Porfolio Management without Probability or Statistics

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- Log-optimality,
- Kelly's rule,
- stationarity,
- evolustionary stability,
- stochastic approximation

PLAN

The setting

The adaptive process

Replicator dynamics

Evolutionary stability in two-person games.

Global convergence.

THE SETTING:

discrete time $t = 0, 1, \ldots$

assets $a \in A$ (finite),

gross return $X^a_{t+1} \geq 0$ per unit at the end of $(t, t+1]$,

accumulated financial savings S_t

proportion p^a_t into a.

Dynamic

$$S_{t+1} := \langle p_t, X_{t+1} \rangle \, S_t := \Big(\sum_{a \in A} p^a_t X^a_{t+1} \Big) S_t.$$

NB! "portfolio" always in standard simplex $\mathbb{P}$.

$X_t = (X_t^a)$, $t = 0, 1, \ldots$ independent $\sim X \Rightarrow p$ constant $\Rightarrow$

$$S_T = \Pi_{t=0}^{T-1} \langle p, X_{t+1} \rangle$$

Strong Law of Large Numbers:

$$\log(S_T^{1/T}) = \frac{1}{T} \sum_{t=1}^{T} \log \langle p, X_{t+1} \rangle \to E \log \langle p, X \rangle \text{ a.s.}$$

Look for Log-optimal portfolio p,

$$\text{maximize} \quad E \log \langle p, X \rangle \quad \text{s.t.} \quad p \in \mathbb{P}, \qquad (1)$$

More generally,

$$\text{maximize} \quad E_t \log \langle p_t, X_{t+1} \rangle \quad \text{s.t.} \quad p_t \in \mathbb{P}, \quad (2)$$

using $E_t := E[\cdot \,|\mathcal{F}_t]$.

BUT! distribution often unknown!

Problem:

Can a scantly informed, statistically unskilled investor maximize long-run growth?

Kuhn-Tucker conditions for problem (2):

$$\left.\begin{array}{lll} E_t\left[X_{t+1}^a/\langle p_t, X_{t+1}\rangle\right] & =1 & \text{whenever } p_t^a>0, \\ E_t\left[X_{t+1}^a/\langle p, X_{t+1}\rangle\right] & \leq 1 & \text{otherwise.} \end{array}\right\} \quad (3)$$

IDEA: p_t^a should be reduced iff

$$X_{t+1}^a/\langle p, X_{t+1}\rangle \ < 1.$$

For each $a \in A$ let

$$p_{t+1}^a = p_t^a + s_t p_t^a\left[X_{t+1}^a/\left\langle p_t, X_{t+1}\right\rangle - 1\right]. \quad (4)$$

$$\left.\begin{array}{ll} \text{(I)} & \sum_t s_t=+\infty, \quad \text{and} \\ \text{(II)} & \sum_t s_t^2<+\infty. \end{array}\right\} \quad (5)$$

NB! not a stochastic gradient method

$$p_{t+1} = \mathsf{Pr}\, oj_{\mathbb{P}}\left[p_t + s_t X_{t+1}/\langle p_t, X_{t+1}\rangle\right].$$

Log-optimal and Evolutionary Stable Strategies

Suppose *return process* stationary. Recall: $\bar{p}$ log-optimal iff

$$\left.\begin{array}{ll} E\left[X^a / \langle \bar{p}, X\rangle\right] = 1 & \text{whenever } \bar{p}^a > 0, \text{ and} \\ E\left[X^a / \langle \bar{p}, X\rangle\right] \leq 1 & \text{otherwise.} \end{array}\right\} \quad (6)$$

Consider 2-person, noncooperative, symmetric game, action space A.

Payoffs to mixed strategy p against $\bar{p}$:

$$\pi(p, \bar{p}) \quad := \quad \sum_{a \in A} p^a \cdot E\left[X^a / \langle \bar{p}, X\rangle\right]. \qquad (7)$$

$(\bar{p}, \bar{p})$ a symmetric Nash equilibrium iff

$$\pi(p, \bar{p}) \quad \leq \quad \pi(\bar{p}, \bar{p}) \text{ for all } p \in \mathbb{P}. \qquad (8)$$

Proposition (Log-optimality and equilibrium) *A portfolio $\bar{p}$ is unconditioned log-optimal iff it constitutes a Nash equilibrium $(\bar{p}, \bar{p})$ in the two-person, symmetric game with payoff* (7). $\square$

Definition $\bar{\mathbb{P}} \subseteq \mathbb{P}$ *evolutionary stable* iff for each $\bar{p} \in \bar{\mathbb{P}}$ $\exists \delta > 0$ such that

$$p \notin \bar{P}, \quad \|p - \bar{p}\| \leq \delta \;\; \& \quad \pi(p, \bar{p}) = \pi(\bar{p}, \bar{p}) \;\Rightarrow\; \pi(\bar{p}, p) > \pi(p$$

Proposition (Log-optimality and evolutionary stability) *The set $\bar{\mathbb{P}}$ of unconditioned log-optimal portfolios is closed convex and evolutionary stable. In fact, it holds for any $p \in \mathbb{P} \diagdown \bar{\mathbb{P}}$ that $\pi(\bar{p}, p) > \pi(p, p)$.*

Proof. $L(\cdot) := E \log \langle \cdot, X \rangle$ concave & usc. Therefore $\bar{\mathbb{P}} := \arg\max L$ closed convex. p and optimal $\bar{p}$ it holds of course that

$$p \;\text{ nonopt. \& } \bar{p} \;\text{ opt.} \quad \Rightarrow L(p) < L(\bar{p}).$$

By concavity,

$$L(\bar{p}) \leq L(p) + \left\langle L'(p), \bar{p} - p \right\rangle.$$

Addition yields $\langle L'(p), \bar{p} - p \rangle > 0$. Finally, $\langle L'(p), \bar{p} - p \rangle = \pi(\bar{p}, p) - \pi(p, p) > 0$. $\Box$

Payoff of pure strategy a against p

$$\pi(a,p) := E\left[X^a / \langle p, X\rangle\right]$$

Replicator dynamics

$$\dot{p}^a = p^a \left[\pi(a,p) - \pi(p,p)\right] \quad \forall a \tag{9}$$

Proposition (Asymptotic stability) *System* (9) *is globally asymptotically stable on the relative interior of the simplex. That is, for any initial* p_0, *having all* $p_0^a > 0$, *it holds that each accumulation point of the resulting trajectory is log-optimal.*

Proof. Recall that the *relative entropy* (alias *Kullback-Leibler distance*)

$$K(\bar{p}, p) := \sum_{a \in A} \bar{p}^a \log \frac{\bar{p}^a}{p^a} \geq 0 \tag{10}$$

"Distance function"

$$\lambda(p) := \min\left\{K(\bar{p}, p) : \bar{p} \ \text{ log-optimal}\right\}. \tag{11}$$

$\lambda(p)$ convex & differentiable. Indeed, when all $p^a > 0$,

$$\nabla \lambda(p) = \frac{\partial}{\partial p} K(\bar{p}, p)|_{\bar{p}=\bar{p}(p)} = -\left[\bar{p}^a / p^a\right] \in \mathbb{R}^A.$$

Thus,

$$\frac{d}{d\tau}\lambda(p(\tau)) = -\sum_{a \in A} \bar{p}^a \left[\pi(a, p) - \pi(p, p)\right]$$

$$= -\left[\pi(\bar{p}, p) - \pi(p, p)\right] < 0. \square$$

Convergence

Euler's direct method on replicator dynamics:

$$p_{t+1}^a = p_t^a + s_t p_t^a \left[E \left[X^a / \langle p_t, X \rangle \right] - 1 \right].$$

Stocahastic approximation theory: DROP E!

Robbins-Monro assumptions:
1) The conditional distribution of X_{t+1} given $\mathcal{F}_t$ depends at most on p_t.

2)

$$\Gamma := \sup \left\{ E \sum_a \{ X^a / \langle p, X \rangle - 1 \}^2 : p \in \mathbb{P} \right\} < \infty. \tag{12}$$

Theorem (Global convergence to log-optimality) *Under the Robbins-Monro assumption and* (12) *each cluster point of iteration* (4) *is almost surely a log-optimal portfolio. In particular, when the latter is unique, convergence to that portfolio obtains a.s.*

Proof uses

Robbins-Monro Lemma *Suppose* $\mathcal{A}_t, B_t, C_t, D_t, t = 0, 1, ...$ *are finite-valued, non-negative random variables, all measurable with respect to a sigma-field* $\mathcal{F}_t \subseteq \mathcal{F}_{t+1}$, *which satisfy*

$$E\left[\mathcal{A}_{t+1} \left| \mathcal{F}_t\right.\right] \leq (1 + B_t)\mathcal{A}_t + C_t - D_t. \tag{13}$$

Then, in the event $\{\sum_t B_t < +\infty, \sum_t C_t < +\infty\}$ *it holds that*

$$\mathcal{A}_t \to \mathcal{A} < \infty \;\; \text{and} \;\; \sum_t D_t < +\infty \;\; \text{a.s.} \tag{14}$$

As said, in the present case, let $\mathcal{F}_t$ be generated by the return vectors $X_0, ..., X_t$ and other relevant information unveiled up to time t. Plainly, $\mathcal{F}_t \subseteq \mathcal{F}_{t+1}$. Posit

$$\mathcal{A}_t := \min\left\{\|p_t - \bar{p}\|^2 / 2 : \bar{p} \;\; \text{log-optimal}\right\}$$

Note that $\mathcal{A}_t = \|p_t - \bar{p}_t\|^2/2 \geq 0$ for a unique $\mathcal{F}_t$-measurable, log-optimal $\bar{p}_t$. Also,

$$\mathcal{A}_{t+1} \leq \|p_{t+1} - \bar{p}_t\|^2/2 = \|(p_t - \bar{p}_t) + (p_{t+1} - p_t)\|^2/2$$
$$\leq \mathcal{A}_t + s_t \sum_{a\in A} (p_t^a - \bar{p}_t^a)\left[X_{t+1}^a / \langle p_t, X_{t+1}\rangle - 1\right] + s_t^2 \sum_{a\in}$$
$$= \mathcal{A}_t + s_t \sum_{a\in A} (p_t^a - \bar{p}_t^a) X_{t+1}^a / \langle p_t, X_{t+1}\rangle + s_t^2 \sum_{a\in A} \left[X_{t+}^a\right.$$

In this string take conditional expectation $E_t := E[\cdot\,|\mathcal{F}_t]$ to get inequality (13) with

$$B_t := 0, \quad C_t := s_t^2 \Gamma,$$

and $D_t := -s_t \left\{ \sum_{a\in A} (p_t^a - \bar{p}^a) E_t\left[X_{t+1}^a / \langle p_t, X_{t+1}\rangle\right] \right\} \geq 0.$

Now, via condition (II) in (5), since the event $\{\sum_t B_t < +\infty, \sum_t C$ carries full probability, (14) follows. Whenever $\mathcal{A} > 0$, condition (I) in (5) implies $\sum D_t = +\infty$. Consequently, $\mathcal{A} = 0$ a.s. and this completes the proof. $\Box$

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