Credit Spreads, Optimal Capital Structure, and Implied Volatility with Endogenous Default and Jump Risk

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1. Introduction

- Credit risk leads to credit spreads
- Endogenous default vs. exogenous default
- Debt/equity ratio will be affected by credit risk
- Credit risk affects values of both defaultable bonds and firm equity.

Stylized Facts

- Non-zero credit spreads, even as the maturity $T \longrightarrow 0$.
- Various shapes of credit spreads. Upward, downward, humped.
- Negative correlations between credit spreads and risk-free rate.
- Link between credit spreads and implied volatility dating back to Black's conjecture, which says that credit spreads $\uparrow$ as implied volatility $\uparrow$.
- Firms with low recovery rate, large jump risk, and high volatility (e.g. some tech firms) tend to have very little debts, even with tax advantage of issuing debt.
- See the books by Bielecki and Rutkowski (2002), Duffie and Singleton (2003), Lando (2004), Schönbucher (2004).

The contribution of the current paper.

(1) Extending the Leland-Toft endogenous default model based on pure diffusion (Leland, 1994, Leland and Toft, 1996).

(2) By adding jumps, the model produces flexible credit spreads (including upward, humps, downward shapes) with non-zero credit spreads.

(3) Upward curve even for highly risky bonds.

(4) It leads to flexible implied volatility smiles for equity options. Differences between exogenuous and endogenuous defaults.

(5) Jumps lead to much lower debt/equity ratios. The model implies that firms with large jump risk, high volatility, and low recovery rate tend to have little debts.

(6) Analytical solutions for debt and equity values are derived.

2. Comparison with existing literature

(1) Zhou (2001), normal jump-diffusion, exogenous default.

(2) Hilberink and Rogers (2002), one-sided (rather than two-sided) Levy process; sometimes using numerical results to verify "smothing paste". One-sided jump cannot generate convex implied volatility smile.

(3) Huang and Huang (2003), Cremers et al. (2005), empirical analysis based on the DE jump diffusion model but with exogenous default.

(4) Collin-Dufresne and Goldstein (2001), exogenous, mean-reverting default barriers.

(5) Dao (2005), behavior finance aspects.

We discuss credit spreads, optimal capital structure, and implied volatility in a unified framework.

Other explanations for credit spreads in the literature.

- Duffie and Lando (2001) shows how imperfect observation of a diffusion model can also explain of the nonzero limit of credit spreads.
- Huang and Huang (2003) suggests that liquidity difference.
- Leland (2004) supports the explanation of jumps.
- Linetsky (2004) proposes a CEV type model, which assumes that the stock process follows the traditional diffusion process until the bankrupt event happens, in which the stock price suddenly drops down to zero.

3. Basic Setting of the Model

3.1 Asset Model

- Total un-leveraged asset market value process is given by

$$\frac{dV_t}{V_{t-}} = (r + \pi - \delta)dt + \sigma dB_t + d\left[\sum_{i=1}^{N_t}(Z_i - 1)\right] - \lambda\xi dt$$

- r is the interest rate, π is the risk premium, δ is the proportional rate at which profit is distributed to both bond and equity investors,

- Z_i's are i.i.d. random variables and $Y = \ln(Z_1)$ has a double-exponential distribution

$$f_Y(y) = p_u\eta_u e^{-\eta_u y}\mathbf{1}_{\{y\geq 0\}} + p_d\eta_d e^{\eta_d y}\mathbf{1}_{\{y<0\}},$$

$$\eta_1 > 1, \eta_2 > 0, \ \xi = E[e^Y - 1] = \frac{p_u \eta_u}{\eta_u - 1} + \frac{p_d \eta_d}{\eta_d + 1} - 1.$$

- We need to compute the first passage time of the process; Kou and Wang (2003).
- By using the rational expectations argument with a HARA type of utility function for the representative agent, one can choose a particular risk-neutral measure P^*

$$\frac{dV_t}{V_{t-}} = (r - \delta)dt + \sigma dB_t^* + d\left[\sum_{i=1}^{N_t^*}(Z_i^* - 1)\right] - \lambda^* \xi^* dt$$

3.2 Debt issuing follows Leland and Toft (1996).

- Within the time interval $(t, t+dt)$, the firm issues new debt with par value pdt, and maturity profile φ, where $\varphi(t) = me^{-mt}$ (i.e. the maturity is chosen randomly according to an exponential distribution with mean $1/m$).
- At time interval $(t, t+dt)$, the total debt due is

$$\left(\int_{-\infty}^{t} p\varphi(t-u)du\right) dt = pdt,$$

which is exactly the par value of the newly-issued debt.

- Thus the par value of all pending debt is a constant $P = p\int_0^{+\infty} e^{-ms}ds = \frac{p}{m}$.

3.3 Coupon Payment and Default Payment

- Two debt liabilities: after-tax coupon payment $(1-\kappa)\rho P dt$ and due principal pdt. The total cash outflow to the bond-holders is $((1-\kappa)\rho P + p)dt$.
- Two cash inflows: $b_t dt$ from selling new debts, where b_t is the price of the total newly issued bonds, and the total asset payout cash flow is $\delta V dt$.
- If $(\delta V + b(t))dt > ((1-\kappa)\rho P + p)dt$, the difference of these two goes to the party of the equity holders as dividends; otherwise, additional equity must be issued to fulfill the due debt liabilities.
- The difference $(((1-\kappa)\rho P + p) - (\delta V + b)dt$ is an infinitesimal quantity. Thus, such financing strategy is possible as long as the equity value remains positive. Limit liability constraint.

Default

- Default time $\tau = \inf\{t \geq 0 : V_t \leq V_B\}$.
- On the default, the firm loses $(1-\alpha)$ of V_τ to reorganize the firm and the bondholders shares the rest of the value left, αV_τ, after reorganization.
- At default time, all bond holders require portion of the remaining asset of the firm, αV_τ.

How to distribute the remaining asset among debt holders?

- We assume recovery at a fraction of the treasury bonds

$$ce^{-r(T-\tau)}.$$

- To determine c, we have by the memoeyless property from the bond maturity profile,

$$P \int_{\tau}^{+\infty} ce^{-r(T-\tau)} \cdot me^{-m(T-\tau)} dT = \alpha V_{\tau}.$$

- $c = \frac{m+r}{m} \frac{\alpha V_{\tau}}{P}$

Therefore, the bond price is given by

$$
\begin{aligned}
& B(V, 0; V_B, T) \\
= \; & e^{-rT} E[\mathbf{1}_{\{\tau \geq T\}}] \\
& \quad + E[e^{-r\tau} \cdot \alpha \frac{V_\tau}{P} \frac{m+r}{m} e^{-r(T-\tau)} \mathbf{1}_{\{\tau \leq T\}}] \\
& \quad + E[\int_0^{\tau \wedge T} \rho e^{-rs} ds]
\end{aligned}
$$

3.4. Analytical Solutions for Debt and Equity Values

Consider a polynomial equation

$$G(x) = r + \beta,$$

$$G(x) = -(r - \delta - \frac{1}{2}\sigma^2 - \lambda\xi)x + \frac{1}{2}\sigma^2 x^2 + \lambda(\frac{p_d \eta_d}{\eta_d - x} + \frac{p_u \eta_u}{\eta_u + x} - 1).$$

From Kou and Wang (2003), it must have four roots, denoted by $\gamma_{1,\beta}, \gamma_{2,\beta}, -\gamma_{3,\beta}, -\gamma_{4,\beta}$.

Lemma 1: The Laplace transform of $B(V,0;V_B,T)$ is

$$\int_0^{+\infty} e^{-\beta T} B(V,0;V_B,T)dT$$

$$= \frac{\rho+\beta}{\beta(r+\beta)} - \frac{\beta+\rho}{\beta(r+\beta)}\left\{d_{1,\beta}\left(\frac{V_B}{V}\right)^{\gamma_{1,\beta}} + d_{2,\beta}\left(\frac{V_B}{V}\right)^{\gamma_{2,\beta}}\right\}$$

$$+\frac{\alpha(m+r)}{mP(\beta+r)}V\left\{c_{1,\beta}\left(\frac{V_B}{V}\right)^{\gamma_{1,\beta}+1} + c_{2,\beta}\left(\frac{V_B}{V}\right)^{\gamma_{2,\beta}+1}\right\},$$

$$c_{1,\beta} = \frac{\eta_d-\gamma_{1,\beta}}{\gamma_{2,\beta}-\gamma_{1,\beta}}\frac{\gamma_{2,\beta}+1}{\eta_d+1},\ c_{2,\beta} = \frac{\gamma_{2,\beta}-\eta_d}{\gamma_{2,\beta}-\gamma_{1,\beta}}\frac{\gamma_{1,\beta}+1}{\eta_d+1},$$

$$d_{1,\beta} = \frac{\eta_d-\gamma_{1,\beta}}{\eta_d}\frac{\gamma_{2,\beta}}{\gamma_{2,\beta}-\gamma_{1,\beta}},\ d_{2,\beta} = \frac{\gamma_{2,\beta}-\eta_d}{\eta_d}\frac{\gamma_{1,\beta}}{\gamma_{2,\beta}-\gamma_{1,\beta}}.$$

In particular the value of total debt is

$$
\begin{aligned}
& D(V; V_B) \\
= \; & P \int_0^{+\infty} m e^{-mT} B(V, 0; V_B, T) dT \\
= \; & \frac{P(\rho + m)}{r + m} [1 - \left\{ d_{1,m} \left(\frac{V_B}{V} \right)^{\gamma_{1,m}} + d_{2,m} \left(\frac{V_B}{V} \right)^{\gamma_{2,m}} \right\}] \\
& + \alpha V_B \left\{ c_{1,m} \left(\frac{V_B}{V} \right)^{\gamma_{1,m}} + c_{2,m} \left(\frac{V_B}{V} \right)^{\gamma_{2,m}} \right\}
\end{aligned}
$$

The formula above has the following interpretation:

- $\frac{P(\rho+m)}{r+m}$ is the present value of the debt with face value P and maturity profile $\phi(t) = me^{-mt}$ and without any bankruptcy.
- The term right afterwards in the first summand is the present value of \$1 contingent on future bankruptcy.
- The second summand is what the bondholders can get from the bankruptcy procedure.
- Due to jumps, in contract to Leland (1994), the remaining asset after bankruptcy is not αV_B any more.

As in Brealey and Myers (1991), the total market value of the firm is the un-leveraged asset value V plus the value of tax benefits less the value of bankruptcy costs.

Lemma 1 (Continued): Total firm value is given by

$$
\begin{aligned}
& v(V;V_B) \\
= \; & V + \frac{P\kappa\rho}{r}\{1 - \left\{ d_{1,0}\left(\frac{V_B}{V}\right)^{\gamma_{1,0}} + d_{2,0}\left(\frac{V_B}{V}\right)^{\gamma_{2,0}} \right\}\} \\
& \quad -(1-\alpha)V\left\{ c_{1,0}\left(\frac{V_B}{V}\right)^{\gamma_{1,0}+1} + c_{2,0}\left(\frac{V_B}{V}\right)^{\gamma_{2,0}+1} \right\}.
\end{aligned}
$$

Total equity value is

$$
\begin{aligned}
& S(V;V_B) = v(V;V_B) - D(V;V_B) \\
= \ & V - \frac{P\kappa\rho}{r}\left\{d_{1,0}\left(\frac{V_B}{V}\right)^{\gamma_{1,0}} + d_{2,0}\left(\frac{V_B}{V}\right)^{\gamma_{2,0}}\right\} \\
& -(1-\alpha)V\left\{c_{1,0}\left(\frac{V_B}{V}\right)^{\gamma_{1,0}+1} + c_{2,0}\left(\frac{V_B}{V}\right)^{\gamma_{2,0}+1}\right\} \\
& +\frac{P\kappa\rho}{r} - \frac{P(\rho+m)}{r+m} \\
& +\frac{(\rho+m)P}{r+m}\left\{d_{1,m}\left(\frac{V_B}{V}\right)^{\gamma_{1,m}} + d_{2,m}\left(\frac{V_B}{V}\right)^{\gamma_{2,m}}\right\} \\
& -\alpha V\left\{c_{1,m}\left(\frac{V_B}{V}\right)^{\gamma_{1,m}+1} + c_{2,m}\left(\frac{V_B}{V}\right)^{\gamma_{2,m}+1}\right\}
\end{aligned}
$$

4. Optimal Capital Structure and Endogenous Default

Two stage optimization.

- The optimal capital structure P should solve $\max_P v(V; V_B)$.
- But the equity holder controls V_B (endogenous default). The equity value should be non-negative whenever $V \geq V_B$. Mathematically, the maximizing problem that the equity holders will face is:

$$\max_{V_B \leq V} S(V; V_B)$$

subject to

$$S(V'; V_B) \geq 0, \quad \forall\, V' \geq V_B \geq 0.$$

- Such two-stage problem was discussed by Leland (1994).

- It is obvious to see that the optimal firm leveraged value of one stage maximizing problem (i.e., to choose leverage P and bankrupt trigger V_B simultaneously) is greater than that of the two stage one.
- Leland (1998) uses the difference of the two values to explain the agency cost, which is the loss due to the conflict between equity and bond holders.
- Final Solution: the initial optimal capital structure should be set up in anticipation of the optimal endogenous default level $V_B(P)$.

Theorem 1: Given the debt level P, the optimal default level is $V_B^* = \epsilon P$, where ϵ is

$$\frac{\frac{\rho+m}{r+m}(d_{1,m}\gamma_{1,m} + d_{2,m}\gamma_{2,m}) - \frac{\kappa\rho}{r}(d_{1,0}\gamma_{1,0} + d_{2,0}\gamma_{2,0})}{(1-\alpha)(c_{1,0}\gamma_{1,0} + c_{2,0}\gamma_{2,0}) + \alpha(c_{1,m}\gamma_{1,m} + c_{2,m}\gamma_{2,m}) + 1}$$

- Leland and Toft (1996), Hilberink and Rogers (2002) use numerical methods to justify a smoothing fit heuristically. Here we prove it mathematically.
- The optimal debt structure is given by the following optimization problem. $\max_P v(V; V_B^*)$ with $V_B^* = \epsilon P$.

Theorem 1 (Continued): $v(V; \epsilon P)$ is a concave function of P. That implies that we can find a unique optimal solution.

5. Credit spreads

Definition: ν (dependent on T) is the *yield to maturity* if it satisfies

$$B(V,0;V_B,T) = e^{-\nu T} + \int_0^T \rho e^{-\nu s} ds = e^{-\nu T} + \frac{\rho}{\nu}(1 - e^{-\nu T})$$

and its *credit spread* is defined as follows: $Y(T) = \nu(T) - r$.

Theorem 2: The credit spread at time 0 for a corporate bond is given by

$$\lambda p_d \left(\frac{V_B}{V}\right)^{\eta_d} \left[1 - \frac{\alpha V_B}{P}\frac{m+r}{m}\frac{\eta_d}{\eta_d + 1}\right] > 0$$

This gives a mathematical proof that jump can generate non-zero credit spreads.

Various shapes of credit spreads.

1. For high grade bonds, upward and sometimes humped shapes.

2. For low grade bonds, empirically it is still debtable. Could be all kind of shapes. Sarig and Warga (1989), Fons (1994), Helwege and Turner (1996), Duffie and Singleton (2003).

3. Our model can generate all these three kinds (upward, humped, and downward shapes), even for low grade bonds.

Collin-Dufresne and Goldstein (2001) use exogeneous mean reverting boundary to generate upward shape for low grade bonds.

7. Numerical Results

- $r = 8\%$, close to the historical average Treasury rate during 1973-1998, the coupon rate is $\rho = 8.162\%$, the pay ratio $\delta = 6\%$, as in Huang and Huang (2003).
- The initial un-leverage value of the firm is $V = 100$ and the number of shares of stocks is 100. one trading year is equal to 252 days.
- Unless otherwise specified, we set $\sigma = 0.2$ and corporate tax rate is 35%. After default, the loss fraction of the firm value is $\alpha = 0.5$, consistent with Leland and Toft (1996).

Three different cases.

- Case A is a pure diffusion process. $\lambda = 0$.
- Case B is with small jump intensity $\lambda = 0.2$ (i.e., one jump per 5 years averagely) and quite large jump sizes, $\eta_u = 3$, $\eta_d = 2$ and $p_u = 0.4$ (i.e., jumps up 33% with probability 0.4 or down 50% with probability 0.6).
- Case C is with moderate jump intensity and moderate jump sizes: $\lambda = 1$ (i.e., one jump per 1 years averagely), $\eta_u = 10$, $\eta_d = 5$ and $p_u = 0.5$ (i.e., when jumps up 10% with probability 0.5 or down 20% with probability 0.5).

			$m^{-1}=0.5$		$m^{-1}=1$		$m^{-1}=2$		$m^{-1}=5$	
			$\sigma=0.2$	$\sigma=0.4$	$\sigma=0.2$	$\sigma=0.4$	$\sigma=0.2$	$\sigma=0.4$	$\sigma=0.2$	$\sigma=0.4$
Case B	$\alpha=5\%$	$\lambda=0$	7.14%	1.11%	11.19%	2.37%	17.56%	5.04%	30.67%	12.91%
		$\lambda=0.5$	0.67%	0.19%	1.63%	0.60%	3.94%	1.86%	11.70%	7.32%
		$\lambda=1$	0.12%	0.04%	0.42%	0.19%	1.45%	0.84%	6.54%	4.79%
		$\lambda=2$	0.01%	0.001%	0.05%	0.003%	0.34%	0.24%	3.16%	2.64%
	$\alpha=25\%$	$\lambda=0$	13.78%	3.21%	18.31%	5.28%	25.08%	9.13%	38.41%	19.07%
		$\lambda=0.5$	2.49%	0.88%	4.35%	1.88%	8.07%	4.28%	18.66%	12.63%
		$\lambda=1$	0.66%	0.28%	1.5%	0.77%	3.67%	2.29%	11.89%	9.22%
		$\lambda=2$	0.067%	0.04%	0.28%	0.18%	1.18%	0.86%	6.87%	5.96%
	$\alpha=50\%$	$\lambda=0$	25.44%	9.22%	30.30%	12.63%	37.29%	18.27%	50.52%	31.19%
		$\lambda=0.5$	8.88%	4.13%	12.39%	6.65%	18.50%	11.56%	33.33%	25.52%
		$\lambda=1$	3.71%	1.97%	6.09%	3.74%	10.95%	7.79%	25.25%	21.25%
		$\lambda=2$	0.87%	0.53%	2.02%	1.42%	5.22%	4.18%	18.45%	16.90%
Case C	$\alpha=5\%$	$\lambda=0$	7.14%	1.11%	11.19%	2.37%	17.56%	5.04%	30.67%	12.91%
		$\lambda=0.5$	4.80%	0.87%	7.96%	1.94%	13.22%	4.33%	24.88%	11.73%
		$\lambda=1$	3.45%	0.69%	6.01%	1.61%	10.48%	3.75%	21.04%	10.73%
		$\lambda=2$	1.95%	0.44%	3.71%	1.12%	7.10%	2.87%	16.09%	9.13%
	$\alpha=25\%$	$\lambda=0$	13.78%	3.21%	18.31%	5.28%	25.08%	9.13%	38.41%	19.07%
		$\lambda=0.5$	10.09%	2.64%	13.96%	4.51%	19.98%	8.10%	32.51%	17.73%
		$\lambda=1$	7.75%	2.20%	11.11%	3.88%	16.54%	7.22%	28.40%	16.56%
		$\lambda=2$	4.92%	1.54%	7.53%	2.92%	12.07%	5.84%	22.92%	14.63%
	$\alpha=50\%$	$\lambda=0$	25.44%	9.22%	30.30%	12.63%	37.29%	18.27%	50.52%	31.19%
		$\lambda=0.5$	20.58%	8.11%	25.16%	11.38%	31.92%	16.90%	45.24%	29.88%
		$\lambda=1$	17.14%	7.17%	21.44%	10.29%	27.96%	15.69%	41.28%	28.70%
		$\lambda=2$	12.50%	5.67%	16.35%	8.51%	22.48%	13.65%	35.82%	26.69%

Table 1: Effect of various parameters on optimal leverage level: interest rate $r = 8\%$, pay ratio $\delta = 6\%$, coupon rate $\rho = 8.162\%$

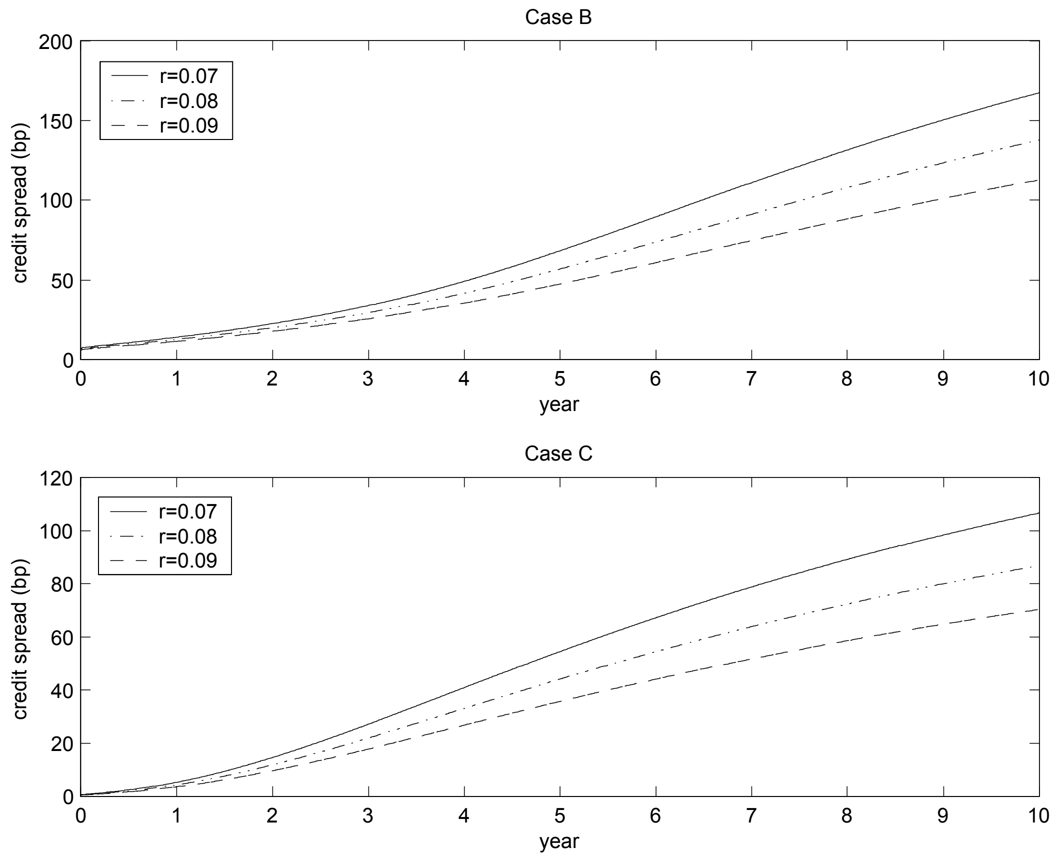

Figure 1: the effect of interest rate on credit spreads: pay ratio $\delta = 6\%$, coupon rate $\rho = 8.162\%$, leverage level $P = 30\%$, corporate tax rate $\kappa = 35\%$, bankrupt loss fraction $\alpha = 50\%$, average bonds maturity $m^{-1} = 5$ years

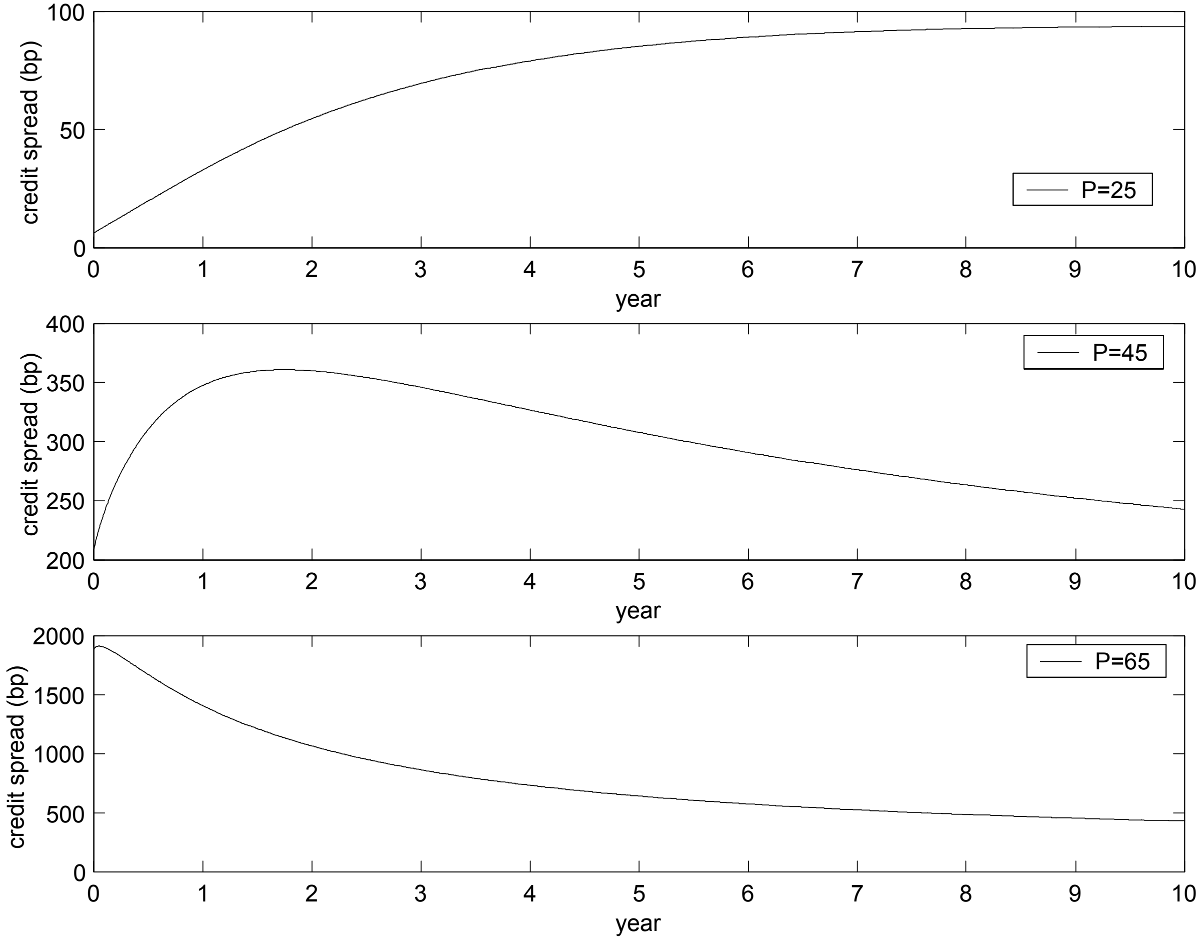

Figure 2: All kinds of shapes of credit spreads: interest rate $r = 8\%$, coupon rate $\rho = 1\%$, pay ratio $\delta = 1\%$, volatility $\sigma = 10\%$, corporate tax rate $\kappa = 35\%$, bankrupt loss fraction $\alpha = 50\%$, average maturity $m^{-1} = 0.5$ years, jump sizes are the same as Case C, jump rates $\lambda = 2$.

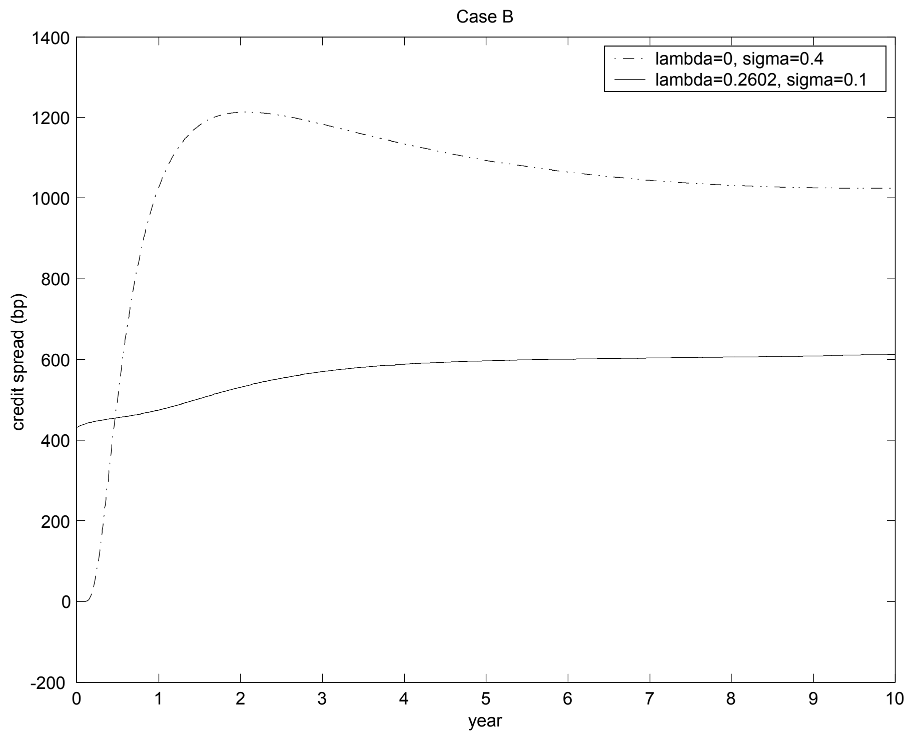

Figure 3: Speculative bonds can have upward-sloping spread curves: interest rate $r = 8\%$, pay ratio $\delta = 6\%$, coupon rate $\rho = 8.162\%$, leverage level $P = 90\%$, total volatility $\sigma = 40\%$,corporate tax rate $\kappa = 35\%$, bankrupt loss fraction $\alpha = 50\%$, average bonds maturity $m^{-1} = 5$ years

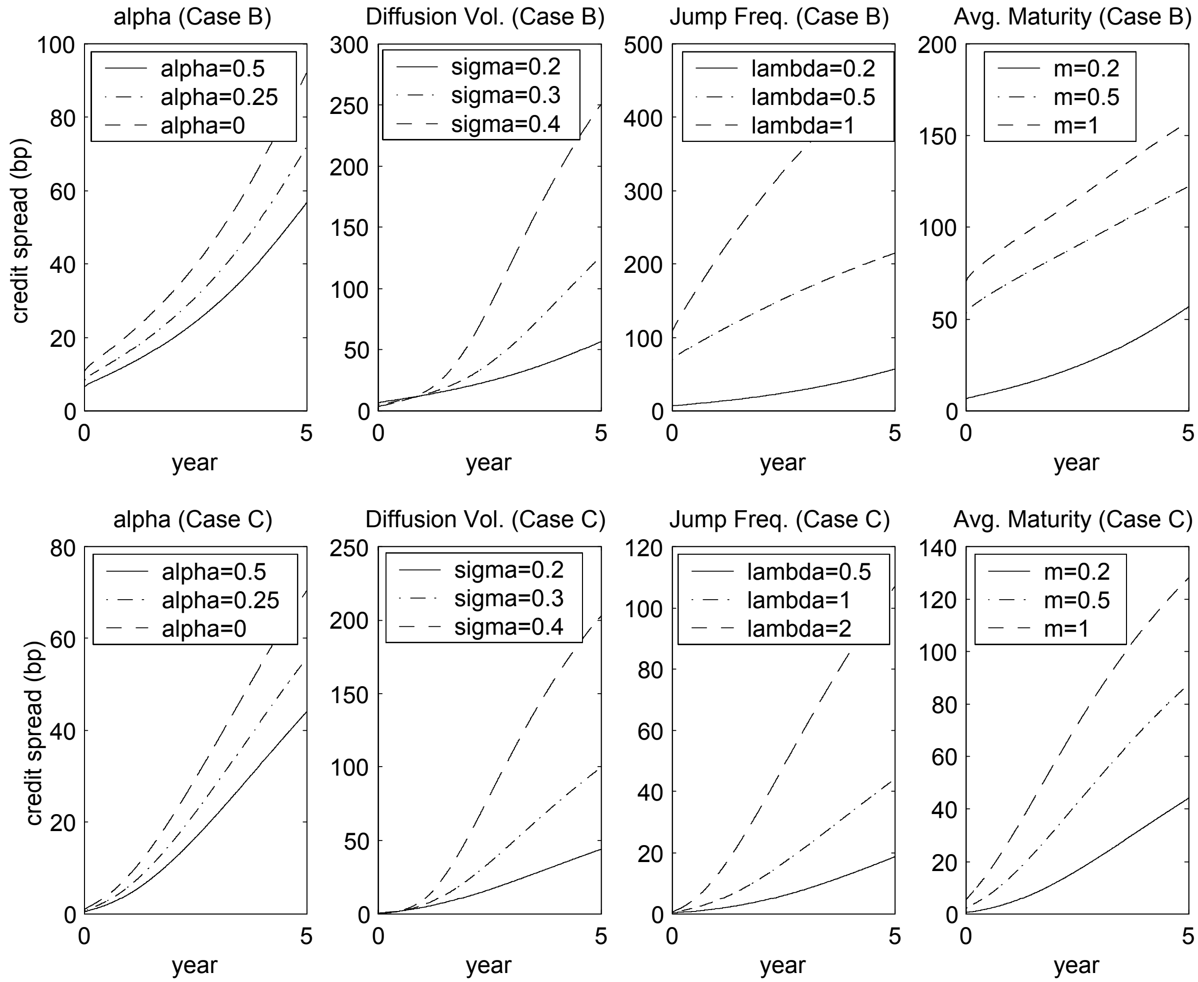

Figure 4: the effect of various parameters on credit spread: interest rate $r = 8\%$, pay ratio $\delta = 6\%$, coupon rate $\rho = 8.162\%$, leverage level $P = 30\%$, corporate tax rate $\kappa = 35\%$

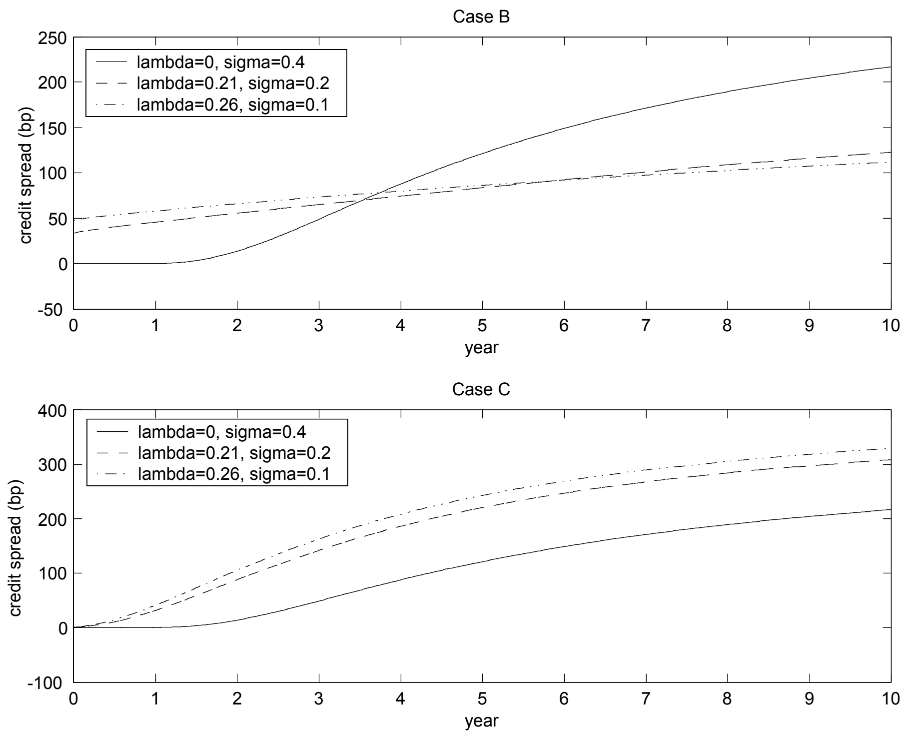

Figure 5: Jump volatility vs Diffusion volatility: interest rate $r = 8\%$, pay ratio $\delta = 6\%$, coupon rate $\rho = 8.162\%$, leverage level $P = 30\%$, total volatility $\sigma = 40\%$,corporate tax rate $\kappa = 35\%$, bankrupt loss fraction $\alpha = 50\%$, average bonds maturity $m^{-1} = 5$ years.

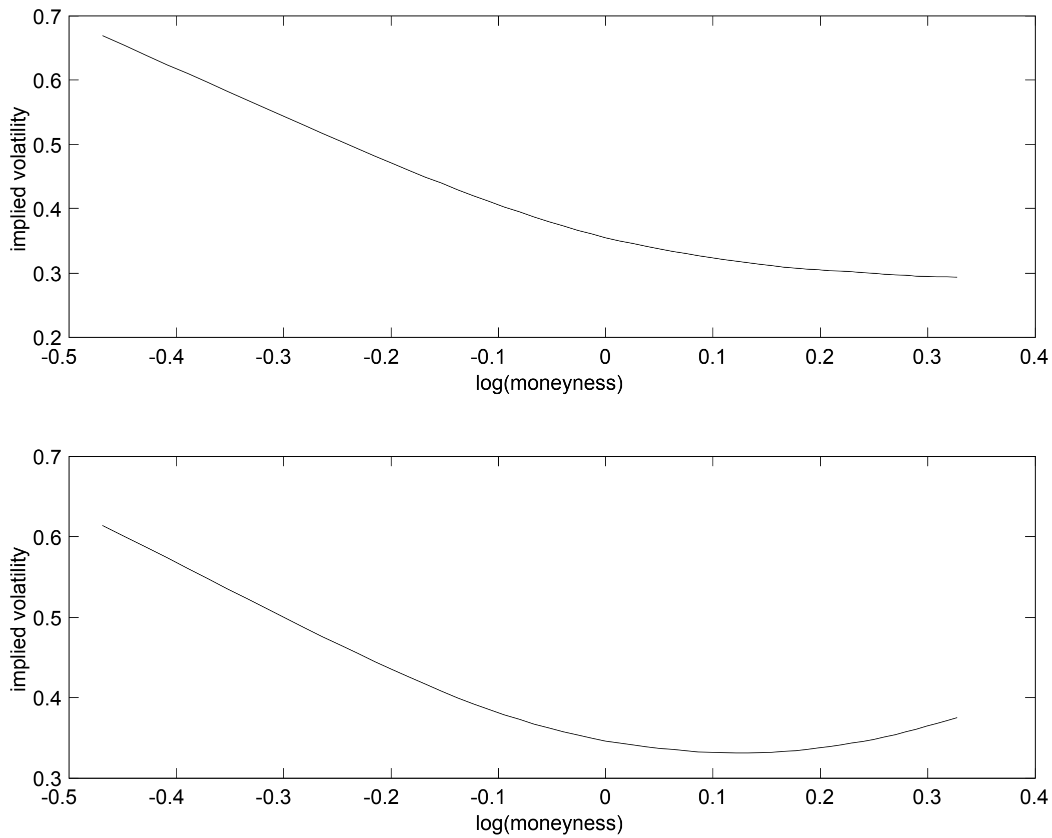

Figure 6: One sided jumps vs. two sided jumps: implied volatility against log(strike price/stock price): interest rate $r = 8\%$, coupon rate $\rho = 8.162\%$, pay ratio $\delta = 6\%$, diffusion volatility $\sigma = 20\%$, corporate tax rate $\kappa = 35\%$, bankrupt loss fraction $\alpha = 50\%$, average bonds maturity $m^{-1} = 5$ years, leverage level $P = 30\%$, call options maturity $T = 0.25$

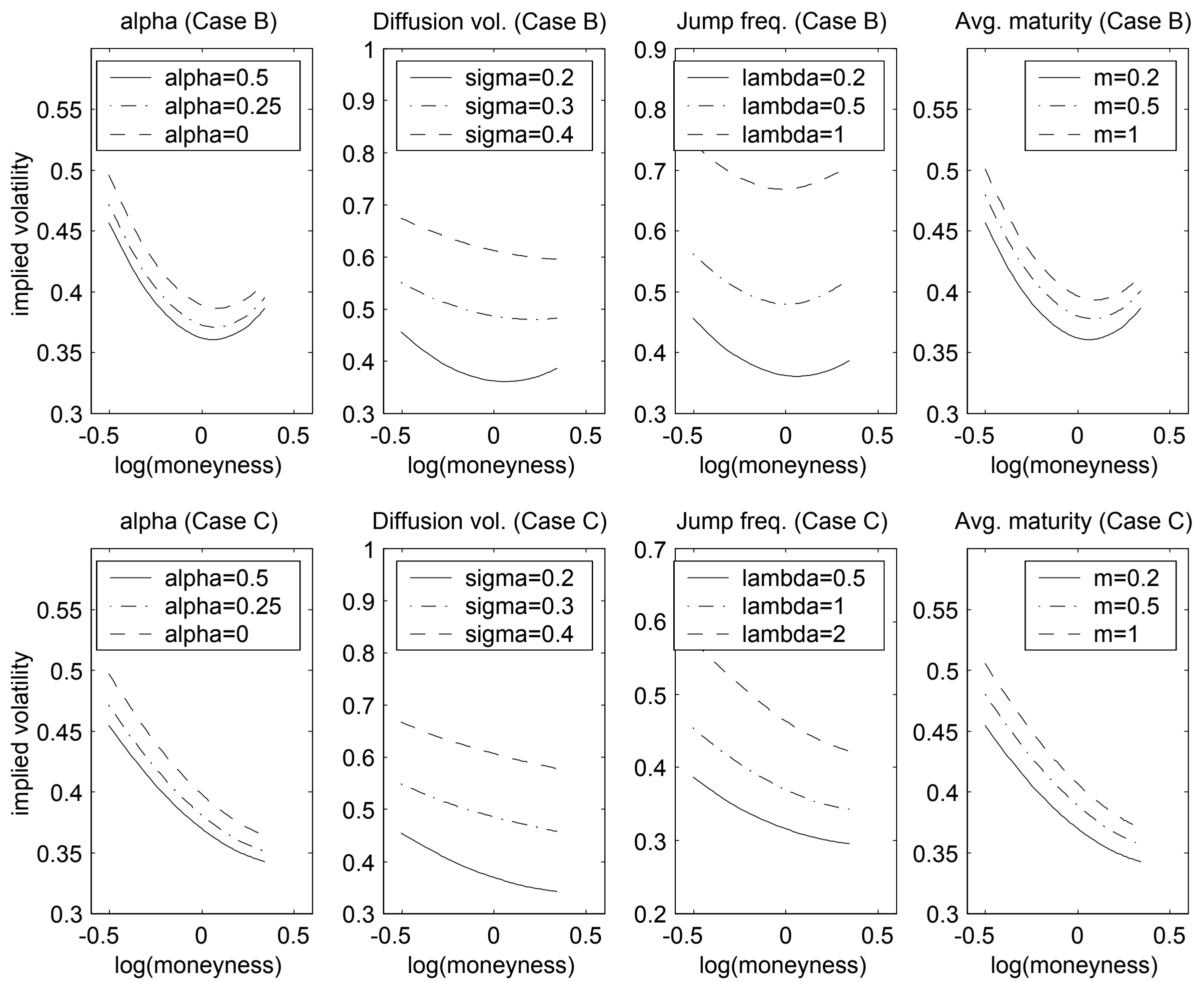

Figure 7: the effect of various parameters on implied volatility implied volatility against log(strike price/stock price): interest rate $r = 8\%$, coupon rate $\rho = 8.162\%$, pay ratio $\delta = 6\%$, leverage level $P = 30\%$, call options maturity $T = 1$

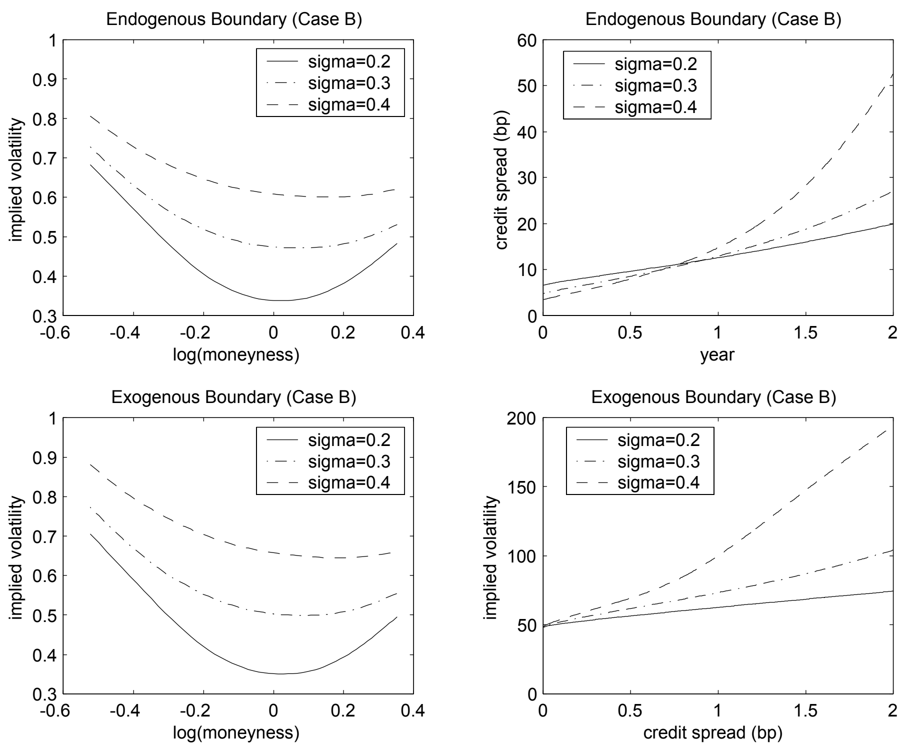

Figure 8: exogenous default boundary and endogenous boundary (Case B): interest rate $r = 8\%$, pay ratio $\delta = 6\%$, coupon rate $\rho = 8.162\%$, leverage level $P = 30\%$, corporate tax rate $\kappa = 35\%$, exogenous default boundary $V_B = P = 30\%$, the maturity of option is $T = 0.25$

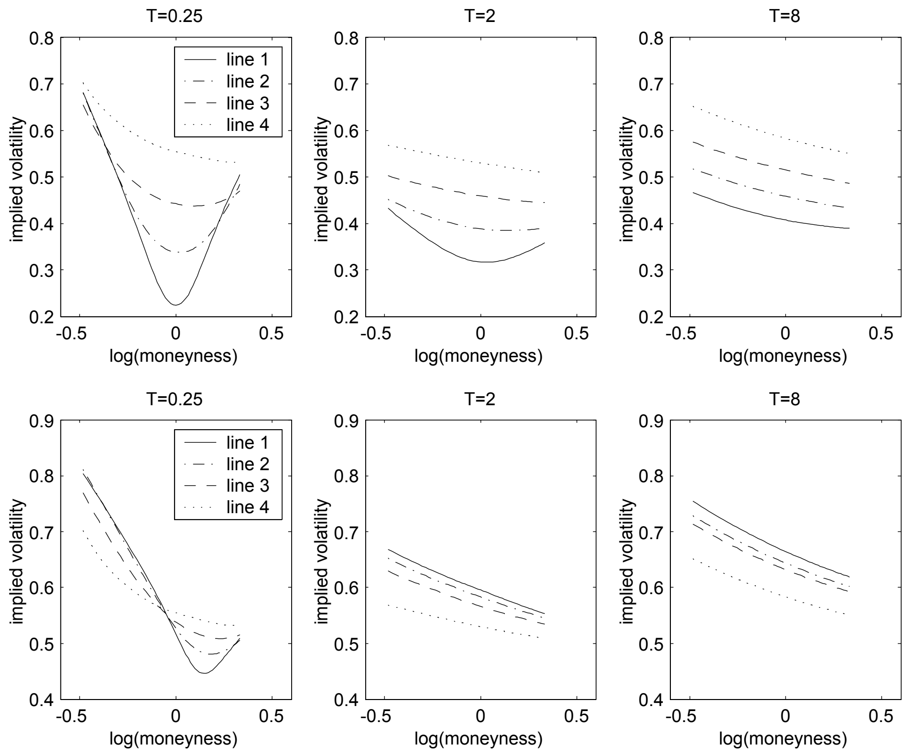

Figure 9: Jump volatility vs diffusion volatility. In the first row of this picture, line 1 is for the case that $\sigma = 0.1, \lambda = 0.26$, line 2 for $\sigma = 0.2, \lambda = 0.20$, line 3 for $\sigma = 0.3, \lambda = 0.12$, line 4 for $\sigma = 0.4, \lambda = 0$. In the second row of this picture, line 1 is for the case that $\sigma = 0.1, \lambda = 4.47$, line 2 for $\sigma = 0.2, \lambda = 3.57$, line 3 for $\sigma = 0.3, \lambda = 2.08$, line 4 for $\sigma = 0.4, \lambda = 0$.