The Curious Incident of the Investment in the Market: Real Options and a Fair Gamble

David Hobson

University of Bath

dgh@maths.bath.ac.uk

http://www.bath.ac.uk/∼masdgh

Joint work with Jonathan Evans (Bath)

and Vicky Henderson (Princeton)

Introduction I

- Our problem - When should a manager sell a real asset ?
  - Decision irreversible
  - Real asset indivisible
  - Incomplete market with a risk-averse agent
- Continuous time, continuous price processes, infinite horizon
- Context: Well studied area of real options - Dixit and Pindyck (1994), McDonald and Siegel (1986) ....
- In complete market (perfect spanning asset) these are standard optimal stopping problems (McKean/Samuelson (1965))
- Alternatively, manager assumed risk-averse to market risks but risk-neutral to idiosyncratic risks (McDonald and Siegel (1986) CAPM argument)
- We will assume the real asset is not traded continuously so manager faces idiosyncratic risks and an incomplete market

Introduction II

- Consider the effect of embedding this optimal asset sale timing problem within a model where there is a financial market in which manager can trade
- Compare two incomplete market situations: in both cases (with or without the financial market) the real asset is not traded

Literature and Motivation

- Real options - Myers (1977), Brennan and Schwartz (1985), McDonald and Siegel (1986), Dixit and Pindyck (1994), ...

- Henderson (2002), Henderson and Hobson (2002), Kahl, Liu, Longstaff (2003)
- Detemple and Sundaresan (1999), Ingersoll (2006), Henderson (2004)

- Kadam, Lakner and Srinivasan (2004)

- Smith and Nau (1995)
- Henderson (2004)
- Miao and Wang (2004)

I. Asset Sale Problem with No Market Asset

- Risk-averse manager wishes to sell indivisible real asset over infinite horizon
- No alternative investment opportunities
- Zero interest rates throughout for simplicity
- Value received on selling real asset is Y, where

$$\frac{dY}{Y} = \mu dt + \sigma dW$$

where μ, σ constants and W Brownian motion.

- Y_t known at t but manager cannot continuously trade $Y \to$ Incomplete
- Let $\gamma = 2\mu/\sigma^2$. If $\gamma > 1$, Y grows to ∞, if $\gamma < 1$, Y tends to zero.
- Manager chooses sale time τ to max expected utility of wealth:

$$V^N(x,y) = \sup_{\tau} \mathbb{E}[U(x+Y_\tau)|Y_0 = y]$$

- Let $U(x) = \ln(x)$ but generalizes to CRRA

Proposition 1 *For* $\gamma \leq 0$, $V^N(x,y) = \ln(x+y)$
For $\gamma \geq 1$, $V^N(x,y) = \infty$
For $0 < \gamma < 1$, *in exercise region* $y \geq x/w^*$

$$V^N(x,y) = \ln(x+y)$$

in the continuation region $y < x/w^*$,

$$V^N(x,y) = \ln x + \left(\frac{yw^*}{x}\right)^{1-\gamma} \ln(1+1/w^*). \tag{1}$$

where optimal exercise ratio $w^* = w^*(\gamma)$ *is the unique solution to*

$$(1-\gamma)\ln(1+1/w) - \frac{1}{1+w} = 0 \tag{2}$$

I. Asset Sale Problem with No Market Asset: Remarks

• Non-degenerate case is $0 < \gamma < 1$ where risk premium $\mu > 0$ but $Y \to 0$. Risk aversion to idiosyncratic risk causes manager to sell if real asset value is large enough proportion of wealth

• If we considered a risk-neutral manager ($U(x) = x$), he either waits indefinitely if positive risk premium or sells immediately if negative risk premium. Hinges on whether $\gamma > 0$

• Risk aversion induces the manager to reduce uncertainty by selling in case $0 < \gamma < 1$

• We use $U(x) = \ln x$ so no explicit discounting.

II. Asset Sale Problem Allowing Trading in the Market

• Extend the problem to include a traded asset (the market) in which the manager may trade continuously

• Market asset with price process

$$\frac{dP}{P} = \eta dB + \nu dt$$

where $dBdW = \rho dt$.

• Trading wealth X_t^θ is self-financing:

$$dX_t^\theta = \theta_t(\eta dB + \nu dt)$$

• Manager chooses selling time τ and investment strategy θ in the market to solve:

$$V^M(x,y) = \sup_\tau \sup_\theta \mathbb{E}[U(X_\tau^\theta + Y_\tau)|X_0 = x, Y_0 = y], \qquad (3)$$

• If market were correlated $\rho \neq 0$ with real asset $\rightarrow$ hedging motive to trade (offset risk). cf. Detemple and Sundaresan (1999), Kahl et al (2003), Viceira (2002)

• If market has positive Sharpe ratio then investment motive to trade.

• We assume traded asset **uncorrelated** with real asset and traded asset has **zero** Sharpe ratio $\rightarrow$ no hedging motive, no investment motive

• X represents a **fair gamble** since a $\mathbb{P}$-martingale

Proposition 2 *Under the strategy specified by the thresholds (ξ, η), the value function is*

$$V^M(x,y) = \begin{cases} \ln(x+y) & x \leq \xi y \\ G(x,y) & \xi y \leq x < \eta y \\ H(x,y) & \eta y < x \end{cases}$$

where

$$G(x,y) = \frac{\xi}{\eta-\xi}\left(\frac{x}{\xi y} - 1\right)(\ln y + \ln \eta + \Theta) + \frac{\eta}{\eta - \xi}\left(1 - \frac{x}{\eta y}\right)(\ln y + \ln(1+\xi))$$

and

$$H(x,y) = \ln x + \left(\frac{\eta y}{x}\right)^{1-\gamma}\Theta$$

with

$$\Theta \equiv \Theta(\eta,\xi) = \left[\frac{\eta - \xi + \eta\left(\ln(1+\xi) - \ln\eta\right))}{\eta + (\eta-\xi)(1-\gamma)}\right]. \qquad (4)$$

Lemma 3 *Let* γ_- *be the unique solution in* $(0,1)$ *of* $\Gamma_-(\gamma) = 0$ *where*

$$\Gamma_-(\gamma) = (1-\gamma)(2-\gamma)\ln\left(\frac{2-\gamma}{1-\gamma}\right) - 1$$

Then $\gamma_- \sim 0.3492$. *Consider now the problem of finding the maximum of* $\eta^{1-\gamma}\Theta$ *over* $-1 < \xi \leq \eta < \infty$. *For* $0 < \gamma \leq \gamma_-$ *the max is attained at* $\eta = w^*$, $\xi = w^*$ *where* $w^* = w^*(\gamma)$ *is the solution to* (2)

For $\gamma_- < \gamma \leq 1$ *the max is attained at* $\eta = \eta^*$ *where*

$$\eta^* = \eta^*(\gamma) = \frac{1-\gamma}{2-\gamma}\left(\frac{1}{1-\gamma} - \ln\left(\frac{2-\gamma}{1-\gamma}\right)\right)^{-1} \tag{5}$$

and $\xi = \xi^*$ *where*

$$\xi^* = \xi^*(\gamma) = \frac{\eta^*(2-\gamma)}{1-\gamma} - 1 = \frac{(1-\gamma)\ln((2-\gamma)/(1-\gamma)) - \gamma}{1-(1-\gamma)\ln((2-\gamma)/(1-\gamma))} \tag{6}$$

Proposition 4 *(i) For* $\gamma \leq 0$, $V^M(x,y) = \ln(x+y)$.
(ii) For $0 < \gamma \leq \gamma_-$ $V^M(x,y) = \ln(x+y)$ *in the exercise region* $y \geq x/w^*$, *and in the continuation region* $y < x/w^*$

$$V^M(x,y) = \ln x + \left(\frac{w^* y}{x}\right)^{1-\gamma} \ln(1+1/w^*)$$

where w^* *solves* (2).
(iii) For $\gamma_- < \gamma < 1$, *in the exercise region* $x \leq y\xi^*(\gamma)$,

$$V^M(x,y) = \ln(x+y)$$

for $y\xi^*(\gamma) < x < y\eta^*(\gamma)$,

$$\begin{aligned} V^M(x,y) &= \frac{\xi^*}{\eta^* - \xi^*}\left(\frac{x}{\xi^* y} - 1\right)(\ln y + \ln \eta^* + \Theta^*) \\ &+ \frac{\eta^*}{\eta^* - \xi^*}\left(1 - \frac{x}{\eta^* y}\right)(\ln y + \ln(1+\xi^*)) \end{aligned}$$

and for $x \geq y\eta^*(\gamma)$,

$$V^M(x,y) = \ln x + \left(\frac{\eta^* y}{x}\right)^{1-\gamma} \Theta^*$$

where η^* *and* ξ^* *are given by (5) and (6) and* $\Theta^* = \Theta(\eta^*, \xi^*)$.

(iv) For $\gamma \geq 1$, $V^M(x,y) = \infty$

Key result is that the manager's improves his expected utility when he can invest in the market, if $\gamma_- < \gamma < 1$

One-Period Model

- We can describe the same features in a one (or perhaps two!)-period binomial model
- Real asset is indivisible; sale is irreversible, agent is risk-averse with decreasing absolute risk aversion.
- Manager offered gambling opportunity
- This opportunity occurs before the movemnets in real asset price.
- Sale decision for the real asset is made ex post.

- Manager has initial wealth x and owns real asset with current value y
- Value of real asset at time 1 is either yu or yd where $u > d$. Take $p = 1/2$ for simplicity
- Take $u + d > 2$ so $\mathbb{E}Y_1 > y$ and $ud < 1$ so $\mathbb{E}\ln Y_1 < \ln y$. Analogous to $0 < \gamma < 1$
- Manager can sell at time 0 or wait till time 1. Expected utility is

$$\ln(x + y) \text{ if sell at } t = 0$$

or

$$\frac{1}{2}\ln(x + yu) + \frac{1}{2}\ln(x + yd) \text{ if wait}$$

- Optimal to wait if $y < x/w^*$ where

$$w^* = (1 - ud)/(u + d - 2)$$

Manager with Access to Fair Gamble

- Now let the manager enter a fair gamble at time 0 before deciding whether to sell
- Gamble pays $\pm\epsilon$ with probability $\frac{1}{2}$ in each case
- The outcome of the gamble is independent of Y
- If the manager did not own the real asset, he would never accept a fair gamble with concave utility
- Suppose $y = x/w^*$ and manager accepts the bet. If bet and win,

$$x \to x + \epsilon, \quad y < (x + \epsilon)/w^* \text{ wait}$$

If bet and lose,

$$x \to x - \epsilon, \quad y > (x - \epsilon)/w^* \text{ sell at time 0}$$

Expected utility is

$$\frac{1}{4}\{\ln(x + \epsilon + xu/w^*) + \ln(x + \epsilon + xd/w^*)\} + \frac{1}{2}\ln(x - \epsilon + x/w^*).$$

Writing $\epsilon = x\tilde{\epsilon}/w^*$ and equation solved by w^*,

$$\begin{aligned} & \ln x + \ln(1 + 1/w^*) \\ + \quad & \frac{1}{4}\left\{\ln\left(1 + \frac{\tilde{\epsilon}}{w^* + u}\right) + \ln\left(1 + \frac{\tilde{\epsilon}}{w^* + d}\right) + 2\ln\left(1 - \frac{\tilde{\epsilon}}{w^* + 1}\right)\right\} \end{aligned}$$

This utility exceeds $\ln x + \ln(1 + 1/w^*)$ if the final bracket is positive, and expanding this term in $\tilde{\epsilon}$ we see that this happens (for some positive ϵ) provided

$$0 < \frac{1}{w^* + u} + \frac{1}{w^* + d} - \frac{2}{w^* + 1} = \frac{(u + d - 2)}{(w^* + 1)^2}$$

which is true since $u + d > 2$

IV. A. Interpretation

• We saw in the one-period model that the manager's value function was not concave in initial wealth $x \rightarrow$ manager is locally risk seeking and accepted a gamble in some situations

• This underlies the continuous time results as well

• The curious behavior occurs when $\gamma_{-} < \gamma < 1$. When there is no market, manager sells when $Y/x > 1/w^*$ as risk aversion outweighs benefits of waiting

• However if there is a market, the manager can reduce the proportion of wealth in the real asset by trading: If successful, proportion in real asset drops and waits to sell. If unsuccessful, sell real asset (and worse off). When γ large enough, the first effect dominates and worthwhile to gamble

B. The Certainty Equivalent Value of the Right to Sell the Real Asset

• Certainty equivalent value of the right to sell the real asset is cash amount manager would accept in place of the right to sell
• Given by p^N in the no-market setting, solves $\ln(x + p^N) = V^N(x, y)$
• Given by p^M in market setting, solves $\ln(x + p^M) = V^M(x, y)$

C. The Probability of Exercise/Selling

• Market enables manager to trade so that real asset only sold when it forms a higher proportion of wealth than in the no-market case. ie. $1/\xi^* > 1/w^*$

• For fixed $\gamma = 0.5$ the graph shows that probability of ever selling decreases when there is a market asset

Proposition 5 *For fixed positive initial wealth, for $\gamma_- < \gamma < 1$ and for any initial value y for the real asset, the probability that the real asset is sold is lower in the model with the market than in the no-market situation*

• Recall, if the manager trades successfully, he holds onto the real asset for longer and the probability of selling is reduced

• However, if he is unsuccessful, he sells the real asset. There are scenarios where manager with market access sells real asset sooner than manager without the market

D. No-Borrowing Constraints

• We have allowed the manager to borrow against holdings in the real asset, so $X_t \geq -Y_t$

• What happens if we constrain the manager to keep his trading wealth X positive ?

• If $\xi^* > 0$ then constraint has no effect. Define γ^+ to be solution of $\Gamma^+(\gamma) = 0$ where

$$\Gamma^+(\gamma) = (1-\gamma)\ln\left(\frac{2-\gamma}{1-\gamma}\right) - \gamma$$

We find $\gamma^+ = 0.5341$. γ^+ is critical γ where constraint has an effect - previously borrowing occurred beyond γ^+

Proposition 6 *For $\gamma^+ < \gamma < 1$, the max of $\eta^{1-\gamma}\Theta(\eta,\xi)$ over $0 \le \xi \le \eta$ is attained at $\xi = 0$ and $\eta = \eta^*$ where $\eta^* = \eta^*(\gamma) = e^{-\gamma/(1-\gamma)}$. Under the no-borrowing constraint the value function becomes*
for $xe^{\gamma/(1-\gamma)} < y$

$$V^M(x,y) = \frac{e^{\gamma/(1-\gamma)}x}{y} \quad \ln y + \frac{1-\gamma}{2-\gamma} \quad + \quad 1 - \frac{e^{\gamma/(1-\gamma)}x}{y} \quad \ln y,$$

and for $y \le xe^{\gamma/(1-\gamma)}$

$$V^M(x,y) = \ln x + \left(\frac{y}{x}\right)^{1-\gamma} \frac{e^{-\gamma}}{(2-\gamma)(1-\gamma)}$$

- If $\gamma^+ < \gamma < 1$, manager only sells when he becomes insolvent, ie. $X = 0$

E. CRRA Preferences and Limiting Cases

• Extends to CRRA preferences $U(x) = x^{1-R}/(1-R);\ \ R > 0$

• For CRRA, the upper threshold (beyond which wait indefinitely) becomes $\min\{R, 1\}$

• Range $(\gamma_-(R), 1)$ is where manager chooses to invest in the market. Largest range is for $R = 1$, log utility. For log, manager chooses to take fair gamble for the widest range of parameter values (Sharpe ratio and volatility of real asset)

• As $R \to 0$, $U(x) = x$, risk-neutral. Here $\gamma_-, \gamma^+ \to 0$ and $\min\{R, 1\} \to 0$. Manager sells if $\gamma < 0$ and waits indefinitely if $\gamma > 0$. Corresponds to special case of McDonald and Siegel (1986)

• As $R \to \infty$, $\gamma_-(R), \gamma^+(R) \to 1$ and market is no use. Optimal strategy corresponds to model without a market. Corresponds to exponential utility - wealth factors out so no-market value is concave in x since U is (cf Henderson (2004))

Conclusions

- We have shown that it can be optimal for a risk-averse manager to accept a fair gamble if he is facing idiosyncratic risk in an incomplete market arising from the right to sell a real asset
- Beware of omitting assets which appear not to alter behaviour ...
- A rationale for gambling?
- Our conclusions were robust to choice of CRRA preferences but do not occur for exponential utility
- Investment in the market or fair gamble hinges on indivisibility of the real asset
- Applications to executive stock options where manager receives restricted stock (or options) and asks when to optimally exercise
- Applications to optimal retirement choice.
- Extension to consumption models.