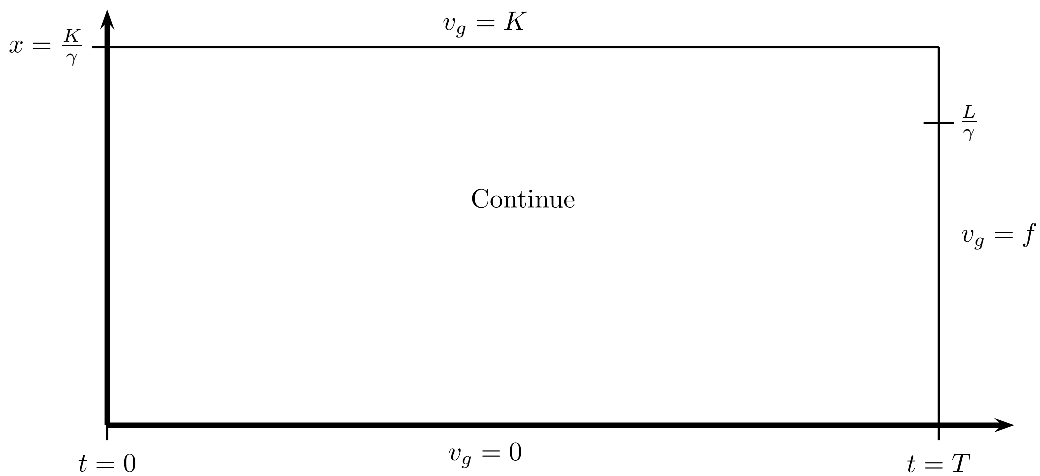
Idea of proof. $h(t, x) = \gamma x$ is a subsolution of (3) so h is dominated by the solution v of this equation.

Overlap of Case I and Case II: $\delta \leq \frac{c}{K} \leq r$.

Call and conversion coincide when firm value reaches $\frac{K}{\gamma}$, and v_g is the unique viscosity solution of

$$-v_t + rv - (rx - c)v_x + \delta(x - g)v_x - \frac{1}{2}\sigma^2 x^2 v_{xx} = c \tag{4}$$

satisfying the boundary conditions (2).



Remark. We have fixed an arbitrary $g \in \mathcal{G}$. The function v_g is the viscosity solution of (4) subject to (2). We have not assumed Hölder continuity of g, so we do not know that v_g is smooth enough to be a classical solution of (4).

Theorem 2 (Fixed point). Let g_1 and g_2 be in $\mathcal{G}$, and let v_{g_1} and v_{g_2} be as described above. Then v_{g_1} and v_{g_2} are in $\mathcal{G}$ and

$$\sup_{t,x} \big| v_{g_1}(t,x) - v_{g_2}(t,x) \big| \le \frac{\delta}{r} \sup_{t,x} \big| g_1(t,x) - g_2(t,x) \big|.$$

In particular, there exists a unique function $g \in \mathcal{G}$ such that $v_g = g$.

Idea of proof (Jensen and Ishii):

Use viscosity solution arguments to bound the function

$$(t,x,y) \mapsto v_{g_1}(t,x) - v_{g_2}(t,y) - \frac{\alpha}{2}|x-y|^2,$$

and then let $\alpha \to \infty$.

Remark. Let v^* be the fixed point of Theorem 2. Prior to call, conversion and bankruptcy, the price of the bond is

$$D(t) = v^*\big(t, X(t)\big),$$

where

$$dX(t) = rX(t)\,dt + \sigma X(t)\,dW(t) - c\,dt - \delta\Big[X(t) - v^*\big(t, X(t)\big)\Big]\,dt.$$

Theorem 3 (Characterization of bond price).

Case I: $r \geq \frac{c}{K}$. Conversion precedes call and v^* is the unique continuous viscosity solution of the equation

$$\min\left\{ -v_t + rv - (rx-c)v_x + \delta(x-v)v_x - \frac{1}{2}\sigma^2 x^2 v_{xx} - c,\ v - \gamma x \right\} = 0$$

satisfying the boundary conditions (2).

Case II: $\delta \leq \frac{c}{K}$. Call precedes conversion and v^* is the unique continuous viscosity solution of the equation

$$\max\left\{ -v_t + rv - (rx-c)v_x + \delta(x-v)v_x - \frac{1}{2}\sigma^2 x^2 v_{xx} - c,\ v - K \right\} = 0$$

satisfying the boundary conditions (2).

Overlap of Case I and Case II: $\delta \leq \frac{c}{K} \leq r$.

Call and conversion coincide when firm value reaches $\frac{K}{\gamma}$, and v^* is the unique viscosity solution of the equation

$$-v_t + rv - (rx-c)v_x + \delta(x-v)v_x - \frac{1}{2}\sigma^2 x^2 v_{xx} = c$$

satisfying the boundary conditions (2).

Asymptotic behavior.

For fixed maturity T, let $v_T^*(t,x)$, $0 \le t \le T$, denote the price of the bond at time t if x is the firm value. This price depends only on the *time to maturity* $\tau = T - t$, i.e., there is a function $f(\tau, x)$ such that

$$f(\tau, x) = v_T^*(t, x).$$

Theorem 4. The limit

$$f(x) = \lim_{\tau \to \infty} f(\tau, x), \quad x \ge 0,$$

exists and is the price of the *perpetual convertible bond.*

Idea of proof. One shows that the convergence in Theorem 4 is uniform in x and that the limiting function $f(x)$ is the unique continuous viscosity solution of the autonomous versions of the differential equations for the two cases in Theorem 3.

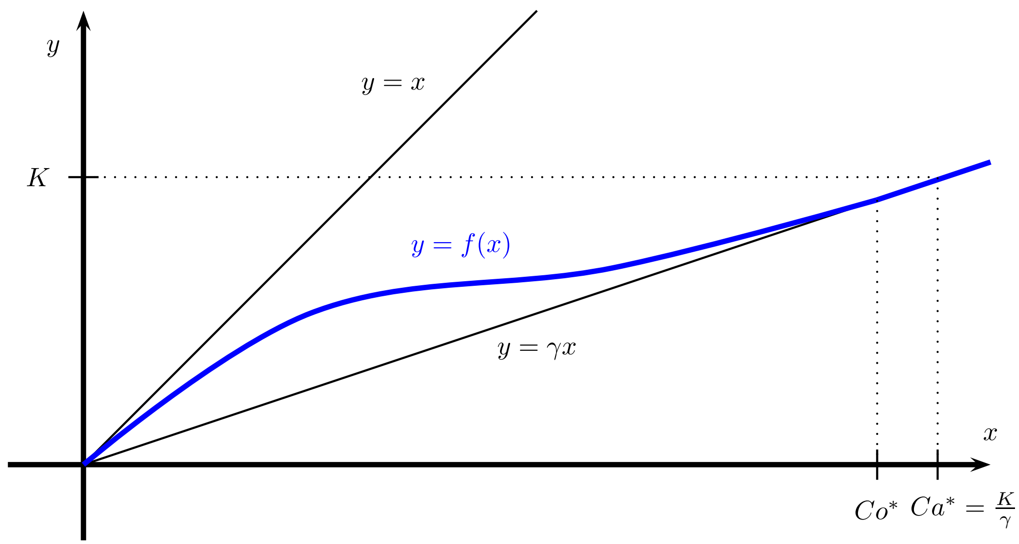
These equations were shown in

M. SÎRBU, I. PIKOVSKY AND S. SHREVE, *Perpetual convertible bonds*, SIAM J. Control Optim. to appear,

to characterize the perpetual convertible bond price.

Asymptotic behavior (continued).

Case I: $r \geq \frac{c}{K}$.

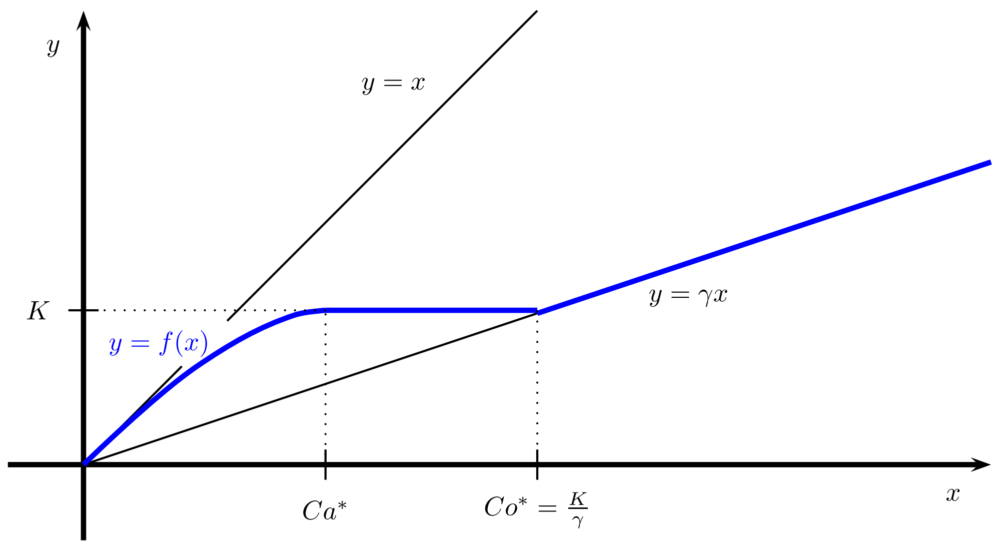


Large call price.

When $r \geq c/K$, the game reduces to the optimal stopping problem of *optimal conversion*. Call occurs at firm value K/γ, which is greater than or equal to the optimal conversion level.

Asymptotic behavior (continued).

Case II: $\delta \leq \frac{c}{K}$.

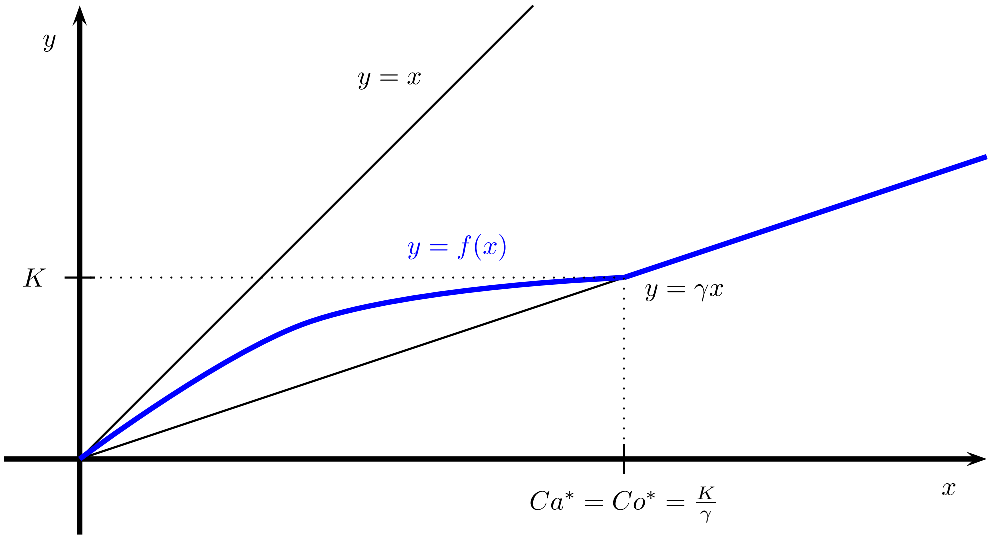


Small call price.

When $\delta \leq c/K$, the game reduces to the optimal stopping problem of *optimal call*. Conversion occurs at firm value K/γ, which is greater than or equal to the optimal call level.

Asymptotic behavior (continued).

Overlap of Case I and Case II: $\delta \leq \frac{c}{K} \leq r$.



Intermediate call price.

When $\delta \leq \frac{c}{K} \leq r$, *call and conversion occur simultaneously* at firm value K/γ.

Additional Literature

P. Asquith, *Convertible bonds are not called late*, J. Finance 50 (1995), 1275–1289.

P. Asquith and D. Mullins, Jr., *Convertible debt: Corporate call policy and voluntary converstion*, J. Finance 46 (1991), 1273–1289.

E. Ayache, P. A. Forsyth and K. R. Vetzal *The valuation of convertible bonds with credit risk*, Center for Advanced Studies in Finance, University of Waterloo 2002.

Barone-Adesi, G., A. Bermudez and J. Hatgioannides, *Two-factor convertible bonds valuation using the method of characteritics/finite elements*, City University of Business, London, 2001.

A. Bensoussan, M. Crouhy, and D. Galai, *Sto- chastic equity volatility and the capital structure of the firm*, Philos. Trans. Roy. Soc. London Ser. A, 347 (1994), 531–541.

M. Brennan and E. Schwartz, *Analyzing convertible bonds*, J. Financial Quantitative Analysis 15 (1980), 907–929.

P. Carayannopoulos, *Valuing convertible bonds under the assump-*

tion of stochastic interest rates: an empirical investigation, Quarterly J. Bus. Economics 35 (1996).

W. Chung and I. Nelken, *Costing converts*, Risk 7(7) (1994), 47–49.

G. Constantinides, *Warrant exercise and bond converstion in competitive markets*, J. Financial Econom. 13 (1984), 371–397.

G. Constantinides and B. Gundy, *Call and conversion of convertible corporate bonds: theory and evidence*, Graduate School of Business, University of Chicago, 1984.

G. Constantinides and R. Rosenthal, *Strategic analysis of the competitive exercise of certain financial options*, J. Economic Theory 32 (1984), 128–138.

M. Davis and F. Lischka, *Convertible bonds with market risk and credit risk*, Department of Mathematics, Imperial College, London, 1999.

K. Dunn and K. Eades, *Voluntary converstion of convertible securites and the optimal call strategy*, J. Financial Economics 23 (1984), 273–301.

A. Harris and A. Raviv, *A sequential model of convertible debt call policy*, J. Finance 40 (1985), 1263–1282.

T. Ho and M. Pteffer, *Convertible bonds: model, value attribution and analytics*, Financial Analysts Journal 52, Sept.–Oct. (1996), 35–44.

J. E. Ingersoll, *An examination of corporate call policies on convertible securities*, J. Finance 32 (1977), 463–478.

F. Longstaff and E. Schwartz, *Valuing risky debt: a new approach*, University of California at Los Angeles, 1993.

B. Loshak, *The valuation of defaultable convertible bonds under stochastic interest rates*, Ph.D. dissertation, Krannert Graduate School of Management, Purdue University, 1996.

J. J. McConnell and E. S. Schwartz, *LYON taming*, J. Finance **41** (1986), 561–577.

K. Tsiveriotas and C. Fernandes, *Valuing convertible bonds with credit risk*, J. Fixed Income 8 (1998), 95–102.

A. B. Yigitbasioglu, *Pricing convertible bonds with interest rate, equity and FX risk*, ISMA Center, University of Reading, UK, 2002.