Credit Derivatives

Summary

- Introduction to exotic credit derivative structures: rationale and basic examples

- The Li /Gaussian Copula model

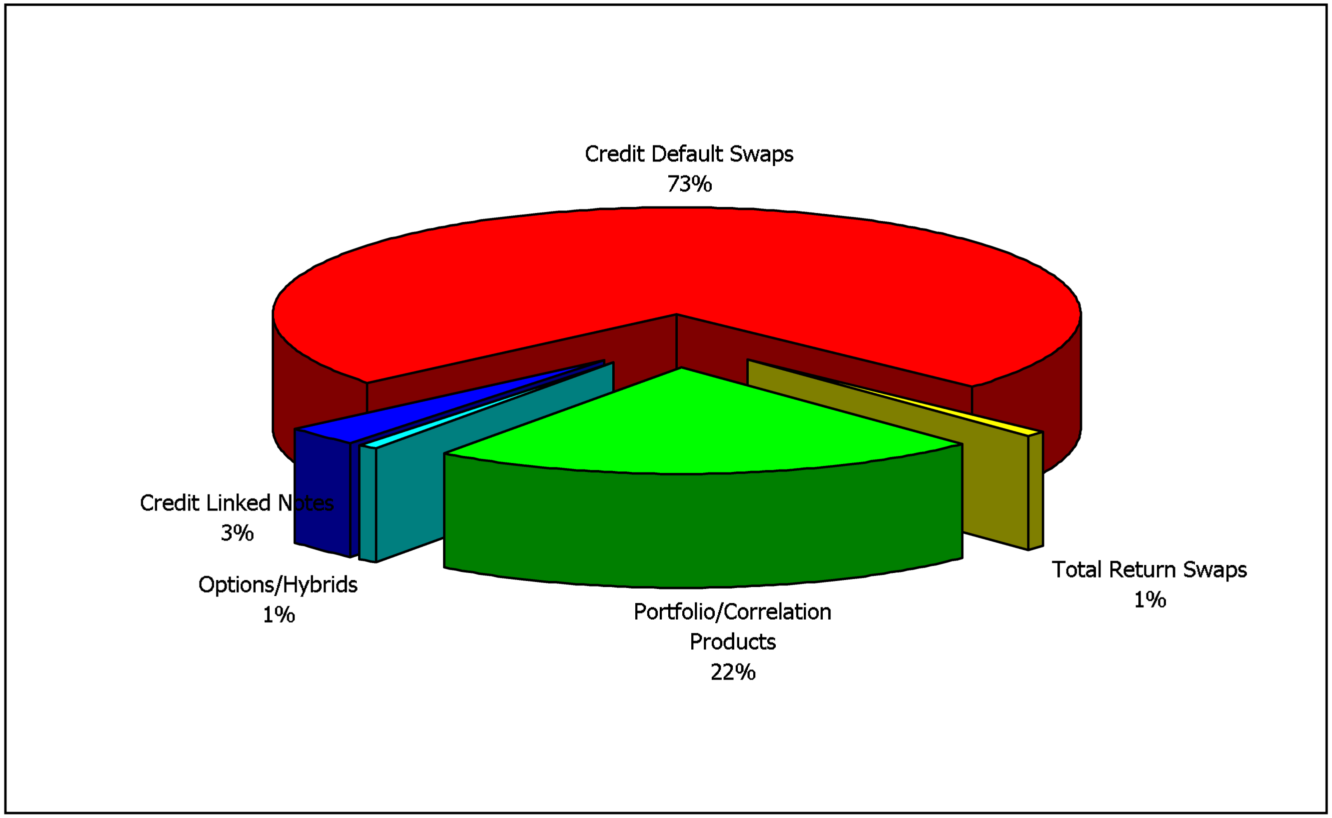
- Importance Sampling

- Likelihood Ratio/Pathwise methods for computing Greeks

- Hedging of nth default swaps

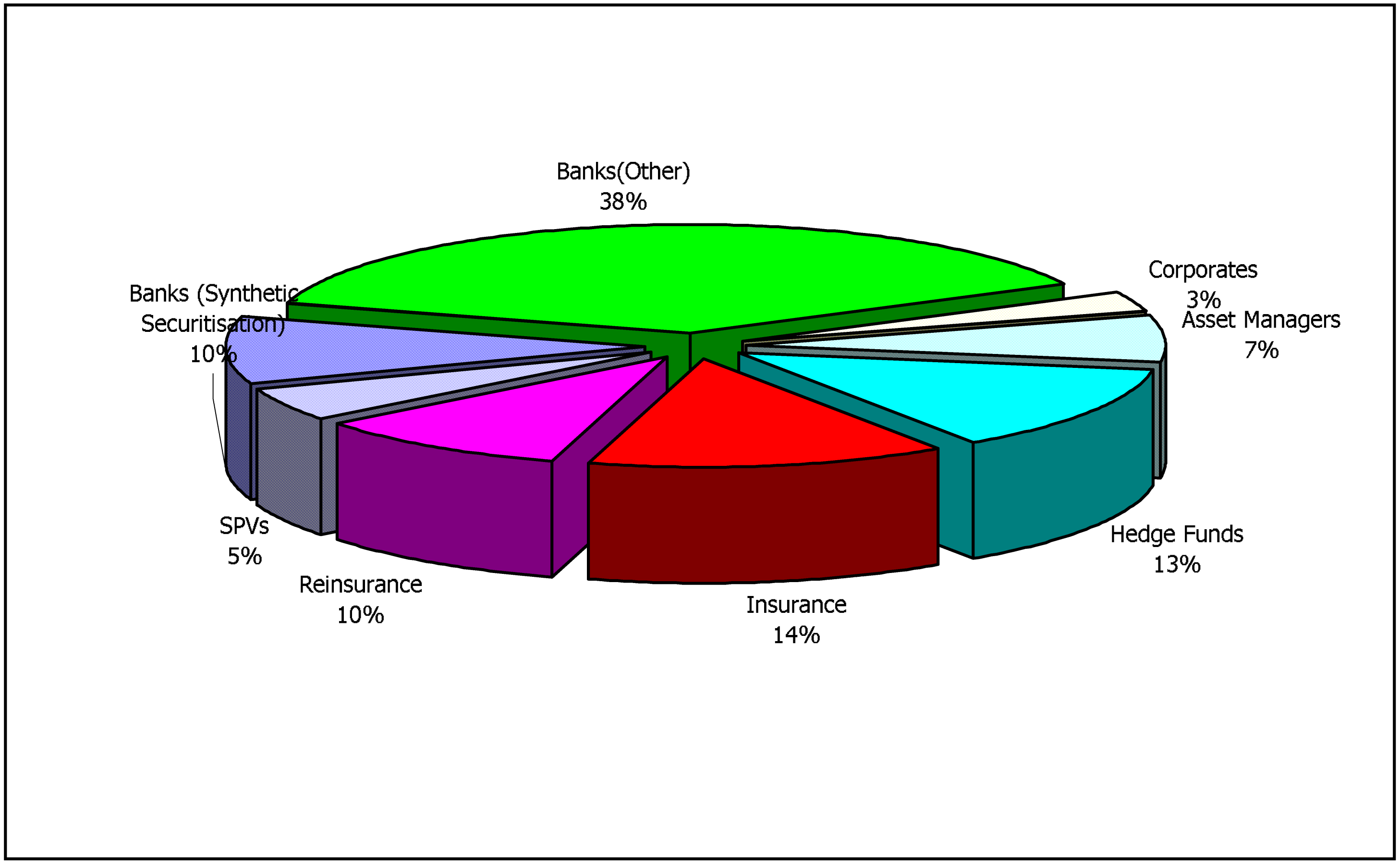
Market Overview

- Total Market Notional: $2.3 trillion (up 50% from 2002).
- Single name CDS dominant : 73 %; sign of a market becoming more mainstream focusing on standardised, liquid contracts. Portfolio products 22 %; symbiotic with single name CDS

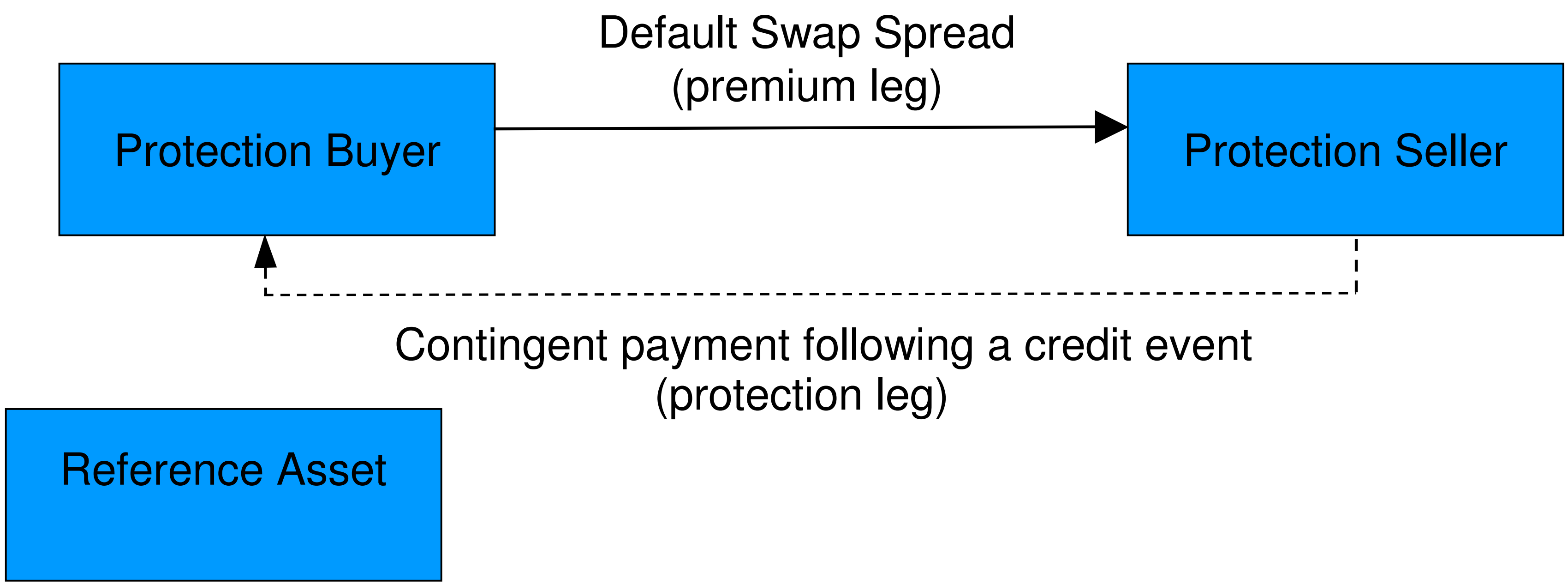


Market Overview

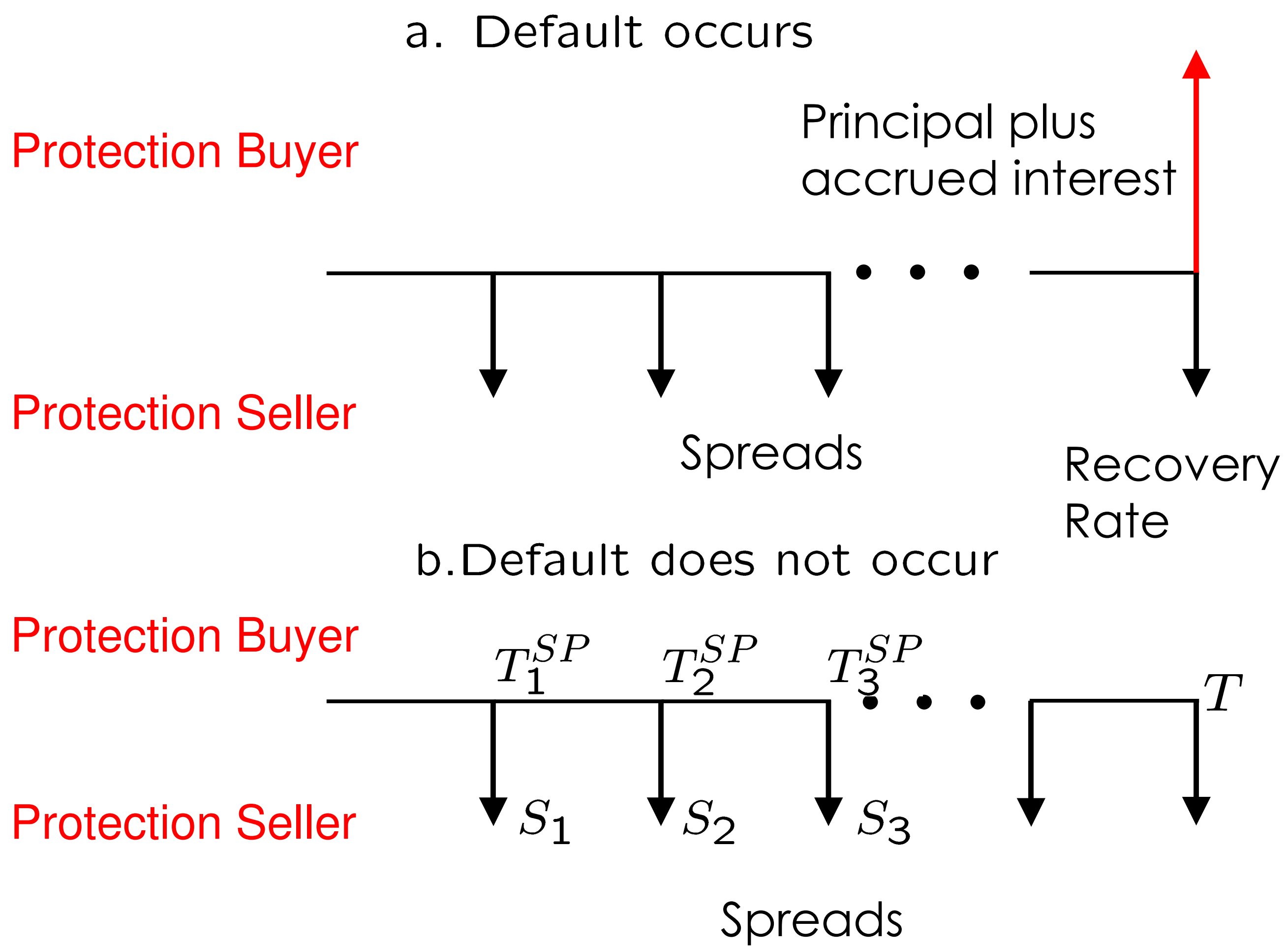
- Base of CD users: principally banks; insurance companies and hedge funds have increased market share significantly in the last two years.



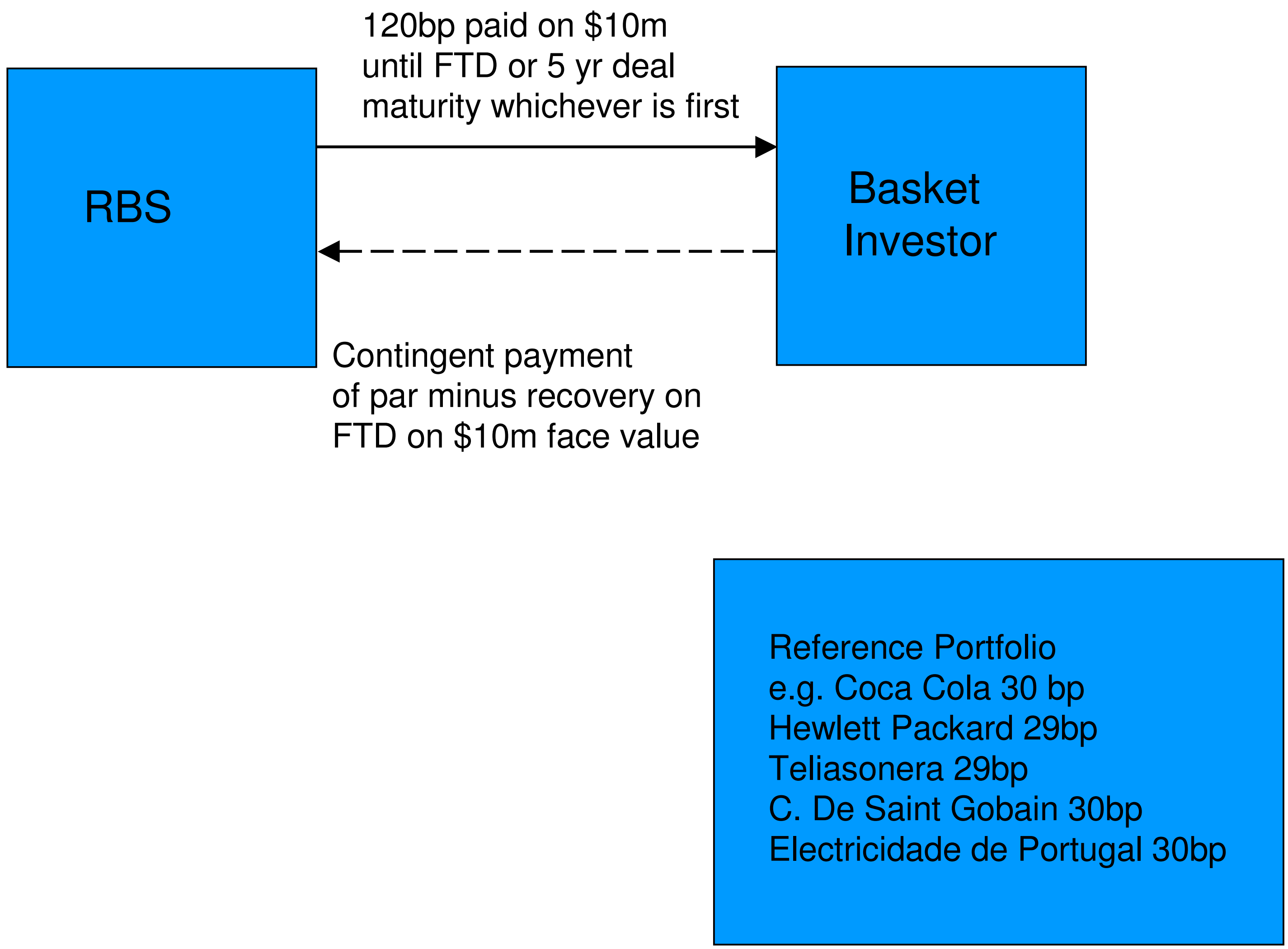
Credit Default Swaps: Mechanics



- Basically an insurance play; CDS is used to transfer credit risk on the reference asset from the protection buyer to the seller
- Protection buyer shorts the credit risk


a. Default occurs
Protection Buyer
Principal plus
accrued interest
Protection Seller
Spreads
Recovery
Rate
b.Default does not occur
Protection Buyer
T_1^{SP}
T_2^{SP}
T_3^{SP}
T
Protection Seller
S_1
S_2
S_3
Spreads

First to Default Baskets



Basket Default Swaps

- Basket default swaps are similar to CDS; trigger now is the nth credit event in a specified basket of reference entities.
- Typically baskets are 5 - 10 entities. e.g., first to default - first asset in basket to default triggers a payment to the protection buyer — usually physical delivery of the defaulted asset in return for par amount in cash.
- Applications:
  - Investors can use default baskets as a means of leveraging: they get a higher yield without increasing their notional at risk.
  - Credit investors can use default baskets to hedge a blow up in a portfolio of credits more cheaply than buying protection on individual credits.
  - Default baskets allow investors to trade default correlation.

Some Observations

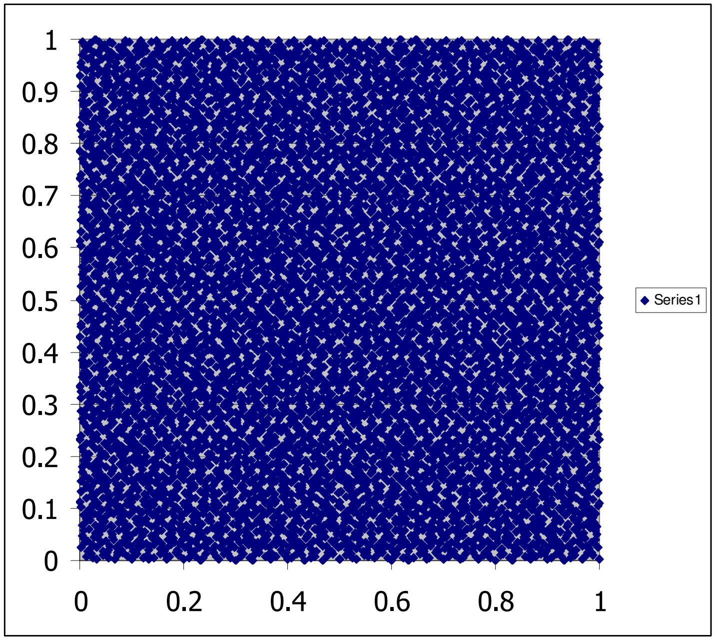
- Sum of all nth default to swaps on a basket equals the sum of all the individual credit default swaps on the names
- Modelling can only move value between the individual nth to default swaps
- Fundamental driver of price is correlation
- How do we correlate defaults? This problem is not yet fully resolved.

Some Definitions

- Consider some security A. We define the default time, τ_A, as the time from today until A defaults.
- We assume the defaults to occur as a Poisson process
- The intensity of this process, $h(t)$, is called the hazard rate.

The Li Model

- Defaults are assumed to occur for individual assets according to a Poisson process with a deterministic intensity called the hazard rate.
- Recovery rates are deterministic.
- This means that default times are exponentially distributed.
- Li: Correlate these default times using a Gaussian copula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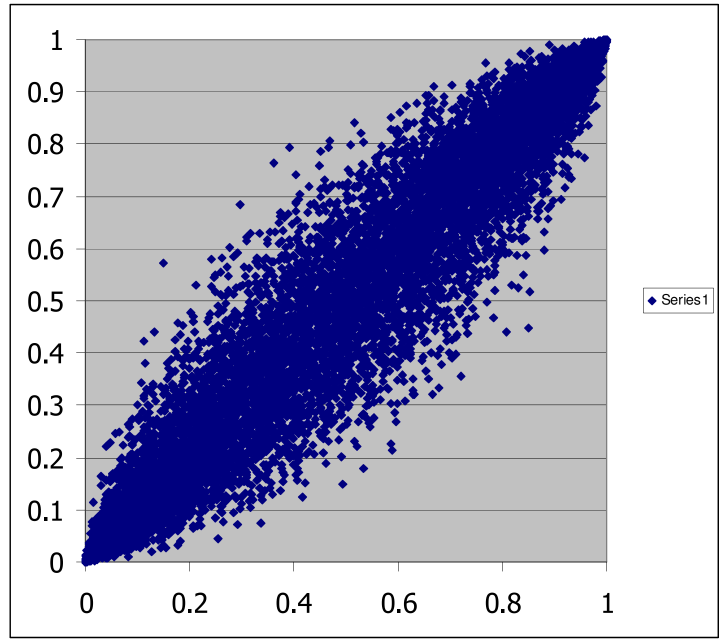
1
Series1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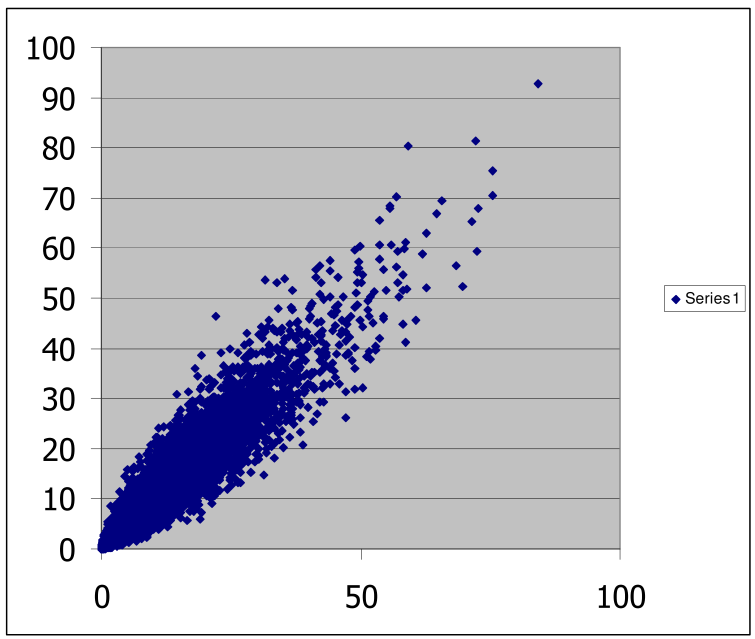
1
Series1


100
90
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10
0
0
50
100
Series1

The Pricing Algorithm: SetUp

Given a correlation matrix C we compute A such that

$$AA^T = C$$

Let $E(\tau, h)$ denote the cumulative exponential distribution function in τ given a fixed h:

$$E(\tau, h) = \mathbb{P}(t < \tau) = 1 - \exp(-\int_0^\tau h_j(t)dt).$$

$E^{-1}(u, h)$ denotes its inverse for fixed h.

The Pricing Algorithm

- Draw a vector of independent normals, ***z***
- Generate a set of correlated Gaussian deviates:

$$\mathbf{w} = A\mathbf{z}.$$

- Map to uniforms:

$$u_i = N(w_i)$$

- Map to default times:

$$\tau_i = E^{-1}(u_i, h)$$

- Compute the cash flow in this scenario; discount back.

$$F(\tau_1, \ldots, \tau_N) = P(D_n(\tau_1, \ldots, \tau_N))[V_{\text{prot}} + (1 - r_n)H(T - D_n(\tau_1, \ldots, \tau_N))].$$

Critique of the Gaussian Copula Model

- The Copula methodology and in particular the Gaussian copula approach has become something of an industry standard for pricing correlation sensitive products. FtD basket prices are quoted in terms of the Gaussian correlation cf. Black Scholes implied vol.
- Copula models will yield proces that are automatically consistent with the prices and the term structures of the vanilla hedging instruments — the CDS.
- We can demonstrate an equivalence between the Gaussian copula approach and multivariate extensions of the Merton firm value (CreditMetrics,KMV) approaches.
- Relatively straightforward; can easily alter multivariate dependence structure; given numerical improvements above can price and hedge in a trading environment.

Critique of the Gaussian Copula Model

- Copula models although a relatively sophisticated approach to the pricing of default correlation products are probably not the final solution; just a good first step.
- There are issues both in terms of the products and the modelling approach used:
  - We have only the credit default swaps to hedge two sources of uncertainity: stochastic spread movements and default events. There is therefore a lack of payoff replicability even in principle.
  - The Li model does not incorporate spreads volatility.
  - The Li model is not time homogeneous: prices of forward starting baskets differ from their prices today.
  - As set up copula models do not allow for default contagion.

Implementation Issues

- For a short team deal, many Monte Carlo paths result in a zero or constant pay-off.

- Example:
  - Basket of 5 names.
  - Each 1% chance of default a year. Defaulting independently for simplicity.
  - One year deal.
  - First to default swap, only 5% of paths are interesting.
  - 5th to default swap, 1e-10 of paths are interesting.

Importance Sampling for 1st to default

Intuitively: want to sample more thoroughly in the regions where defaults occur.

We want at least one default per path.

We want each asset to have equal probability of default.

We want first defaults reasonably distributed across time.

$$\int f(x)\phi(x)\,dx = \int f(x)\frac{\phi(x)}{\psi(x)}\psi(x)\,dx$$

$$\mathbf{E}_\phi(f(x)) = \mathbf{E}_\psi\left(f(x)\frac{\phi(x)}{\psi(x)}\right)$$

Designing the importance density when *i* =1

- Make the *i* th asset default before T with probability:

$$\frac{1}{(n+1)-i}$$

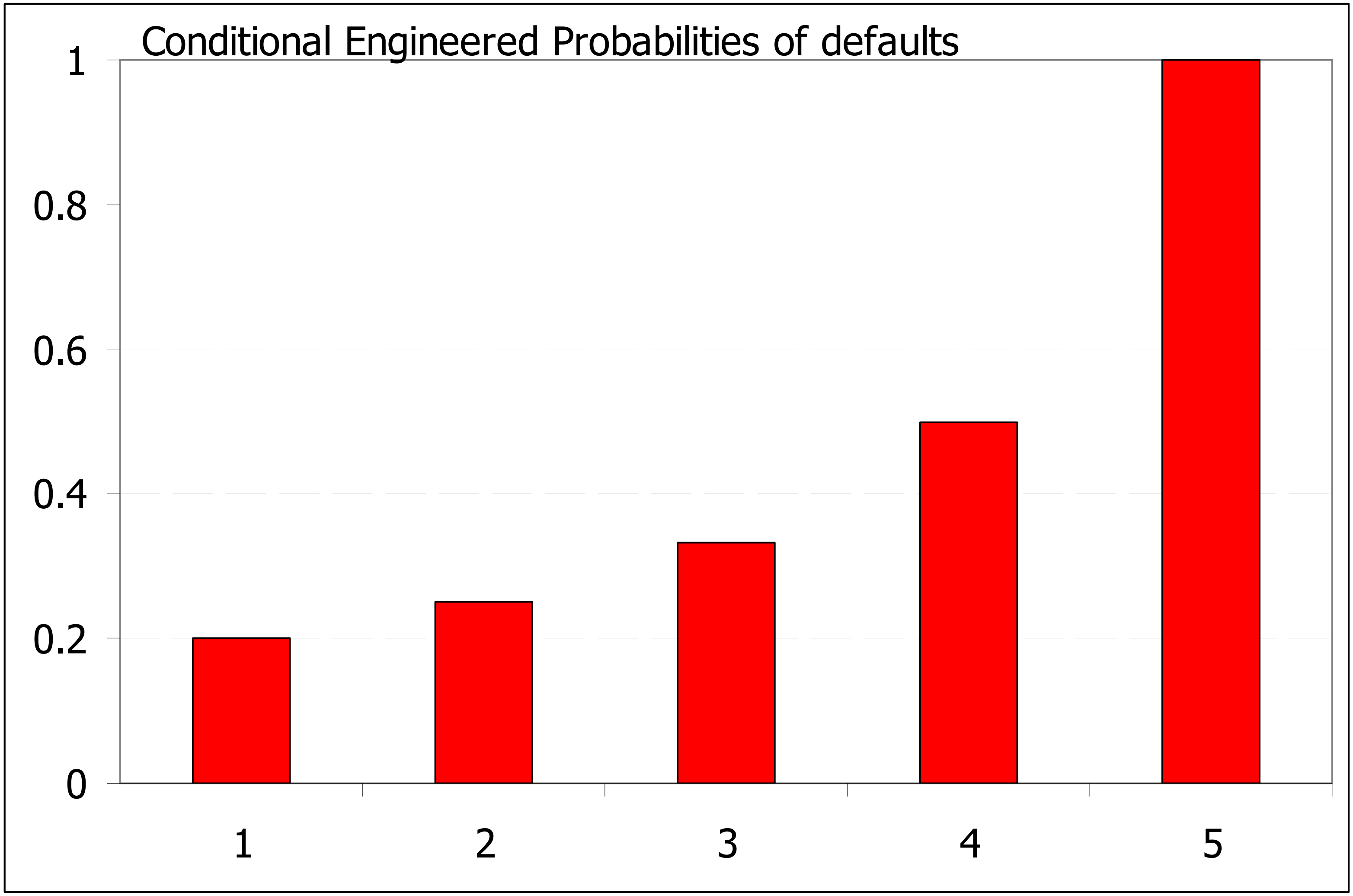
- Why? After *i* non defaults want all the remaining credits to have an equal chance of default

- Pick a uniform u_i. If:

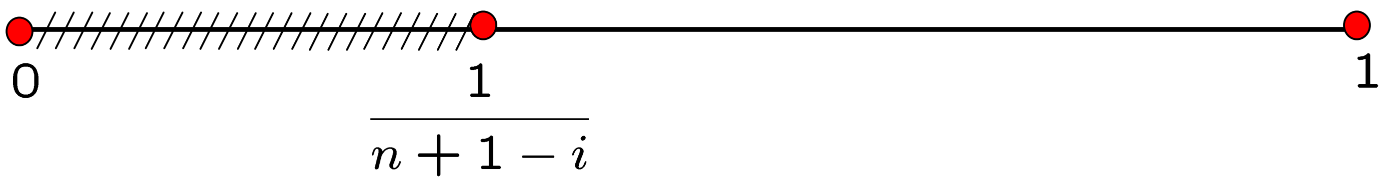
$u_i < \frac{1}{n+1-i}$ map u_i to a region where asset *i* defaults.

$u_i > \frac{1}{n+1-i}$ map u_i a region where asset *i* doesn't default.

Designing the importance density when $i = 1$



Designing the importance density



- Look at the original default region for asset *i*

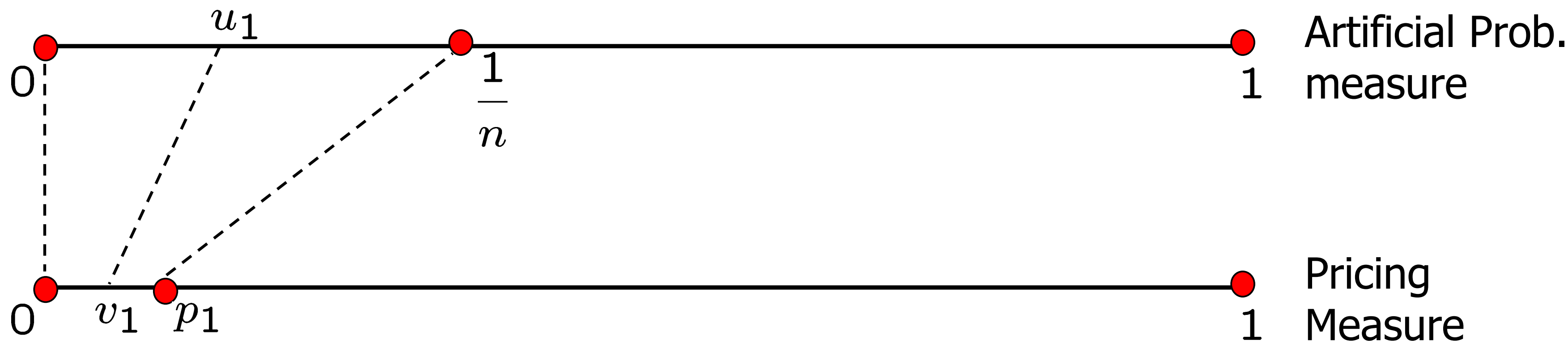
$$\tau_i < T \quad \longrightarrow \quad w_i < x \quad \longrightarrow \quad \sum_j A_{ij} z_j < x$$

w_i: Correlated Gaussian

z_j: $N(0,1)$

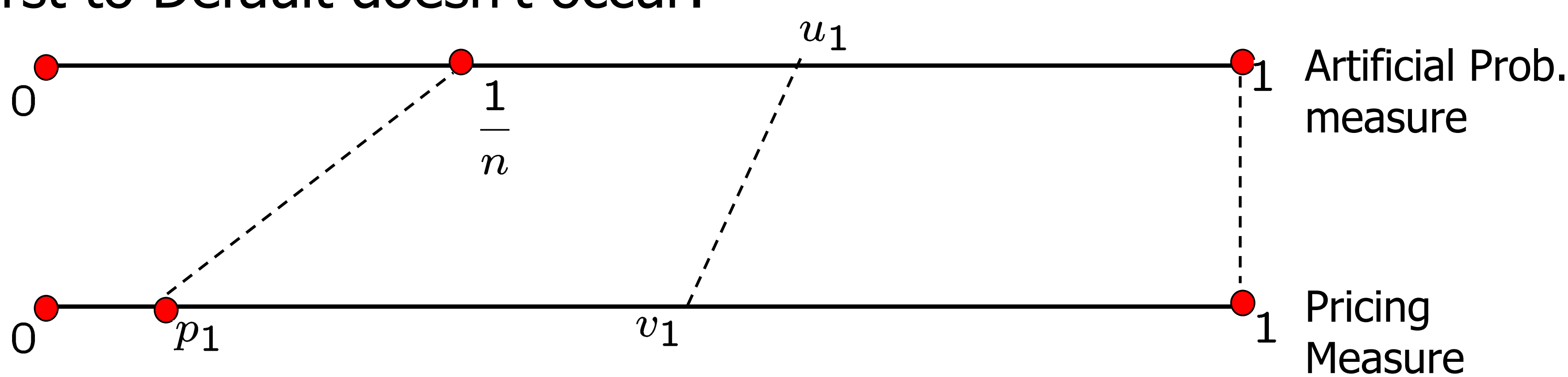
- For our first to default case: $a_{11} z_1 < x \implies z_1 < \frac{x_1}{a_{11}}$
- Translate to uniforms: $p_1 = N\left(\frac{x_1}{a_{11}}\right)$

First to Default occurs:



- u_1 maps to v_1 where: $\frac{v_1}{p_1} = u_1 n \implies v_1 = u_1 n p_1$

First to Default doesn't occur:



- u_1 maps to v_1 where: $v_1 = p_1 + \frac{1 - p_1}{1 - \frac{1}{n}}(u_1 - \frac{1}{n})$

We need to scale the contributions of these paths

First asset defaults: weight by np_1

Doesn't default: weight by $\dfrac{1-p_1}{1-\frac{1}{n}}$

Suppose that we have dealt with the first (*j* -1) assets. The unmassaged default probability now depends on Z:

$$W_j < x_j \text{ if and only if } \sum_{i<j} a_{ij} Z_i + a_{jj} Z_j < x_j.$$

However, as A is lower triangular we have

$$p_j = \frac{x_j - \sum_{i<j} a_{ij} Z_j}{a_{jj}}$$

And repeat as before.

Greeks

- The sensitivities to credit spreads are as important as price. It is customary to "delta hedge" them.
- They can be computed naïvely by bumping credit spreads. However, this results in very slow convergence.
- The reason is that when finite differencing on a path by path basis, it is the paths where a default goes from being after maturity to before it that result in the most change.
- A small number of paths giving a large amount of value causes a large variance.

Value CDS =

$$\int P(D_n(\tau_1,\ldots,\tau_N))[(1-r_n)H(T-D_n(\tau_1,\ldots,\tau_N))\psi(\tau_1,\ldots,\tau_N)]d\tau_1\ldots d\tau_N.$$

- When we differentiate the payoff w.r.t the hazard rates we get a δ function.

Improving the Greeks

- Importance sampling removes much of the problem - implicit likelihood ratio.
- Standard techniques are the likelihood ratio method and the pathwise method. (Broadie-Glasserman).
- Application of likelihood ratio method is straightforward. Compute log of density and differentiate.
- Pathwise method involves differentation of the pay-off. The pay-off involves step functions so derivative contains delta distributions.
- Delta distributions can be integrated out analytically so pathwise method can be applied.

The Likelihood Ratio Method

Value of an option:

$$V = \mathbb{E}^{\mathbb{Q}}[F(S_T)] = \int F(S)\psi(S,\theta)\,dS$$

We can write the sensitivity w.r.t θ:

$$\frac{\partial V}{\partial \theta} = \int F(S)\frac{\partial}{\partial \theta}\psi(S,\theta)\,dS$$

No longer integrating against our Monte Carlo density! However, we can reintroduce it:

$$\frac{\partial V}{\partial \theta} = \int F(S)\frac{\partial \psi(S,\theta)}{\partial \theta}\frac{1}{\psi(S,\theta)}\psi(S,\theta)\,dS$$

$$= \int F(S)\frac{\partial}{\partial \theta}\log \psi(S,\theta)\psi(S,\theta)\,dS$$

The Likelihood Ratio Method for nth Default Swaps

- Value of the CDS:

$$\int P(D_n)(1-r_n)H(T-D_n)\psi(\tau_1,\ldots,\tau_N)d\tau_1\ldots d\tau_N.$$

- Differentiate w.r.t. *i* th hazard rate :

$$\frac{\partial V}{\partial h_i}=\int_0^T P(D_n)(1-r_n)H(T-D_n)\frac{\partial\psi(\tau_1,\ldots,\tau_N)}{\partial h_i}d\tau_1\ldots d\tau_N.$$

- Applying Broadie/Glasserman's trick:

$$\frac{\partial V}{\partial h_i}=\int_0^T P(D_n)(1-r_n)H(T-D_n)\frac{\partial\log\psi(\tau_1,\ldots,\tau_N)}{\partial h_i}\psi(\tau_1,\ldots,\tau_N)d\tau_1\ldots d\tau_N.$$

The Likelihood Ratio Method for nth Default Swaps

- The calculation is straightforward for Gaussian copula and flat hazard rates:

$$\frac{\partial \log \psi(\tau_1, \ldots, \tau_n)}{\partial h_i} = -(\rho^{-1} - \mathbf{1})_{ij} \eta_j \frac{\partial \eta_i}{\partial u_i} \frac{\partial u_i}{\partial h_i} + \frac{1}{h_i} - \tau_i$$

where ρ is the correlation matrix and

$$\eta_i = \phi^{-1}(u_i) \qquad \frac{\partial \eta_i}{\partial u_i} = \sqrt{2\pi} e^{\frac{1}{2}\phi^{-1}(u_i)^2}$$

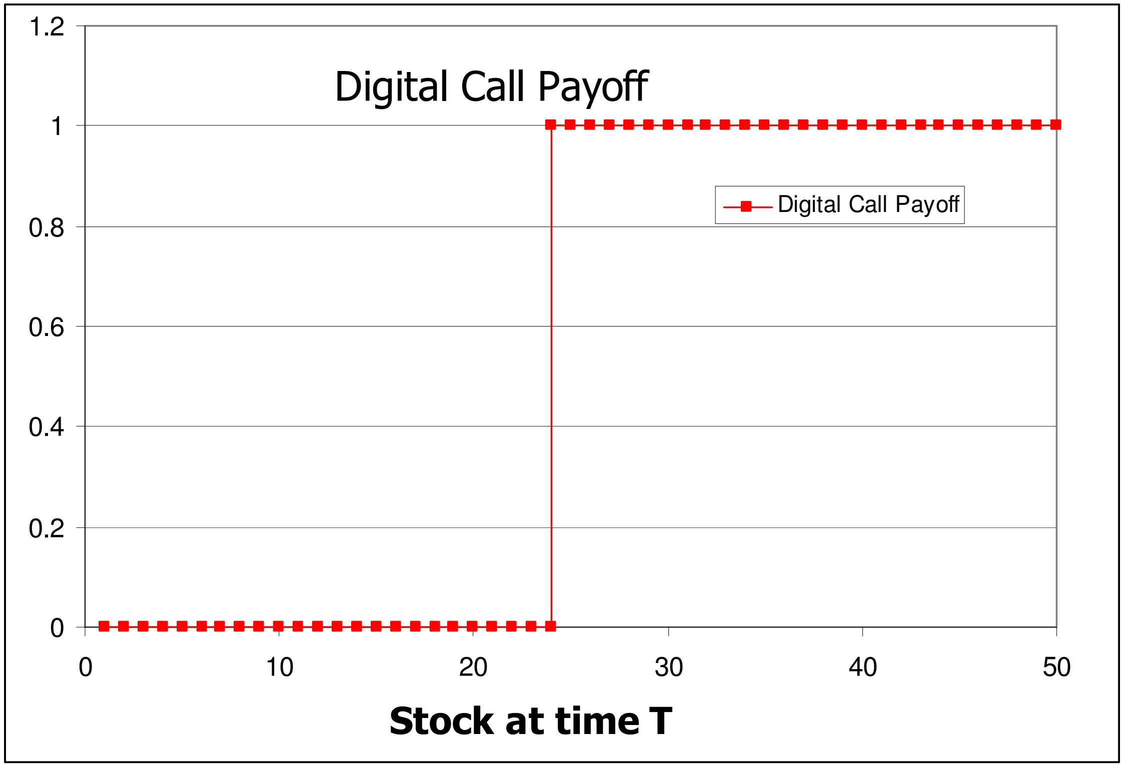
The Pathwise Method

Rewrite integral so dependence on parameter is in the pay-off and not in the density i.e. We are now differentiating the *payoff* !

•Suppose we have a digital option:

$$f(S_T) = H(S_T - K)$$

•Differentiate and we get a δ function



The Pathwise Method for nth Default Swaps

- We differentiate the *discounted pay-off* w.r.t h_j (ignore the spreads for the moment):

$$F(\tau_1, \ldots, \tau_N) = P(D_N(\tau_1, \ldots, \tau_N))[(1 - r_n)H(T - D_n(\tau_1, \ldots, \tau_N))]$$

$$\frac{\partial F}{\partial h_j} = \frac{\partial F}{\partial \tau_j}\frac{\partial \tau_j}{\partial h_j}$$

where if the jth asset is the nth to default

$$\frac{\partial F}{\partial \tau_j} = \frac{\partial P}{\partial t}(\tau_j)[H(T - \tau_j)(1 - r_N)] \\ - P(\tau_j)[\delta(\tau_j - T)(1 - r_n) + H(\tau_j - T)\ \frac{\partial}{\partial t}(1 - r_n)|_{t=\tau_j}]$$

And zero otherwise.

The Pathwise Method for nth Default Swaps

The important terms are the second and third terms.

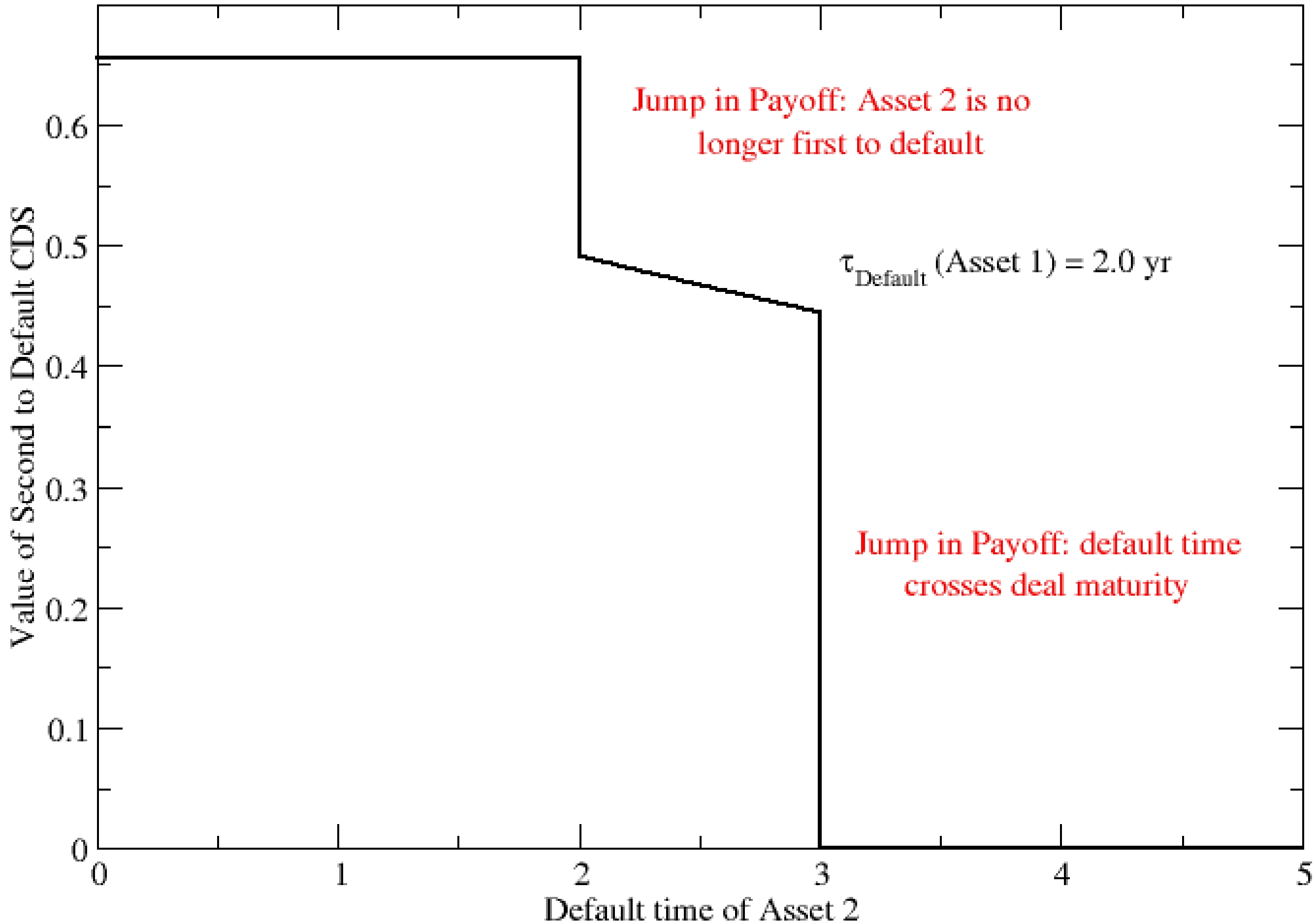
They correspond to:

a. default time of *j* th asset crosses final maturity of the product.

b. Upon bumping the *jth* hazard rate we alter which asset is the *nth* to default

Both result in a jump in value and hence a
Delta function in the derivative.

The Pathwise Method for nth Default Swaps



- When differentiated these jumps in the payoff give rise to delta functions !

The Pathwise Method for nth Default Swaps

The delta functions make a bumped Monte Carlo converge very slowly. However, we can integrate these *analytically* to obtain

$$-P(T)\frac{\partial E^{-1}}{\partial h_j}\int \psi(\tau_1,\dots,\tau_{j-1},T,\tau_{j+1},\dots,\tau_N)d\tau_1\dots d\tau_{j-1}d\tau_{j+1}\dots d\tau_n.$$

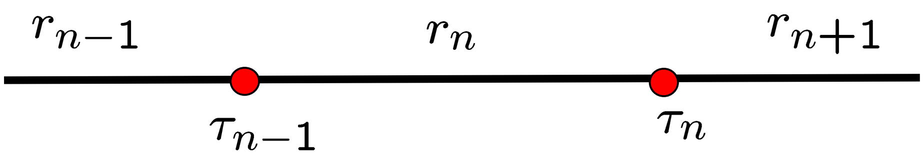
As before we simply reintroduce it, the second term is now

$$\int \frac{(I\psi(\tau_1,\dots,\tau_{j-1},T,\tau_{j+1},\dots,\tau_N))}{\psi_{n-1}(\tau_1,\dots,\tau_{j-1},\tau_{j+1},\dots,\tau_N)} \psi_{n-1}(\tau_1,\dots,\tau_{j-1},\tau_{j+1},\dots,\tau_N)d\tau_1\dots d\tau_{j-1}d\tau_{j+1}\dots d\tau_n,$$

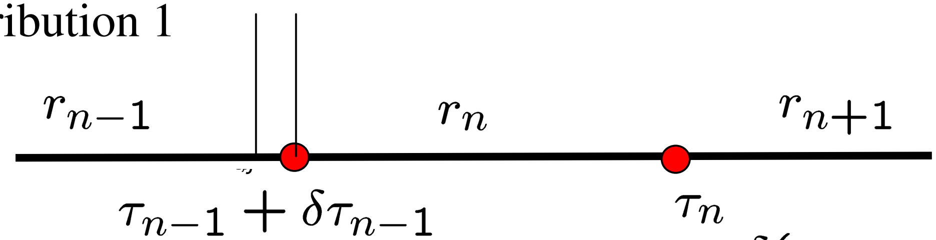
where I = 1 if t_j is the nth default time and zero otherwise.

Delta contributions from recovery rates

Two possible contributions: after sorting *j* th bond becomes (*n-1*)th or *n* th default.

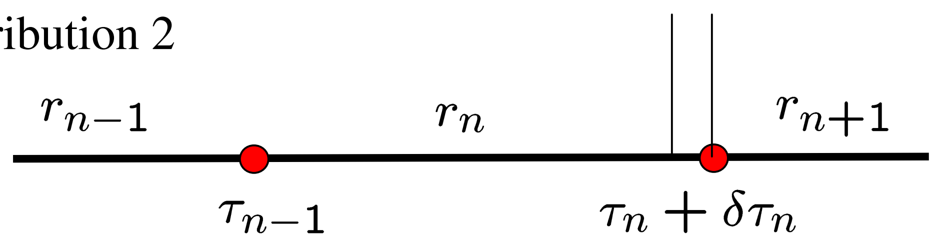


Contribution 1

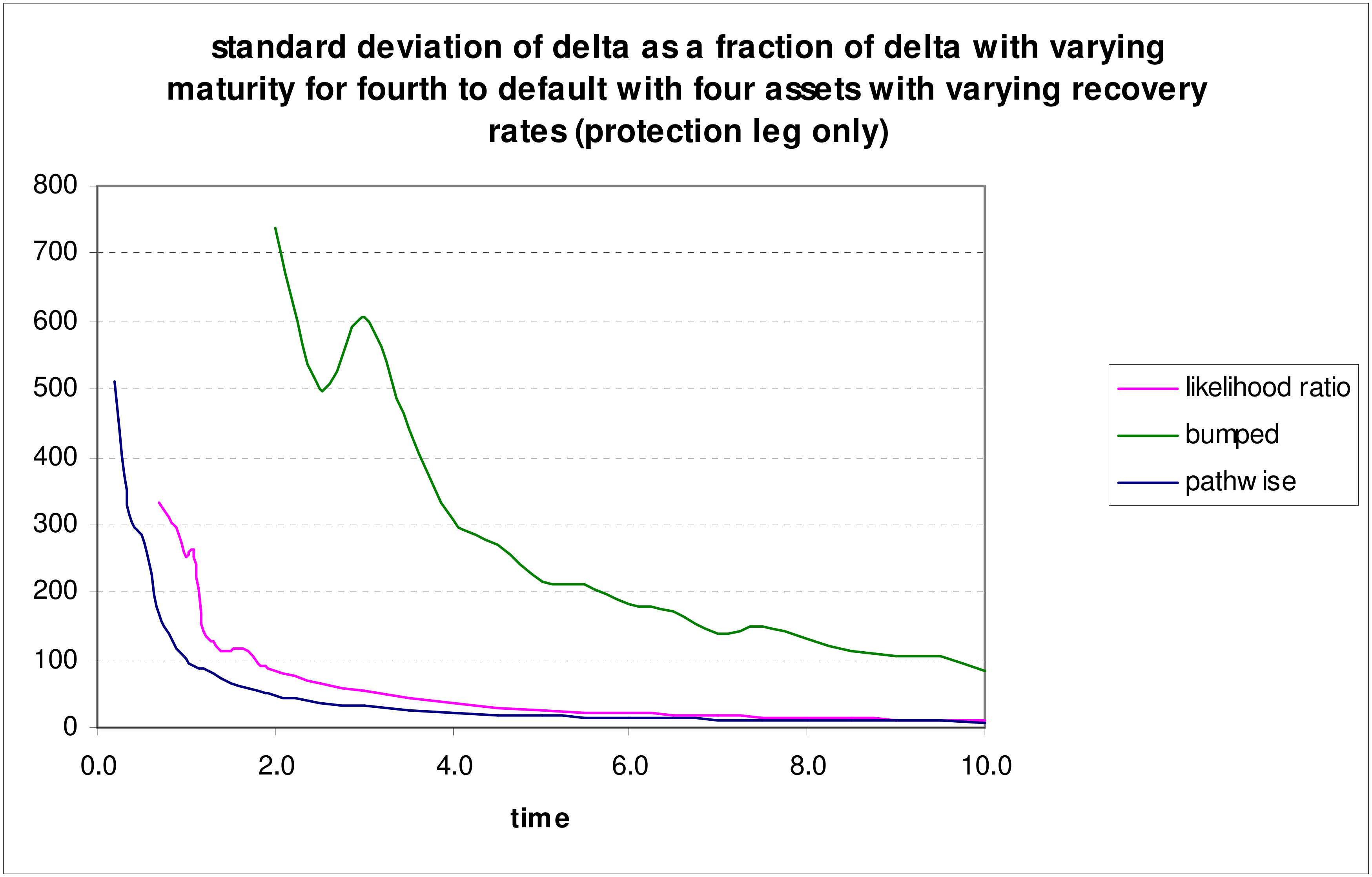


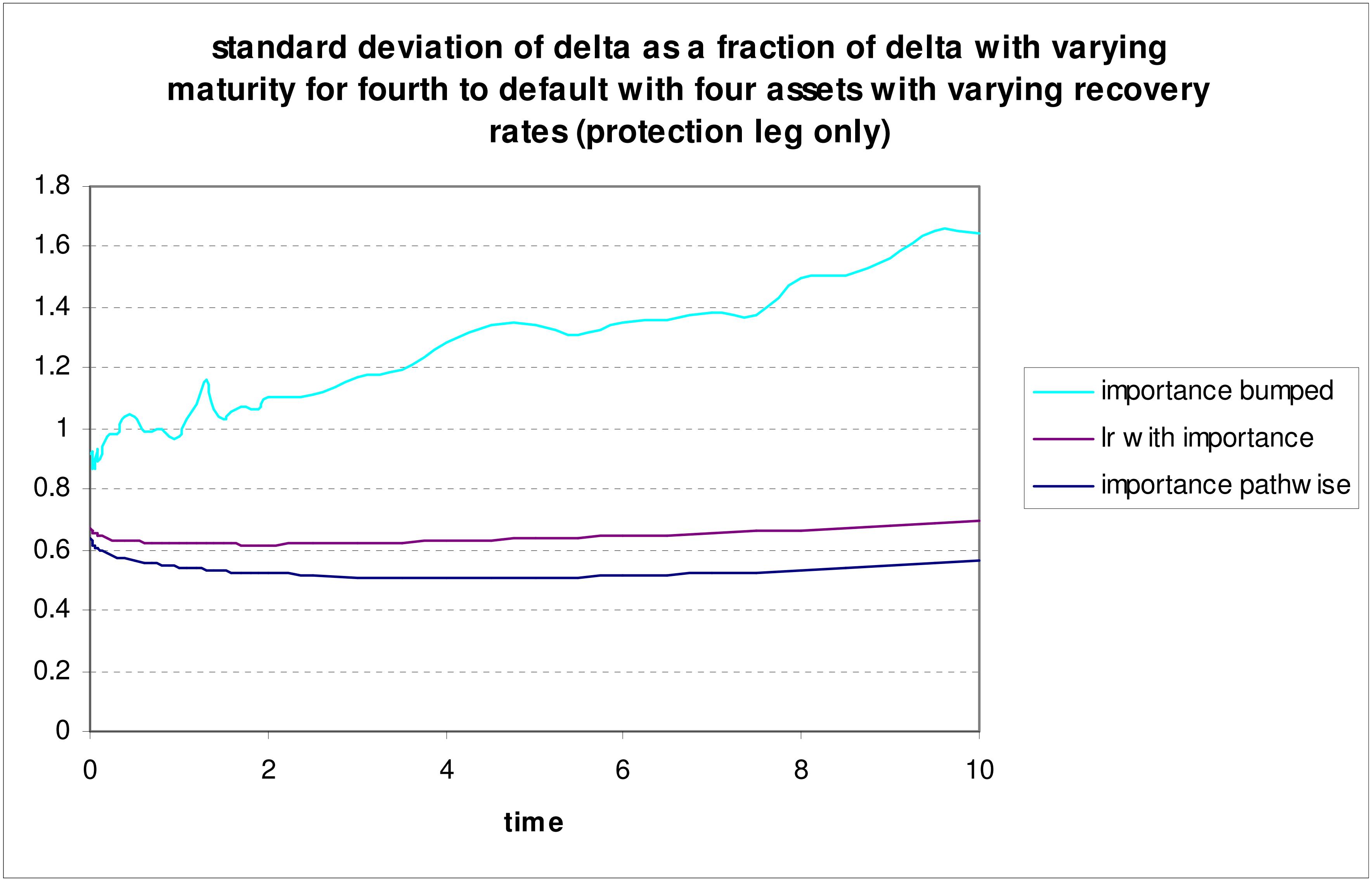
$$\delta(\tau_{n-1} - T)[((1 - r_n) - (1 - r_{n-1}))P(T)]$$

Contribution 2



$$\delta(\tau_n - T)[((1 - r_{n+1}) - (1 - r_n))P(T)]$$


standard deviation of delta as a fraction of delta with varying maturity for fourth to default with four assets with varying recovery rates (protection leg only)
800
700
600
500
400
300
200
100
0
0.0
2.0
4.0
6.0
8.0
10.0
time
likelihood ratio
bumped
pathwise


standard deviation of delta as a fraction of delta with varying maturity for fourth to default with four assets with varying recovery rates (protection leg only)
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
2
4
6
8
10
time
importance bumped
lr w ith importance
importance pathw ise

Does delta-hedging make sense?

- Li model -- deterministic intensities
- Typically products are delta-hedged
  - Compute sensitivity to hazard rate
  - Hedge that sensitivity with individual CDS
  - Instantaneously insensitive to credit spread moves

- Outside the model hedging

The Investigation Methodology

- Price and parameter hedge (possibly with different frequencies) the nth to default basket. We will use the Gaussian copula model.

- Introduce spread volatility: simulate the real world evolution of spreads. We will discuss some of the issues, but basically by means of

  1. Pure drift

  2. Pure diffusion

  3. Pure Jumps

- Analyse the terminal variability of the returns.

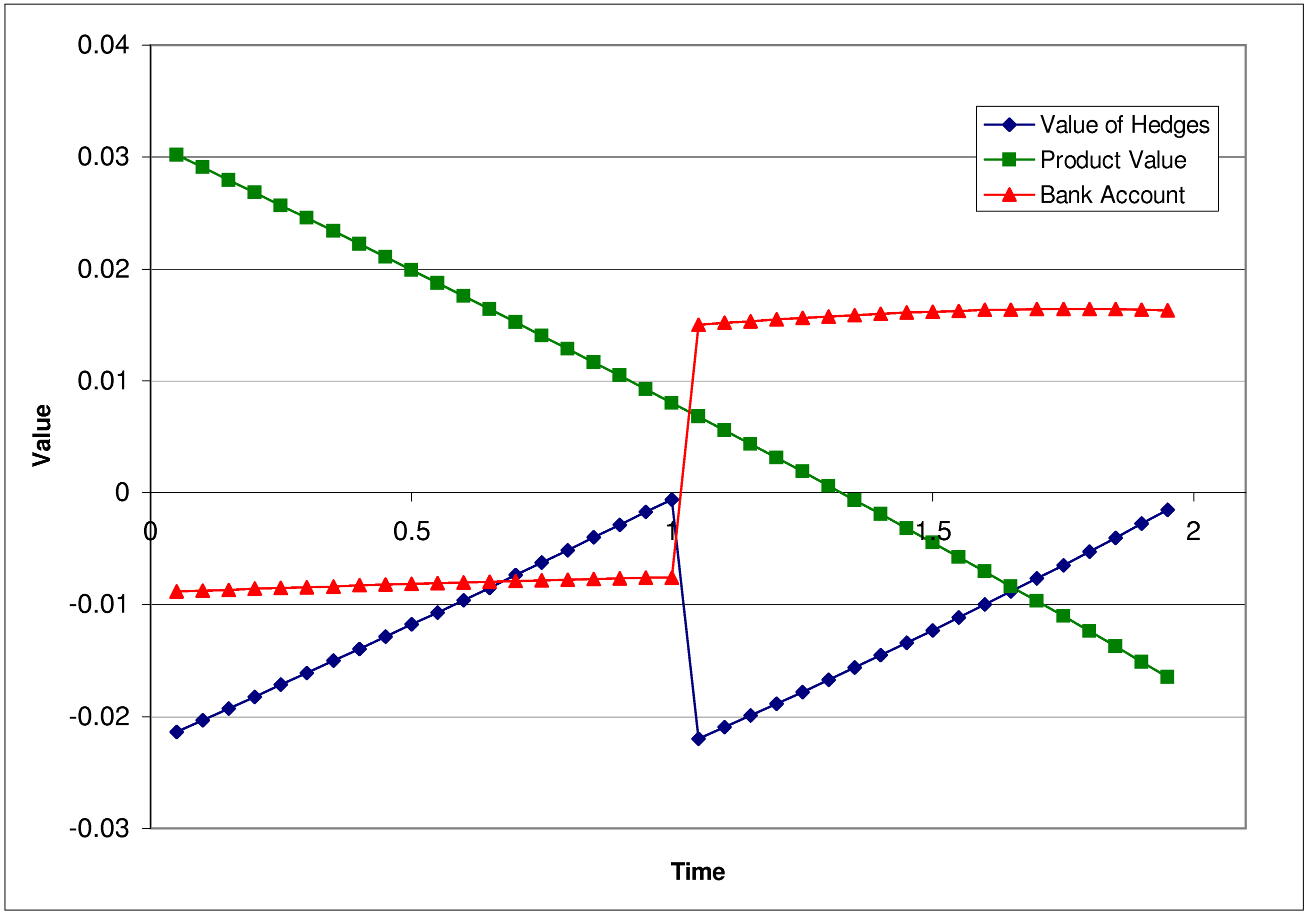
Hedging Algorithm

Parameter Hedging

- Two sources of price variation that we must hedge against:
  - Stochastic variation in spreads
  - Default events

Parameter Hedging cont.

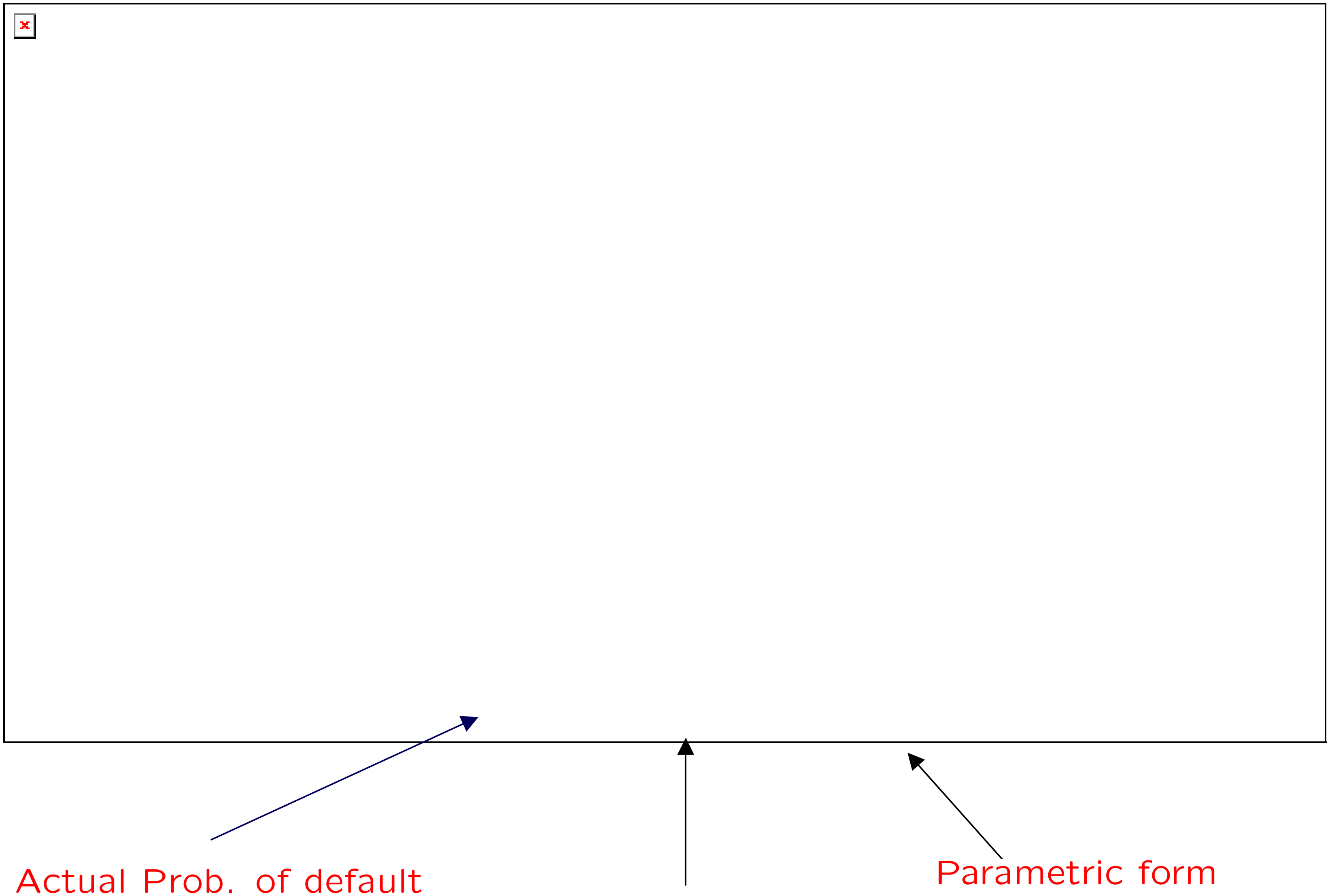
Algorithm



- Plots of the values of the hedges, the hedgee and the residual cash through the course of a hedging simulation, (constant spreads through time). Product is first to default on a homogeneous basket of 5 names. The discontinuity in the 1 year point in the bank account balance is due to receiving an upfront spread.

Hedging Simulations: Modelling Spreads.

Hedging Simulations: Modelling Spreads.

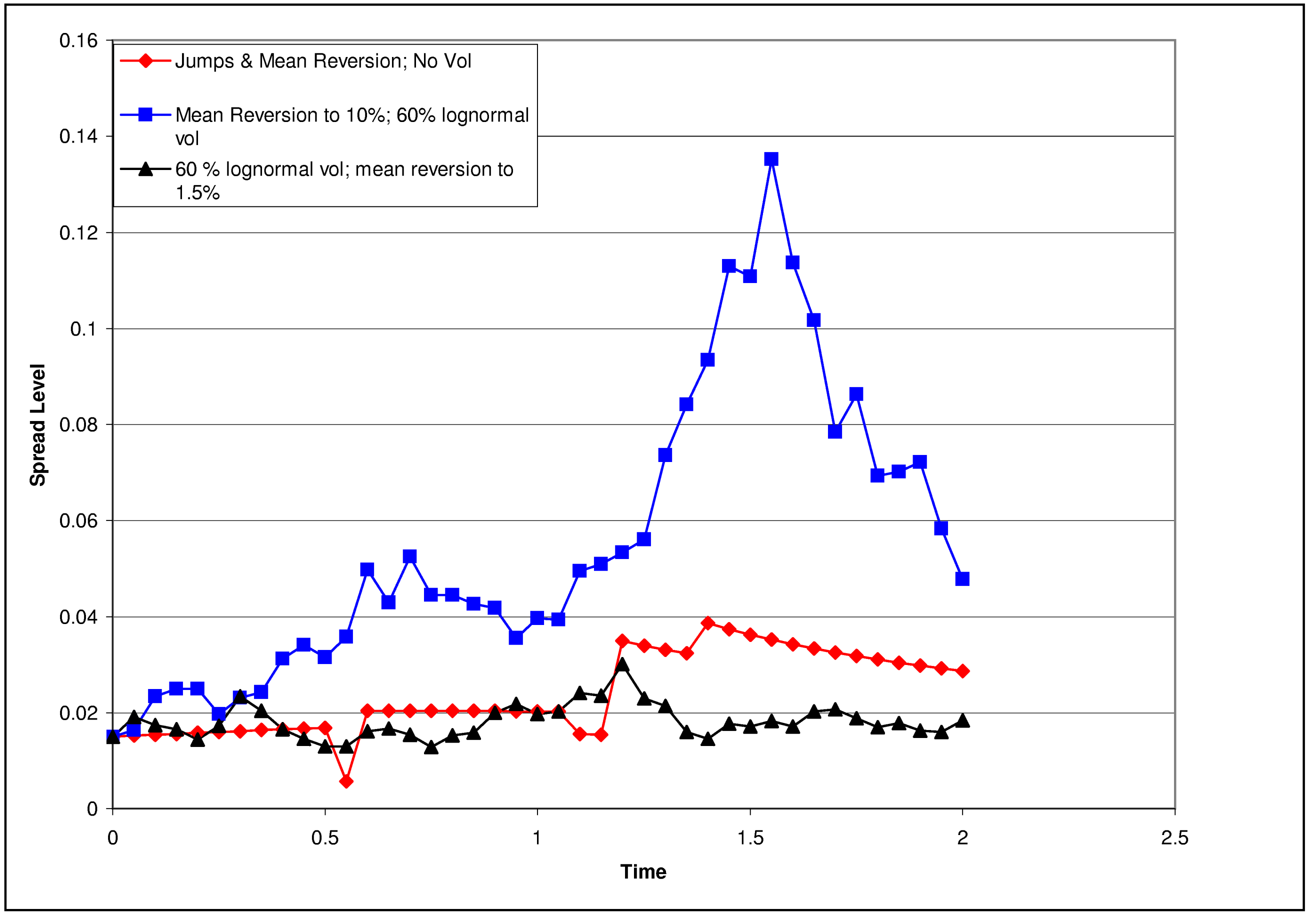


Actual Prob. of default

Market quoted spread

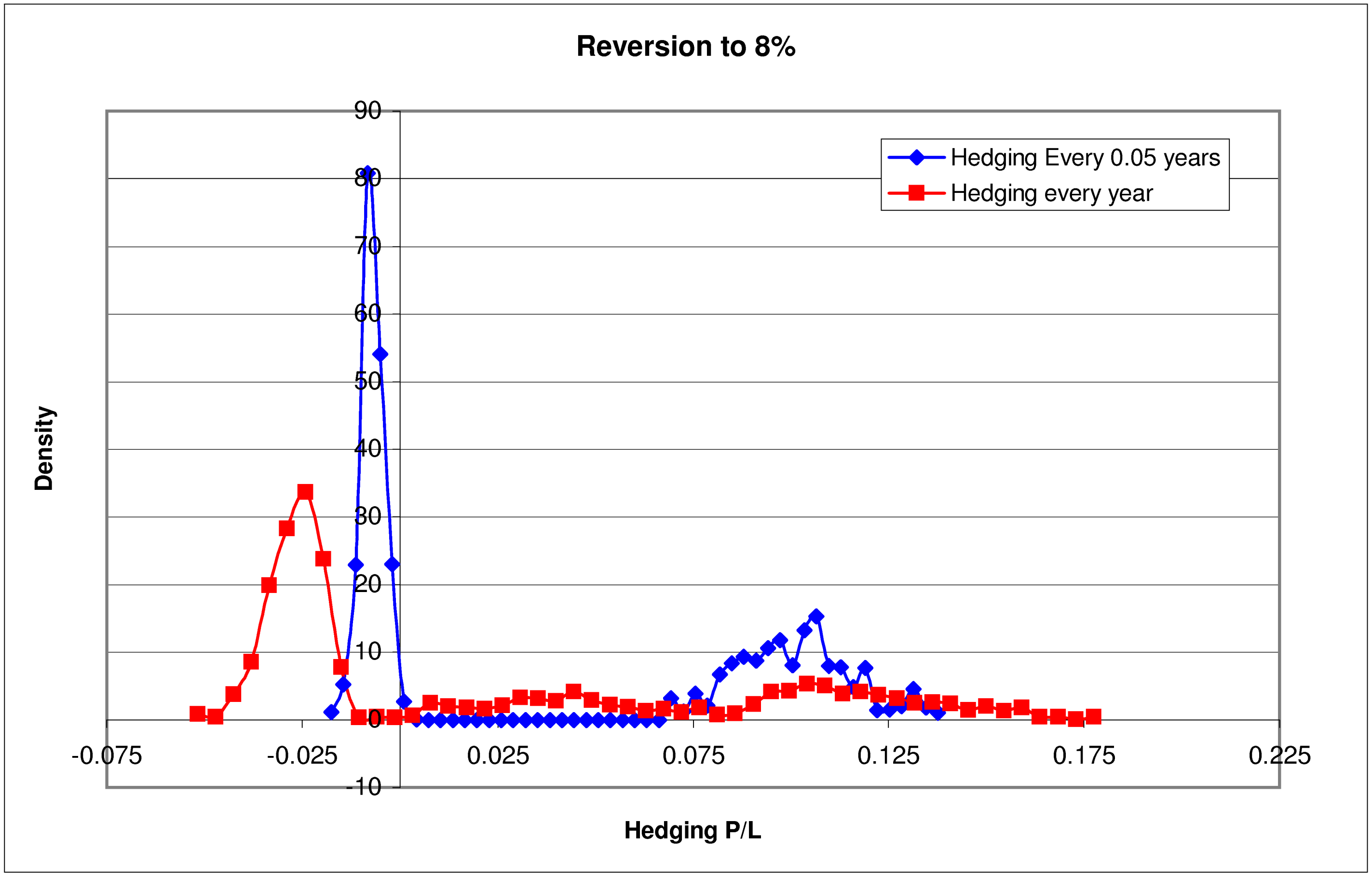
Parametric form
tax, illiquidity, information

Sample Paths.



- Sample evolutions of spreads for diffusive, mean reverting and jumpy random walks. These sorts of evolutions are “typical” of the spread movements produced by our hedging simulator.

Results:P/L as a result of Drift in Spreads.



- A comparison between the variability in income as a result of spreads drifting up to a mean of 0.08 over a period of two years. We have shown two different hedging frequencies here: one with a rehedge every 3 weeks or so and the other extreme, hedging once every year.

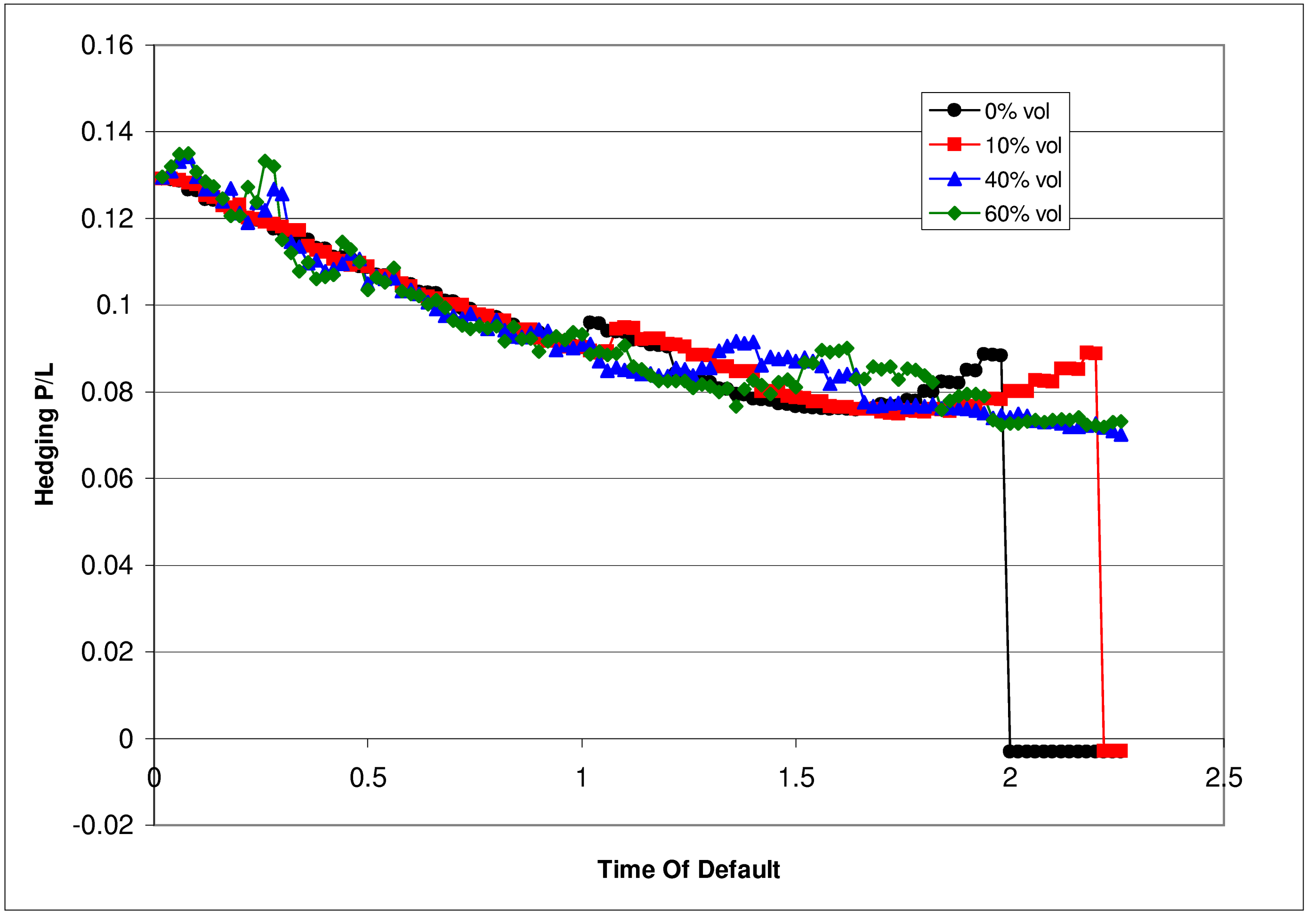
P/L as a result of Drift in Spreads.

- Empirical spreads do drift systematically upwards/downwards over a period of time. Spreads have blown out to 12% for a number of significant firms post World Com/Sep. 11.

- Comparing hedging at a reasonable frequency with hedging infrequently (every year !) certainly leads to a reduction in the variance of the P/L in the no defaults case.

- Variability in P/L for the single default case is driven by the time of default.

P/L assuming 1 default.



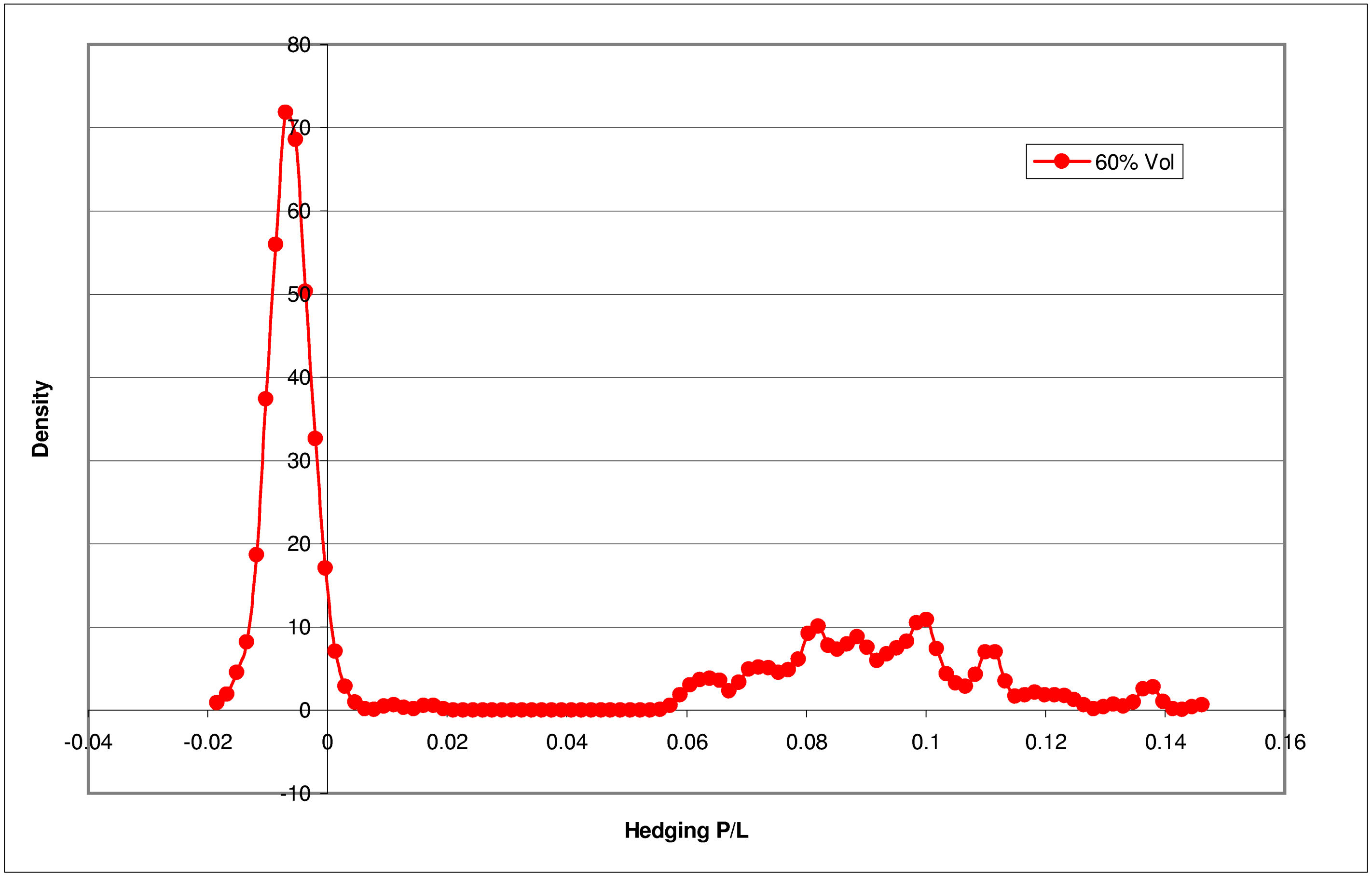
- Assuming that the spreads of the assets alive in the basket do not alter on default of an asset in the basket, we have plotted the P/L as a function of time of default.

- Different lines — different spread volatilities. Clear that the P/L is principally a function of time of default.

Results:P/L as a result of Spread Volatility.

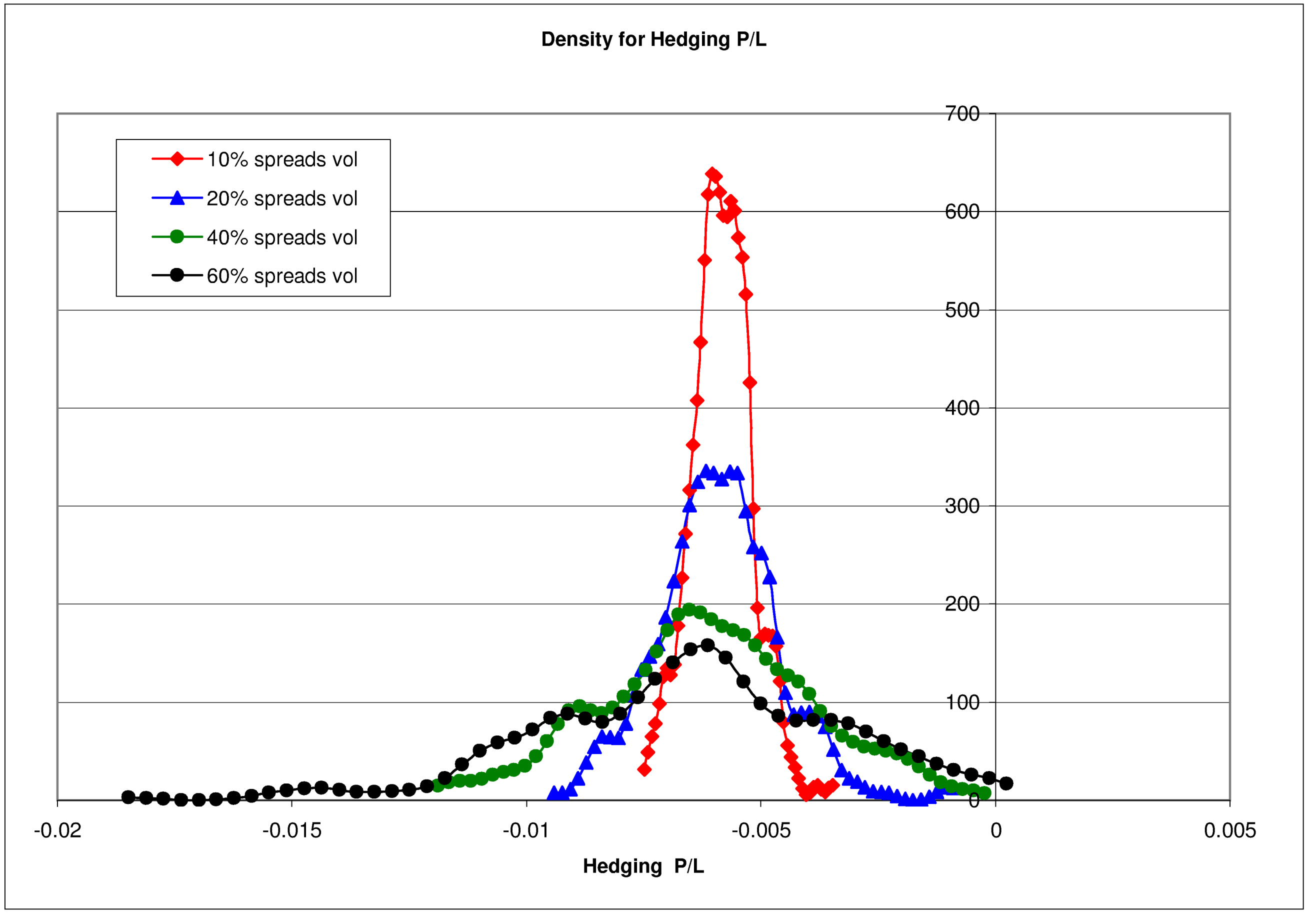
- Li's copula model does not explicitly incorporate the spread vol. Can the pricing and the hedging strategy, as dictated by the copula model adequately hedge volatile spread movements ? Analysed P/L assuming that the spreads are weakly mean reverting, with no jumps; we increase the log-normal volatility of the spreads to see the effects on the variance of the hedged portfolio.

Results:P/L as a result of Spread Volatility.



- A plot of the density of the profit/loss generated by our hedging strategy, for a first to default basket. The spreads in this case were taken to be weakly mean reverting, with no jumps and a lognormal volatility of 60%.

Results:P/L as a result of Spread Volatility.



- Density plot of the P/L generated by our hedging strategy, for a first to default basket. Consider only the cases where there are no defaults. Spreads are taken to be weakly mean reverting, with no jumps with log-normal volatilities of 10%, 20%, 40% and 60%.