Asset Allocation for CARA Utility with Multivariate Lévy Returns

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ABSTRACT. We apply a signal processing technique known as independent component analysis (ICA) to multivariate financial time series. The main idea of ICA is to decompose the observed time series into statistically independent components (ICs). We further assume that the ICs follow the variance gamma (VG) process. The VG process is Brownian motion evaluated with drift at a random time given by a gamma process. We build a multivariate VG portfolio model and analyze empirical results of the investment.

1. Introduction

The relevance of higher moments for investment design has long been recognized in the finance literature and we cite [**Rubinstein (1973)**], and [**Krauss (1976)**] from the earlier literature investigating the asset pricing implications of higher moments. More recently we refer to [**Harvey (2000)**] for the investigation of coskewness in asset pricing. Additionally we note that there appear to be many investment opportunities yielding non-Gaussian returns in the shorter term. This is evidenced by the ability to construct portfolios with return distributions that in fact display very high levels of kurtosis, a typical measure of non-Gaussianity [**Cover (1991)**]. The shorter term perspective is also appropriate for professional investors who can rebalance positions with a greater frequency. Furthermore, we also recognize that there are many ways to construct return possibilities with the same mean and variance but differing levels of skewness and kurtosis. Investment analysis based on traditional Mean-Variance preferences [**Markowitz (1952)**] will not vary the investment across these alternatives, but the presence of skewness preference and kurtosis aversion suggests that the optimal levels of investment should vary across these alternatives. We therefore consider accounting for higher moments in investment analysis.

Key words and phrases. independent component analysis, variance gamma process, utility function.

The theoretical advantages of accounting for higher moments notwithstanding, we note that the computational burdens of portfolio theory using multivariate distributions with long-tailed marginals are extensive in both dimensions of model estimation and the subsequent portfolio design. This lies in sharp contrast to the relative ease with which mean-variance analysis may be executed for large portfolios. We enhance the computational efficiency of such a portfolio theory by bringing to bear some of the fast Fourier transform advances made in the derivative pricing literature as for example [**Carr (1998)**] and [**Carr (2002)**] to facilitate the univariate estimation problems. We adopt the perspective of independent components analysis (ICA)[**Hyvärinen (2001)**] to build dependence and regard returns as linear mixtures of independent processes that are modeled by the Lévy processes used in the derivatives literature. The bulk of the estimation is done by univariate methods after adopting the methods of fast ICA [**Hyvärinen (1999)**] to identify the independent components. The tractability of the final portfolio design is also accomplished by reduction to univariate investment problems for investment in the independent components. The univariate investment problems are solved for in closed form for certainty equivalents related to exponential utility. The final package is thereby rendered quite efficient and we present a back test comparison with mean variance or Gaussian investment.

The outline of the paper is as follows. Section 2 briefly presents results for skewness preference and kurtosis aversion in investment design. The distributional models used for the independent components are described in Section 3. The univariate component investment problem is solved in closed form in Section 4. The portfolio problem is reduced to a sequence of univariate problems in Section 5. Section 6 briefly describes the procedures of ICA for identifying the independent components. The results of the back test are provided in Section 7. Section 8 concludes.

2. Non Gaussian Investment

We begin with fourth order approximations to a general utility function. The reason for going up to the fourth order is that for investments which agree on mean and variance, the third order recognizes a higher order reward statistic for utilities displaying skewness preference, but no risk statistic is then accounted for once we have conditioned on common variances. The next higher order risk statistic is kurtosis, and to account for both reward and risk we consider fourth order approximations to utility. We therefore write for utility $U(x)$ the approximation

$$U(x) \approx U(\mu)+U'(\mu)(x-\mu)+\frac{1}{2}U''(\mu)(x-\mu)^2+\frac{1}{6}U'''(\mu)(x-\mu)^3+\frac{1}{24}U''''(\mu)(x-\mu)^4.$$

Define the skewness s and the kurtosis k [**Karr (1993)**] by

$$\begin{aligned} s &= \frac{E[(x-\mu)^3]}{\sigma^3} \\ k &= \frac{E[(x-\mu)^4]}{\sigma^4} \end{aligned}$$

and write an approximation for the expected utility as

$$E[U(x)] \approx U(\mu) + \frac{1}{2}U''(\mu)\sigma^2 + \frac{1}{6}U'''(\mu)s\sigma^3 + \frac{1}{24}U''''(\mu)k\sigma^4.$$

We also approximate

$$U(\mu) \approx U(0) + U'(0)\mu$$

and assume that $U(0) = 0$ and $U'(0) = 1$. We therefore write

$$E[U(x)] \approx \mu + \frac{1}{2}U''(\mu)\sigma^2 + \frac{1}{6}U'''(\mu)s\sigma^3 + \frac{1}{24}U''''(\mu)k\sigma^4.$$

Employing the exponential utility as a typical approximation to utility with risk aversion parameter η, we get

$$E[U(x)] \approx \mu - \frac{\eta}{2}\sigma^2 + \frac{\eta^2}{6}s\sigma^3 - \frac{\eta^3}{24}k\sigma^4.$$

Consider the question of investing y dollars in a non-Gaussian return with mean μ, variance σ^2, skewness s, and kurtosis k.

The expected utility from this investment on a financed basis with interest rate r is approximately

$$(\mu - r)y - \frac{\eta}{2}\sigma^2 y^2 + \frac{\eta^2}{6}s\sigma^3 y^3 - \frac{\eta^3}{24}k\sigma^4 y^4.$$

The first order condition for the optimal level of investment is

$$\mu - r - \eta\sigma^2 y + \frac{\eta^2}{2}s\sigma^3 y^2 - \frac{\eta^3}{6}k\sigma^4 y^3 = 0. \tag{2.1}$$

We may write equation (2.1) as

$$\frac{\mu - r}{y^*} = \eta\sigma^2 - \frac{\eta^2}{2}s\sigma^3 y^* + \frac{\eta^3}{6}k\sigma^4 (y^*)^2. \tag{2.2}$$

For positive excess return and skewness the optimal y is given by the intersection of a parabola and a hyperbola. This will occur at some positive level for y^*.

We may observe that increased excess returns raise the hyperbola and so raise the level of y^*. Also an increase in σ raises the parabola and so leads to a decrease in y^*. An increase in skewness decreases the slope of the parabola at 0 and shifts the intersection with the hyperbola out further thus increasing y^*, while an increase in kurtosis has the opposite effect.

For a formal analysis of the comparative statistics, we evaluate the differential of the first order condition with respect to y^*, s, and k as our particular interest. This yields the following equation:

$$\left(\frac{\mu - r}{(y^*)^2} - \frac{\eta^2}{2} s\sigma^3 + \frac{\eta^3}{3} k\sigma^4 y^*\right) dy^* = \frac{\eta^2}{2}\sigma^3 y^* ds - \frac{\eta^3}{6}\sigma^4 (y^*)^2 dk.$$

We then have

$$\frac{dy^*}{ds} = \frac{\eta^2\sigma^3 y^*}{2}\left(\frac{\mu - r}{(y^*)^2} - \frac{\eta^2}{2} s\sigma^3 + \frac{\eta^3}{3} k\sigma^4 y^*\right)^{-1},$$

$$\frac{dy^*}{dk} = -\frac{\eta^3\sigma^4 (y^*)^2}{6}\left(\frac{\mu - r}{(y^*)^2} - \frac{\eta^2}{2} s\sigma^3 + \frac{\eta^3}{3} k\sigma^4 y^*\right)^{-1}.$$

The effects of skewness and kurtosis on investment are, respectively, positive and negative, provided the term in the denominator is positive. We may also write that

$$\frac{dy^*}{ds} = \frac{\eta^2\sigma^3 (y^*)^2}{2}\left(\frac{\mu - r}{y^*} - \frac{\eta^2}{2} s\sigma^3 y^* + \frac{\eta^3}{3} k\sigma^4 (y^*)^2\right)^{-1}, \tag{2.3}$$

$$\frac{dy^*}{dk} = -\frac{\eta^3\sigma^4 (y^*)^3}{6}\left(\frac{\mu - r}{y^*} - \frac{\eta^2}{2} s\sigma^3 y^* + \frac{\eta^3}{3} k\sigma^4 (y^*)^2\right)^{-1}. \tag{2.4}$$

Substituting equation (2.2) into equations (**??**) and (2.3) , we obtain:

$$\frac{dy^*}{ds} = \frac{\eta^2\sigma^3 (y^*)^2}{2}\left(\eta\sigma^2 - \eta^2 s\sigma^3 y^* + \frac{\eta^3}{2} k\sigma^4 (y^*)^2\right)^{-1}, \tag{2.5}$$

$$\frac{dy^*}{dk} = -\frac{\eta^3\sigma^4 (y^*)^3}{6}\left(\eta\sigma^2 - \eta^2 s\sigma^3 y^* + \frac{\eta^3}{2} k\sigma^4 (y^*)^2\right)^{-1}. \tag{2.6}$$

Hence for the signs of equations (**??**) and (2.5) to be positive and negative, respectively, we need that

$$1 - \eta s\sigma y^* + \frac{\eta^2}{2} k\sigma^2 (y^*)^2 > 0.$$

The second derivative of expected utility evaluated at the optimum is

$$-\eta\sigma^2 + \eta^2 s\sigma^3 y^* - \frac{\eta^3}{2} k\sigma^4 (y^*)^2.$$

For a maximum, the above expression must be negative. This gives us

$$\eta s\sigma y^* < 1 + \frac{\eta^2}{2} k\sigma^2 (y^*)^2$$

or equivalently,

$$1 - \eta s\sigma y^* + \frac{\eta^2}{2} k\sigma^2 (y^*)^2 > 0.$$

Hence we observe that investment is positively responsive to skewness and negatively responsive to kurtosis.

3. Modelling Distributions

We could build investment strategies based on equation (2.2), which we refer to as *mvsk* investment or investment based on the mean, variance, skewness, and kurtosis. This is essentially a higher moment analog to the classical mean-variance investor. One works with the moments up to the fourth order with no explicit distributional assumptions and a fourth order expansion of utility.

Higher moments like the fourth power are quite volatile with a variance that is at the level of the eighth moment. Accurate estimates require large data series from a common distribution. Though long financial time series are available, it is not clear that distributions are stationary over such long periods. The approach we take is to employ Lévy process distributions that easily accommodate higher moments like skewness and kurtosis, but one may employ estimation strategies which do not get to the eighth moment for the variance. With some process knowledge, one may use lower order powers to estimate the underlying higher moments. The specific Lévy process that we employ is the Variance Gamma (VG) model that has an elementary characteristic function and an elementary Lévy density [**Madan (1998)**]. The utility function we shall work with is exponential utility.

First we briefly define the Variance Gamma Lévy process and its use in modelling the stock price distribution at various horizons. The Variance Gamma process $(X_{VG}(t), t \geq 0)$ evaluates Brownian motion with drift at a random time change given by a gamma process $(G(t), t \geq 0)$. Let

$$Y(t;\sigma,\theta) = \theta t + \sigma W(t)$$

where $W(t)$ is a standard Brownian motion. The process $Y(t;\sigma,\theta)$ is a Brownian motion with drift θ and volatility σ.

Our time change gamma process $G(t;\nu)$ is a Lévy process whose increments $G(t+h;\nu) - G(t;\nu) = g$ have the gamma density with mean h and variance νh [**Rohatgi (2003)**]:

$$f_h(g) = \frac{g^{h/\nu - 1}\exp(-g/\nu)}{\nu^{h/\nu}\Gamma(h/\nu)}.$$

Its characteristic function is [**Billingsley (1995)**]:

$$\phi_g(u) = \left(\frac{1}{1 - iu\nu}\right)^{h/\nu},$$

and for $x > 0$, its Lévy density is:

$$k_g(x) = \frac{\exp(-x/\nu)}{\nu x}.$$

The Variance Gamma process $X_{VG}(t;\sigma,\nu,\theta)$ is defined by

$$\begin{aligned} X_{VG}(t;\sigma,\nu,\theta) &= Y(G(t;\nu);\sigma,\theta) \\ &= \theta G(t;\nu) + \sigma W(G(t;\nu)). \end{aligned}$$

The characteristic function of the VG process may be evaluated by conditioning on the gamma process. This is because, given $G(t;\nu)$, $X_{VG}(t)$ is Gaussian. A simple calculation shows that the characteristic function of the Variance Gamma is

$$\begin{aligned}\phi_{X_{VG}}(t;u) &= E[\exp(iuX_{VG})] \\ &= \left(\frac{1}{1 - iu\theta\nu + \sigma^2\nu u^2/2}\right)^{t/\nu}. \end{aligned} \tag{3.1}$$

The density $p_{VG}(x)$ of the Variance Gamma process at unit time may be obtained in terms of the modified Bessel function $K_a(u)$ and is given by

$$p_{VG}(x) = \frac{\sqrt{2/\pi}}{\nu^{1/\nu}\Gamma(1/\nu)} x^{-(1/2-1/\nu)} \exp\left(\frac{\theta x}{\sigma^2}\right) K_{(1/2-1/\nu)}\left(\frac{x}{\sigma^2}\left(\theta^2 + \frac{2\sigma^2}{\nu}\right)\right). \tag{3.2}$$

The density of the Variance Gamma process for time increments other than the unit time may be obtained by making the substitution in equation (3.2) as follows:

$$\begin{aligned}\sigma &\to \sigma\sqrt{t}, \\ \nu &\to \nu/t, \\ \theta &\to \theta t.\end{aligned}$$

The Variance Gamma process is a Lévy process with infinitely divisible distributions. Thus the characteristic function of the process may be written as the Lévy-Khintchine form [**Sato (1999)**], and the Lévy measure K_{VG} is given by [**Carr (2002)**]

$$K_{VG}(x) = \frac{C}{|x|} \exp\left(\frac{G-M}{2}x - \frac{G+M}{2}|x|\right) \tag{3.3}$$

where

$$\begin{aligned}C &= \frac{1}{\nu}, \\ G &= \sqrt{\frac{2}{\sigma^2\nu} + \frac{\theta^2}{\sigma^4}} + \frac{\theta}{\sigma^2}, \\ M &= \sqrt{\frac{2}{\sigma^2\nu} + \frac{\theta^2}{\sigma^4}} - \frac{\theta}{\sigma^2}.\end{aligned}$$

The density for the Variance Gamma process can display both skewness and excess kurtosis. The density is symmetric when $\theta = 0$ and $G = M$, and the kurtosis is $s(1+\nu)$ in this case. The parameter θ generates skewness and we have a negatively skewed density for $\theta < 0$ and a positively skewed one when $\theta > 0$.

We may accommodate a separate mean by considering the process

$$\begin{aligned}H(t) &= \mu t + \theta(G(t) - t) + \sigma W(G(t)) \\ &= (\mu - \theta)t + X_{VG}(t)\end{aligned}$$

with the characteristic function

$$\begin{aligned}\phi_{H(t)}(u) &= E[e^{iuH(t)}] \\ &= e^{iu(\mu-\theta)t}\phi_{X_{VG(t)}}(u).\end{aligned}$$

This gives us a four parameter process capturing the first four moments of the density.

4. Exponential Utility and Investment in Zero Cost VG Cash Flows

Suppose we invest y dollars in a zero cost cash flow with a VG distribution for the investment horizon of length h with mean $(\mu - r)h$. We may write the zero cost cash flow accessed as X

$$X = (\mu - r)h + \theta(g-1) + \sigma W(g), \tag{4.1}$$

where g is gamma distributed with unit mean and variance ν, and $W(g)$ is Gaussian with zero mean and variance g. We suppose the VG parameters are for the holding period h as the unit period. We also suppose that μ and r have been adjusted for the length of the period and take this to be unity in what follows.

The final period wealth is

$$W = yX.$$

We employ exponential utility and write

$$U(W) = 1 - \exp(-\eta W), \tag{4.2}$$

where η is the coefficient of risk aversion. The certainty equivalent CE solves

$$E(U(W)) = 1 - \exp(-\eta CE).$$

The goal of the investment is to maximize the expected utility function. The expected utility is

$$\begin{aligned}E(U(W)) &= E(1 - \exp(-\eta W)) \\ &= 1 - E(\exp(-y\eta X)).\end{aligned} \tag{4.3}$$

To determine y which maximizes the expected utility is equivalent to minimizing the following expression with respect to y:

$$E(\exp(-y\eta X)).$$

THEOREM 4.1. *Suppose we invest y dollars in a zero cost cash flow with a VG distribution described in equation 4.1 for the investment horizon of length h. And suppose that we employ the exponential utility function as in equation (4.2). The optimal solution for the investment is*

$$\begin{aligned}\tilde{y} &= \left(\frac{\theta}{\sigma^2} - \frac{1}{(\mu - r - \theta)\nu}\right) \\ &\quad + \text{sign}(\mu - r)\sqrt{\left(\frac{\theta}{\sigma^2} - \frac{1}{(\mu - r - \theta)\nu}\right)^2 + \frac{2(\mu - r)}{(\mu - r - \theta)\nu\sigma^2}}\end{aligned}$$

where $\tilde{y} = \eta y$ and η is the risk aversion coefficient.

PROOF. See Appendix A. □

When $\mu > r$, y is positive and we have a long position. Likewise for $\mu < r$, y is negative and we have a short position.

5. Multivariate VG Portfolio

We take an investment horizon of length h and wish to study optimal portfolio for investment in a vector of assets whose zero cost excess returns or financed returns over this period are $R - rh$. Once again we suppose all parameters are adjusted for the time horizon and take this to be unity in what follows.

Let the vector y denote the dollar investment in the collection of assets. We suppose the mean excess return is $\mu - r$ and hence that

$$R - r = \mu - r + x,$$

where x is the zero mean random asset return vector.

Our structural assumption is that there exist a vector of independent zero mean VG random variables s of the same dimension as x and a matrix A such that

$$x = As. \tag{5.1}$$

The law of s_i is that of

$$s_i = \theta_i(g_i - 1) + \sigma_i W_i(g_i),$$

where the $W_i' s$ are independent Brownian motions, and the g_i are gamma variates with unit mean and variance ν_i.

The strategy for estimating this structure is to linearly transform the observed data x into a new vector $\tilde{x}$ so that the observed data is whitened; that is, the expectation

$$E(\tilde{x}\tilde{x}') = I.$$

In other words, the components of $\tilde{x}$ are uncorrelated, and the variances are equal to unity. One way to whiten the data is to use the eigenvalue decomposition of the covariance matrix C of x. That is,

$$C = E(xx') = EDE',$$

where E is the orthogonal matrix of the eigenvectors of matrix C, and D is diagonal matrix of the corresponding eigenvalues [**Anderson (1958)**]. We denote $D = diag(d_1, ..., d_n)$, and that $D^{-\frac{1}{2}} = diag(d_1^{-\frac{1}{2}}, ..., d_n^{-\frac{1}{2}})$. Whitening gives us

$$\tilde{x} = ED^{-\frac{1}{2}}E'x. \tag{5.2}$$

It follows that

$$E(\tilde{x}\tilde{x}') = I.$$

From equations (5.1) and (5.2), we then have

$$\tilde{x} = ED^{-\frac{1}{2}}E'x = ED^{-\frac{1}{2}}E'As = \tilde{A}s.$$

Note that

$$E(\tilde{x}\tilde{x}') = \tilde{A}E(ss')\tilde{A}' = \tilde{A}\tilde{A}' = I,$$

so that the new mixing matrix $\tilde{A}$ is an orthogonal matrix. Estimating the orthogonal matrix $\tilde{A}$ is computationally simpler than estimating the original mixing matrix A. Under the assumption that the observed data x is whitened, the goal of ICA is to find a demixing matrix W such that

$$\begin{aligned} z &= Wx \\ &= WAs. \end{aligned}$$

If $W = A^{-1}$, then $z = s$. It is hard to find the perfect separation $z = s$. In general, it's possible to find W such that $WA = PD$ where P is a permutation matrix, and D is a diagonal matrix. We apply ICA to get the demixing matrix W [**Amari (1996)**]. We then construct the data for the independent components s by

$$s = Wx.$$

The VG parameters can be estimated on these series by univariate methods.

Theorem 5.1. *Let the vector y denote the dollar investment in the collection of assets. We suppose the mean excess return is $\mu - r$ and the zero cost excess return is $R - r$, hence that*

$$R - r = \mu - r + x,$$

where x is the zero mean random asset return vector and assume that $E[xx'] = I$. Let

$$x = As$$

and assume the law of s_i is

$$s_i = \theta_i(g_i - 1) + \sigma_i W_i(g_i),$$

where A is the mixing matrix, the $W_i's$ are independent Brownian motions, and the g_i are gamma variates with unit mean and variance ν_i. Denote

$$\zeta = A^{-1}\frac{\mu - r}{\eta} - \frac{\theta}{\eta}$$

and

$$y = \frac{1}{\eta}A^{-1}\tilde{y},$$

where $y = (y_1, y_2, \cdots, y_n)'$, $\tilde{y} = (\tilde{y_1}, \tilde{y_2}, \cdots, \tilde{y_n})'$, *and* η *is the risk aversion coefficient. Then the solution of* $\tilde{y_i}$, *for* $i = 1, 2, \cdots, n$, *is given by*

$$\begin{aligned}\tilde{y_i} &= \frac{|\zeta_i|\theta_i\nu_i - sign(\zeta_i)\frac{\sigma_i^2}{\eta}}{|\zeta_i|\sigma_i^2\nu_i} \pm \\ &\quad \frac{\sqrt{\left(|\zeta_i|\theta_i\nu_i - sign(\zeta_i)\frac{\sigma_i^2}{\eta}\right)^2 + 2\left(|\zeta_i| + sign(\zeta_i)\frac{\theta_i}{\eta}\right)|\zeta_i|\sigma_i^2\nu_i}}{|\zeta_i|\sigma_i^2\nu_i} \\ &= \frac{\theta_i}{\sigma_i^2} - \frac{1}{\eta\zeta_i\nu_i} \pm \sqrt{\left(\frac{\theta_i}{\sigma_i^2} - \frac{1}{\eta\zeta_i\nu_i}\right)^2 + 2\frac{\zeta_i + \frac{\theta_i}{\eta}}{\zeta_i\sigma_i^2\nu_i}}, \qquad (5.3)\end{aligned}$$

PROOF. See Appendix B. □

We take the positive or the negative root depending on the sign of $\left(\zeta_i + \frac{\theta_i}{\eta}\right)$ which is essentially the implied component mean.

6. ICA in Finance

There are many factors that drive the movements of asset returns. It is not unusual to assume that a set of different asset returns are driven by some common factors. ICA is a process which takes a set of multivariate observed data, x, and extracts from them a new set of statistically independent components, s [**Cardoso (1998)**]. ICA assumes that the observed data vectors x are the result of a mixing process

$$x_i(t) = \sum_{j=1}^{n} a_{ij} s_j(t).$$

$s_j(t)$ are assumed to be statistically independent. They can be sources from wide range which affect the asset returns. Using matrix notation, the model can be written as

$$x = As,$$

where A is the unknown mixing matrix.

Another key assumption of ICA is that the independent components s are non-Gaussian [**Hyvärinen (2001)**]. Option pricing theory was introduced by Fischer Black and Myron Scholes [**Black (1973)**]. In order to value options, Black and Scholes derived a partial differential equation via a hedging argument. The Black-Scholes model evolves a stock price p from the geometric Brownian motion model [**Björk (1998)**]. The stock price model is given by

$$dp = \mu p dt + \sigma p dw,$$

where μ denotes the drift rate, σ is the volatility, and w is the standard Brownian motion. While the Black-Scholes model remains the most widely used in the financial world, it has known shortcomings, such as volatility smiles.

The Variance Gamma model proposed in [**Madan (1998)**] replaces the diffusion process in the Black-Scholes model by a pure jump process. The VG process is a non-Gaussian process. While ICA requires no knowledge about the distributions of the independent components s, we assume that s follows the non-Gaussian VG process. We then estimate VG parameters on these series by univariate methods [**Carr (1998)**]. We use ICA to decompose the multivariate stock return data into statistically independent components. We hope to investigate the common factors for the multivariate stock price returns. The VG model provides the information of higher order statistics. The Gaussian model gives only second order statistics. Non-gaussian models may usefully employ ICA algorithms to process the vector of returns to univariate components amenable to the modelling of higher order statistics using Lévy models.

7. Empirical Results

We apply ICA to the multivariate financial time series. The goal is to decompose the observed multivariate time series into a linear combination of statistically independent components [**Hyvärinen (1999)**]. We assume that the independent components follow the non-Gaussian VG process. We use daily adjusted closing prices from five companies in the S&P 500. Note that

$$x(t) = \frac{p(t) - p(t-1)}{p(t-1)}$$

where $p(t)$ is the stock price for time t. The five stocks chosen are 3M Company, Boeing Company, IBM, Johnson & Johnson, McDonald's Corp., Merck & Co. We take the first 1000 time series data since January 1990 for our first analysis. We then move forward one month to get the second set of rolling 1000 day time series data for our second time period analysis. We repeat the same methodology for 125 time periods of our analysis from January 1990 to May 2004. Thus, we have 125 different 5 by 1000 matrices x_i, $i = 1, 2 \ldots, 125$ of the relative daily returns.

Performing an ICA analysis on the data of these matrices yields 125 sets of 5 non-Gaussian independent components on which we estimate the VG process by 5 univariate applications done 125 times. To appreciate the degree of non-Gaussianity attained by the ICA we present a table with the average level of kurtosis attained for each of the five independent components. We also did such an ICA analysis on a Monte Carlo vector of truly Gaussian returns and found no ability to generate any excess kurtosis. We conjecture that actual investment returns provide considerable access to informative or kurtotic return scenarios that would be of interest to preferences reflecting a concern for these higher moments.

We study investment design by using equation (5.3) to compute the vector of dollars, y, invested in each stock under the hypothesis of returns being a linear mixture of independent VG processes. We also compute dollar amounts invested for the Gaussian process for comparison [**Elton (1991)**].

	VG	Gauss
Sharpe Ratio	0.2548	0.2127
CE ($\eta = .0005$)	47.6883	0.0230
Gain-Loss Ratio	2.3909	1.4536

TABLE 1. Performance Measures

	mean	minimum	maximum
1st IC	15.3388	4.2466	54.1112
2nd IC	12.9027	3.9871	49.4759
3rd IC	8.6070	3.9973	41.8942
4th IC	6.3648	3.7159	18.5333
5th IC	5.4536	3.5134	12.0329

TABLE 2. Summary of the Kurtosis for the Five ICs

At the end of each investment time period, we invest an amount of money y according to our analysis. When an element of y is positive, we take a long position. When an element of y is negative, a short position is taken. We look forward in the time series by one month and calculate the cash flow CF at the end of the month for each time period. The formula is as follows:

$$CF = y \cdot \left(\frac{p(t+21) - p(t)}{p(t)} - r \right)$$

where $p(t)$ is the initial price of the investment, $p(t+21)$ is the price at the maturity, and r is the 3-month treasury bill monthly interest rate. Note that we used $p(t+21)$ as the maturity price, because there are 21 trading days in a month on average. Table 1 presents the three performance measures, the Sharpe ratio, the certainty equivalent (CE), and the gain-loss ratio of both the VG and the Gaussian processes [**Farrell (1997)**]. Table 2 displays the summary of the kurtosis of the five independent compone

Figure 1 and Figure 2 plot the cumulated cash flows through the 125 investment time periods of our analysis for the VG and the Gaussian processes.

8. Conclusion

We present and back test an asset allocation procedure that accounts for higher moments in investment returns. The allocation procedure is made computationally efficient by employing independent components analysis and in particular the fast ICA algorithm to identify long-tailed independent components in the vector of asset returns. Univariate methods based on the fast Fourier transform then analyze these components using models

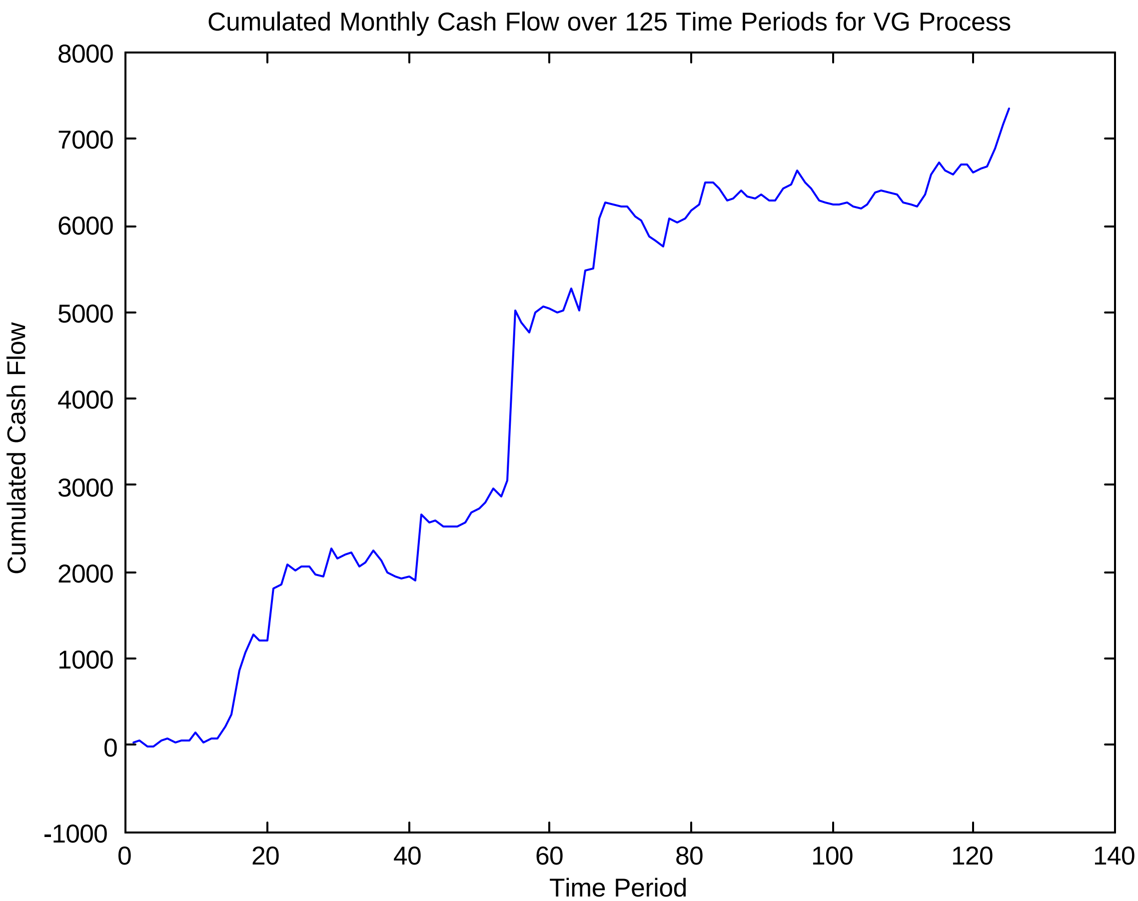

FIGURE 1. VG Cumulated Cash Flow

popularized in the literature on derivative pricing. The multivariate portfolio allocation problem is then reduced to univariate problems of component investment and the latter are solved for in closed form for exponential utility.

The back test shows that the resulting allocations are substantially different from the Gaussian approach with an associated cumulated cash flow that can outperform Gaussian investment. The packaging of fast ICA, the fast Fourier transform and the wide class of Lévy process models now available make higher moment asset allocation a particularly attractive area of investment design and future research.

Appendix A. Proof of Theorem 4.1

PROOF. To find the optimal solution for the investment, our goal is to maximize the expected utility function as in equation (4.3). It is equivalent

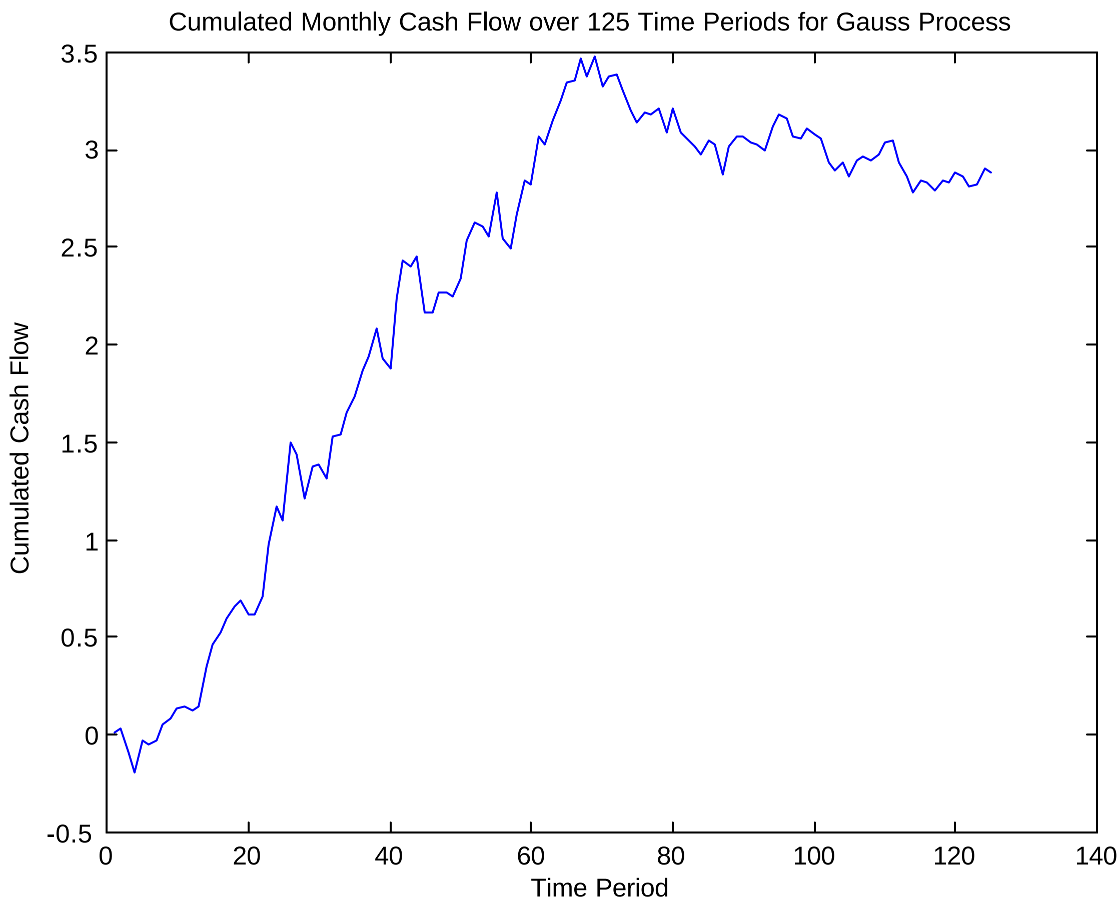

FIGURE 2. Gaussian Cumulated Cash Flow

to minimizing

$$E(\exp(-y\eta X))$$

over y.

$$\begin{aligned}
&E(\exp(-y\eta X)) \\
&= \exp(-y\eta(\mu - r - \theta))E\left(\exp\left(-\left(y\eta\theta - \frac{y^2\eta^2\sigma^2}{2}\right)g\right)\right) \\
&= \exp\left(-y\eta(\mu - r - \theta) - \frac{1}{\nu}\ln\left(1 + \nu\left(y\eta\theta - \frac{y^2\eta^2\sigma^2}{2}\right)\right)\right).
\end{aligned}$$

Minimizing the above expression is equivalent to maximizing

$$z(y) = y\eta(\mu - r - \theta) + \frac{1}{\nu}\ln\left(1 + \nu\left(y\eta\theta - \frac{y^2\eta^2\sigma^2}{2}\right)\right).$$

Suppose $\alpha, \beta \in R$ and $\alpha < 0 < \beta$. Let

$$q(y) = 1 + \nu \left(y\eta\theta - \frac{y^2\eta^2\sigma^2}{2} \right),$$

and $q(\alpha) = q(\beta) = 0$. The function $q(y) > 0$ for $y \in (\alpha, \beta)$ and q is differentiable on (α, β) and continuous on $[\alpha, \beta]$. We have

$$\begin{aligned} z(0) &= 0 \\ \lim_{y \to \alpha^+} z(y) &= -\infty \\ \lim_{y \to \beta^-} z(y) &= -\infty \end{aligned}$$

so that a maximum of z(y) exists on the interval (α, β). The first order condition with respect to y leads to

$$z'(y) = \eta(\mu - r - \theta) + \frac{\eta\theta - \eta^2\sigma^2 y}{1 + \nu\eta\theta y - \nu\eta^2\sigma^2 y^2/2}.$$

Furthermore, assume y_1 and y_2 are two roots for $z'(y) = 0$, and $y_1 < 0$, $y_2 > 0$. That is, $z'(y_1) = z'(y_2) = 0$. Setting $z'(y) = 0$, we obtain

$$\begin{aligned} &(\mu - r - \theta)\left(1 + \nu\eta\theta y - \frac{\nu\eta^2\sigma^2}{2} y^2\right) + \theta - \eta\sigma^2 y \\ \text{(A.1)} \qquad = \; &\mu - r + ((\mu - r - \theta)\nu\theta - \sigma^2)\eta y - (\mu - r - \theta)\frac{\nu\eta^2\sigma^2}{2} y^2 \end{aligned}$$

Observe that $z'(0) > 0$ if $\mu > r$. We have $z(y_1) < 0$ and $z(y_2) > 0$. According to the mean value theorem, y_2 is the root which gives the optimal solution. Similarly, if $\mu < r$, then $z'(0) < 0$. We have $z(y_1) > 0$ and $z(y_2) < 0$ so that y_1 gives the optimal solution in this condition. Let $\tilde{y} = y\eta$ and solve for this magnitude, noting that y is then $\tilde{y}/\eta$. Hence we rewrite equation (A.1) as

$$\begin{aligned} &\tilde{y}^2 - 2\frac{(\mu - r - \theta)\nu\theta - \sigma^2}{(\mu - r - \theta)\nu\sigma^2}\tilde{y} - \frac{2(\mu - r)}{(\mu - r - \theta)\nu\sigma^2} \\ = \; &\tilde{y}^2 - 2\left(\frac{\theta}{\sigma^2} - \frac{1}{(\mu - r - \theta)\nu}\right)\tilde{y} - \frac{2(\mu - r)}{(\mu - r - \theta)\nu\sigma^2} \\ = \; &0. \end{aligned}$$

Hence we have

$$\begin{aligned} \tilde{y} = \; &\left(\frac{\theta}{\sigma^2} - \frac{1}{(\mu - r - \theta)\nu}\right) \\ &+ \text{sign}(\mu - r)\sqrt{\left(\frac{\theta}{\sigma^2} - \frac{1}{(\mu - r - \theta)\nu}\right)^2 + \frac{2(\mu - r)}{(\mu - r - \theta)\nu\sigma^2}}. \end{aligned}$$

□

Appendix B. Proof of Theorem 5.1

Proof. We choose the investment vector y to maximize expected exponential utility for a risk aversion coefficient η. The objective is therefore that of maximizing

$$1 - e^{-\eta y'(\mu - r)} E[e^{-\eta y' x}] = 1 - e^{-\eta y'(\mu - r)} E[e^{-\eta y' A s}].$$

The expectation is then given by

$$E[e^{-\eta y' A s}] = \exp\left(\sum_{i=1}^{n} \eta (y'A)_i \theta_i - \frac{1}{\nu_i} \ln \left(1 + \theta_i \nu_i \eta (y'A)_i - \frac{\sigma_i^2 \nu_i}{2} \eta^2 (y'A)_i^2 \right) \right).$$

It follows that the certainty equivalent is

$$CE = y'(\mu - r) + \sum_{i=1}^{n} (-y'A)_i \theta_i + \frac{1}{\eta \nu_i} \ln \left(1 + \theta_i \nu_i \eta (y'A)_i - \frac{\sigma_i^2 \nu_i}{2} \eta^2 (y'A)_i^2 \right).$$

We may write equivalently

$$CE = \eta (y'A) \left(A^{-1} \frac{\mu - r}{\eta} - \frac{\theta}{\eta} \right) + \sum_{i=1}^{n} \frac{1}{\eta \nu_i} \ln \left(1 + \theta_i \nu_i \eta (y'A)_i - \frac{\sigma_i^2 \nu_i}{2} \eta^2 (y'A)_i^2 \right).$$

Now define

$$\begin{aligned} \tilde{y}' &= \eta y' A, \\ \zeta &= A^{-1} \frac{\mu - r}{\eta} - \frac{\theta}{\eta}, \end{aligned}$$

and write

$$\begin{aligned} CE &= \sum_{i=1}^{n} \left[\zeta_i \tilde{y}_i + \frac{1}{\eta \nu_i} \ln \left(1 + \theta_i \nu_i \tilde{y}_i - \frac{\sigma_i^2 \nu_i}{2} \tilde{y}_i^2 \right) \right] \\ &= \sum_{i=1}^{n} \psi(\tilde{y}_i). \end{aligned}$$

We have additive functions in the vector $\tilde{y}_i$ and these may be solved for using univariate methods in closed form. We then determine

$$y = \frac{1}{\eta} A^{-1} \tilde{y}.$$

First observe that the argument of the logarithm is positive only in a finite interval for $\tilde{y}_i$. Hence the CE maximization problem has an interior solution for $\tilde{y}_i$.

The first order condition yields

$$\psi'(\tilde{y}_i) = \zeta_i + \frac{\frac{\theta_i}{\eta} - \frac{\sigma_i^2}{\eta} \tilde{y}_i}{1 + \theta_i \nu_i \tilde{y}_i - \frac{\sigma_i^2 \nu_i}{2} \tilde{y}_i^2} = 0.$$

It is clear that

$$\psi'(0) = \zeta_i + \frac{\theta_i}{\eta}$$

and the optimal value for $\tilde{y_i}$ is positive when $\psi'(0) > 0$ and negative otherwise.

We may write the condition as

$$|\zeta_i| + \frac{\operatorname{sign}(\zeta_i)\left(\frac{\theta_i}{\eta} - \frac{\sigma_i^2}{\eta}\tilde{y_i}\right)}{1 + \theta_i\nu_i\tilde{y_i} - \frac{\sigma_i^2\nu_i}{2}\tilde{y_i}^2} = 0.$$

The argument of the logarithm must be positive and so we write

$$|\zeta_i|\left(1 + \theta_i\nu_i\tilde{y_i} - \frac{\sigma_i^2\nu_i}{2}\tilde{y_i}^2\right) + \operatorname{sign}(\zeta_i)\left(\frac{\theta_i}{\eta} - \frac{\sigma_i^2}{\eta}\tilde{y_i}\right) = 0.$$

We may rewrite this expression as the quadratic

$$\left(|\zeta_i| + \operatorname{sign}(\zeta_i)\frac{\theta_i}{\eta}\right) + \left(|\zeta_i|\theta_i\nu_i - \operatorname{sign}(\zeta_i)\frac{\sigma_i^2}{\eta}\right)\tilde{y_i} - \frac{|\zeta_i|\sigma_i^2\nu_i}{2}\tilde{y_i}^2 = 0,$$

or equivalently that

$$\frac{|\zeta_i|\sigma_i^2\nu_i}{2}\tilde{y_i}^2 - \left(|\zeta_i|\theta_i\nu_i - \operatorname{sign}(\zeta_i)\frac{\sigma_i^2}{\eta}\right)\tilde{y_i} - \left(|\zeta_i| + \operatorname{sign}(\zeta_i)\frac{\theta_i}{\eta}\right) = 0.$$

The solution for $\tilde{y_i}$ is given by

$$\begin{aligned}
\tilde{y_i} &= \frac{|\zeta_i|\theta_i\nu_i - \operatorname{sign}(\zeta_i)\frac{\sigma_i^2}{\eta}}{|\zeta_i|\sigma_i^2\nu_i} \\
&\pm \frac{\sqrt{\left(|\zeta_i|\theta_i\nu_i - \operatorname{sign}(\zeta_i)\frac{\sigma_i^2}{\eta}\right)^2 + 2\left(|\zeta_i| + \operatorname{sign}(\zeta_i)\frac{\theta_i}{\eta}\right)|\zeta_i|\sigma_i^2\nu_i}}{|\zeta_i|\sigma_i^2\nu_i} \\
&= \frac{\theta_i}{\sigma_i^2} - \frac{1}{\eta\zeta_i\nu_i} \pm \sqrt{\left(\frac{\theta_i}{\sigma_i^2} - \frac{1}{\eta\zeta_i\nu_i}\right)^2 + 2\frac{\zeta_i + \frac{\theta_i}{\eta}}{\zeta_i\sigma_i^2\nu_i}}.
\end{aligned}$$

□

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