Optimal Dynamic Asset Allocation for Defined Contribution Pension Plans

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Outline for talk

- Problem: Accumulation phase of a DC plan
- Model formulation
- Optimal investment strategy
- Qualitative characteristics
- Quantitative characteristics
  - Comparison of optimal strategy with commercial strategies

Using a toy model: how much room for improvement?

The problem

- Identify sources of risk to investor:
  - investment risk
  - interest-rate risk
  - salary risk
- risk assessment
- guidance for plan members, advisers, regulators

How well does a DC plan match a DB benchmark:

$$\text{Replacement Ratio} = \frac{\text{DC pension}}{\text{final salary}}$$

"Model" Occupational DC plan

- Contributions = fixed % of salary
- choice of "commercial" investment strategies
- various asset classes
- static versus dynamic

Typical *default* strategies

Static strategies

- Pension Fund Average

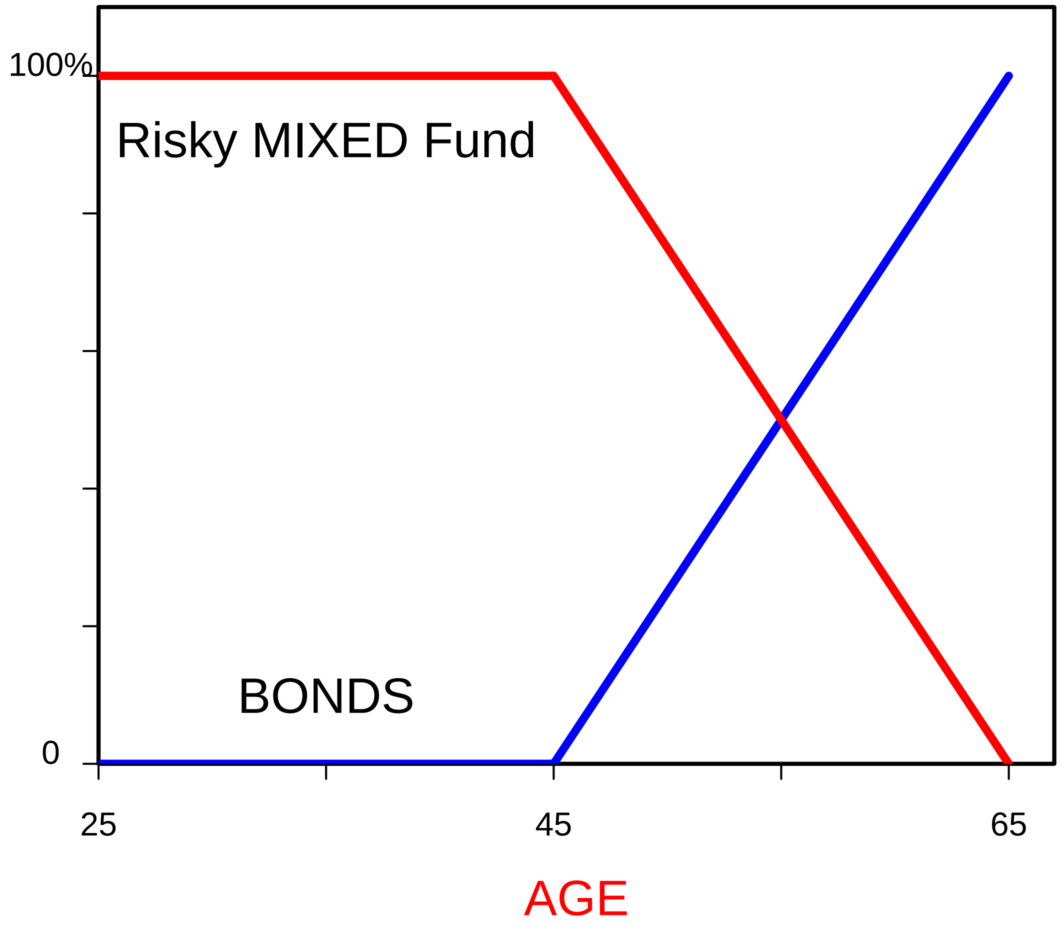
  typical mixed fund ($\sim$ 70% in UK/int'l equities)

- Mixed Bonds (50/50)

  50% long bonds; 50% cash

  $\Rightarrow$ minimum variance of Replacement Ratio

Deterministic Lifestyle strategy



Initially MIXED fund. Then switch gradually into BONDS.

Default “commercial” strategies:

- Static
- Deterministic lifestyle

Are these strategies the best that we can do?

By how much can they be improved?

- theoretical best
- practical best (not this seminar!)

The model

State variables:

$$
\begin{aligned}
Y(t) &= \text{Salary} \\
W(t) &= \text{Accumulated pension wealth} \\
r(t) &= \text{Risk-free interest rate (one-factor model)}
\end{aligned}
$$

The model: Assets

$n+1$ sources of risk: $Z_0(t),\ Z_1(t),\ \ldots,\ Z_N(t)$

Cash account, $R_0(t)$:

$$dR_0(t) \;=\; r(t)R_0(t)dt$$

$$dr(t) \;=\; \mu_r(r(t))dt + \sum_{j=1}^{N} \sigma_{rj}(r(t))dZ_j(t)$$

The model: Assets

Risky assets, $R_1(t), \ldots, R_N(t)$:

$$dR_i(t) = R_i(t)\left[\left(r(t) + \sum_{j=1}^{N} \sigma_{ij}\xi_j\right) dt + \sum_{j=1}^{N} \sigma_{ij} dZ_j(t)\right]$$

$$C \;=\; \left(\,\sigma_{ij}\,\right) \;=\; \text{volatility matrix } (N \times N)$$

(non-singular)

$$\xi \;=\; \left(\,\xi_j\,\right) \;=\; \text{market prices of risk } (N \times 1)$$

The model: Salary and contributions

$$dY(t) = Y(t)\Big[(r(t)+\mu_Y(t))\,dt + \sum_{j=1}^{N}\sigma_{Yj}\,dZ_j(t) + \sigma_{Y0}\,dZ_0(t)\Big]$$

$\mu_Y(t)$ deterministic

Plan member contributes continuously into DC pension plan at the rate $\pi Y(t)$ for constant π.

The model: Pension wealth, $W(t)$:

$$p(t) = \big(p_1(t), \ldots, p_N(t)\big)$$
$$= \text{proportion of wealth in risky assets}$$

$$dW(t) = W(t)\left[(r(t) + p(t)'C\xi)\,dt + p(t)'C\,dZ(t)\right] + \pi Y(t)dt$$

The model: The pension:

Retirement at a fixed date T.

At T the cost of \$1 for life is

$$a(r(T)) = \sum_{u=0}^{\infty} p(65,u)P(T,T+u,r(T))$$

$$p(65,u) = \text{survival probability from 65 to } 65+u$$

$$P(T,\tau,r) = \text{price at } T \text{ for \$1 at } \tau$$

$$\text{given } r(T) = r$$

Replacement ratio:

$$\text{Repl. Ratio} = \frac{Pension(T)}{Y(T)} = \frac{W(T)/a(r(T))}{Y(T)}$$

Teminal utility: $=$ function of replacement ratio

$$u(w, y, r) = \frac{1}{\gamma}\left(\frac{w}{y.a(r)}\right)^{\gamma}$$

($\Rightarrow$ type of habit formation)

Reduction of state space:

Sufficient to model $r(t)$ and $X(t) = W(t)/Y(t)$

$$
\begin{aligned}
dX(t) &= \pi dt \\
+X(t)\Big[\Big(&- \mu_Y(t) + p(t)'C(\xi - \sigma_Y) + \sigma_{Y0}^2 + \sigma_Y'\sigma_Y\Big)dt \\
&-\sigma_{Y0} dZ_0(t) + \Big(p(t)'C - \sigma_Y'\Big)dZ(t)\Big]
\end{aligned}
$$

Optimisation: Given strategy $p(t)$

Expected terminal utility is $J(t, x, r; p) =$

$$E\left[\gamma^{-1}\left(\frac{X_p(T)}{a(r(T))}\right)^{\gamma} \middle| \; X(t) = x,\; r(t) = r\right]$$

$X_p(t) =$ path of $X(t)$ given strategy p.

Objective:

Maximise expected terminal utility

over $p = \{p(t) : 0 \leq t \leq T\}$

$$V(t,x,r) = \sup_p J(t,x,r;p)$$

HJB equation $\Rightarrow$ nonlinear PDE

$$\begin{aligned}
& V_t \\
& +\mu_r(r)V_r \\
& +\big(\pi - \tilde{\mu}_Y(t)x + \sigma_Y'(\xi - \sigma_Y)x\big)V_x \\
& +\frac{1}{2}\sigma_r(r)'\sigma_r(r)V_{rr} \\
& +\frac{1}{2}\sigma_{Y0}^2 x^2 V_{xx} \\
& -\frac{1}{2}(\xi - \sigma_Y)'(\xi - \sigma_Y)\frac{V_x^2}{V_{xx}} \\
& -(\xi - \sigma_Y)'\sigma_r(r)\frac{V_x V_{xr}}{V_{xx}} \\
& -\frac{1}{2}\sigma_r(r)'\sigma_r(r)\frac{V_{xr}^2}{V_{xx}} \quad = \quad 0.
\end{aligned}$$

Model $\Rightarrow$ many assets

Optimisation $\Rightarrow$ we require only 3 mutual funds

A Minumum risk fund to match salary risk

B Minimum risk fund to match salary $\times$ annuity risk

C Efficient, risky fund

A: Minumum risk fund to match salary risk

Mainly cash

adjusted for correlation between salaries and

other assets

Used to minimise short-term risk

B: Minimum risk fund to match salary × annuity risk

Mainly bonds

to minimise *immediate* annuity purchase risk

adjusted for correl. between salaries and other assets

C: Efficient, risky fund

Traditional efficient, risky portfolio with respect to a

salary *numeraire*

Qualitative remarks

- Investment in Fund C $= 1\big/$local RRA
- Investment in Fund A $\longrightarrow 0$ as $t \to T$
- Conjecture:

  As $T - t \nearrow$, investment in Fund B $\longrightarrow 0$

Problem components:

$$
\begin{aligned}
& V_t \\
& +\mu_r(r)V_r \\
& +\left(\pi - \tilde{\mu}_Y(t)x + \sigma_Y'(\xi - \sigma_Y)x\right)V_x \\
& +\frac{1}{2}\sigma_r(r)'\sigma_r(r)V_{rr} \\
& +\frac{1}{2}\sigma_{Y0}^2 x^2 V_{xx} \\
& -\frac{1}{2}(\xi - \sigma_Y)'(\xi - \sigma_Y)\frac{V_x^2}{V_{xx}} \\
& -(\xi - \sigma_Y)'\sigma_r(r)\frac{V_x V_{xr}}{V_{xx}} \\
& -\frac{1}{2}\sigma_r(r)'\sigma_r(r)\frac{V_{xr}^2}{V_{xx}} \quad = \quad 0.
\end{aligned}
$$

Complete market: $\sigma_{Y0} = 0$

Main conclusions: optimal strategy

- Effective assets at t are

$$\overline{W}(t) \;=\; \text{actual pension wealth, } W(t) \\ + \text{risk-adjusted value of future} \\ \text{premiums, } RAVFP$$

Borrow $RAVFP$ in units of mutual fund A

Main conclusions: optimal strategy

- Investment in risky fund C

  $= \text{constant \% of } \overline{W}(t)$

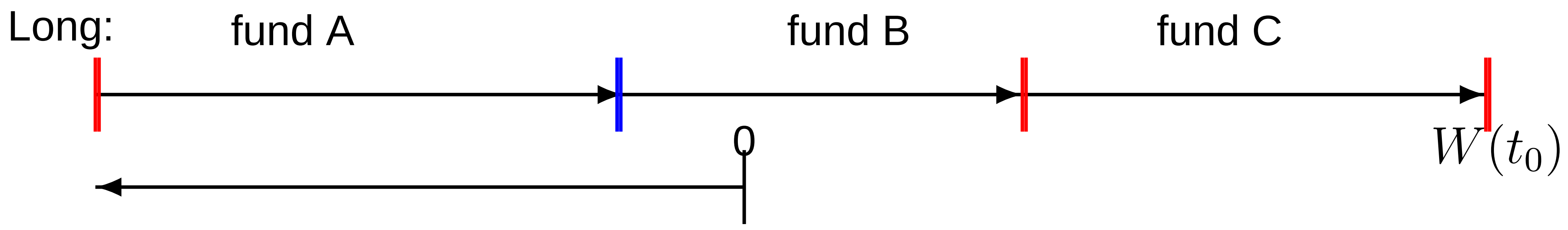
  constant % depends upon plan member's relative risk aversion, RRA

- As a percentage of $\overline{W}(t)$

  investment in mutual fund B grows over time

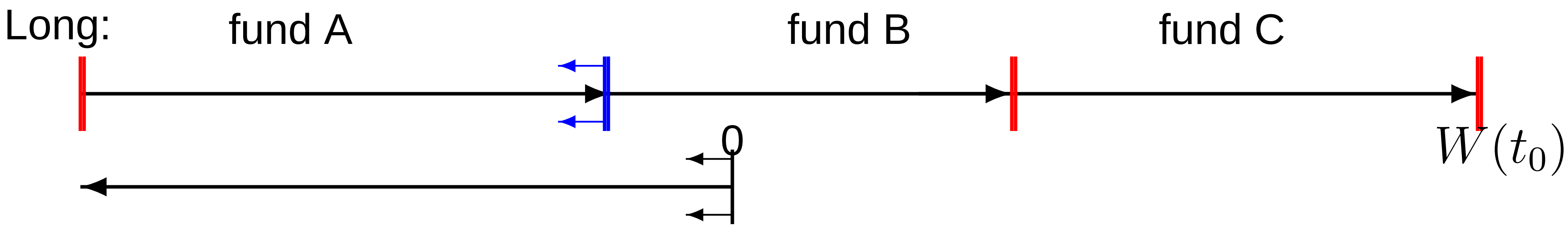
Investment in Mutual Funds A, B, C:

small t_0, some wealth, $W(t_0)$, accumulated



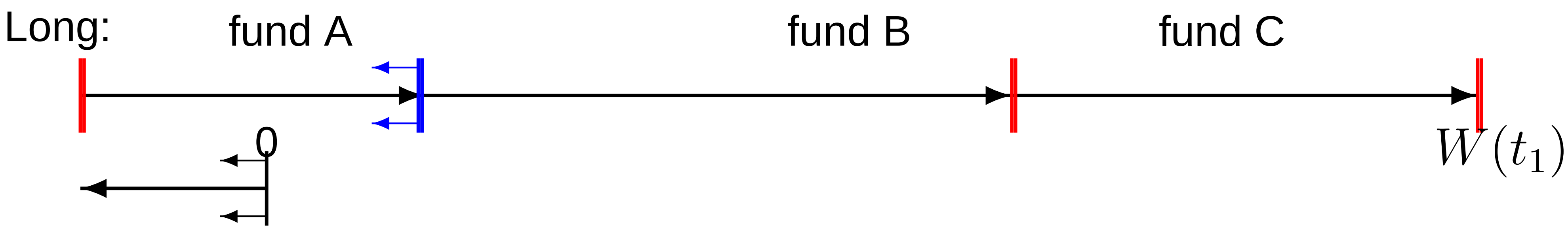
Short in A: future premiums

Small t_0 (as before)



Short in A: future premiums

Large t_1



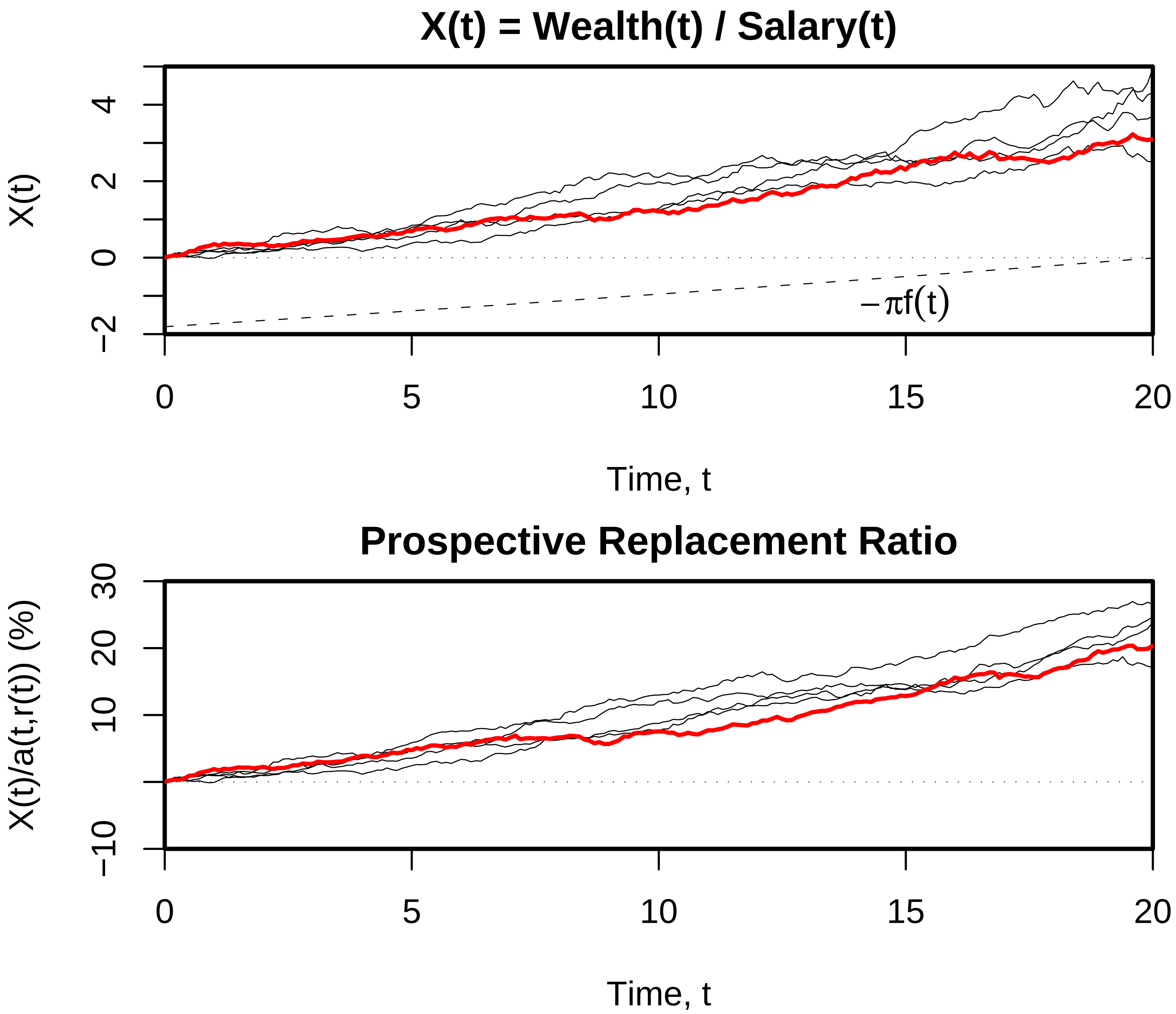
Short in A: future premiums

Numerical example: $r(t) \sim$ Vasicek

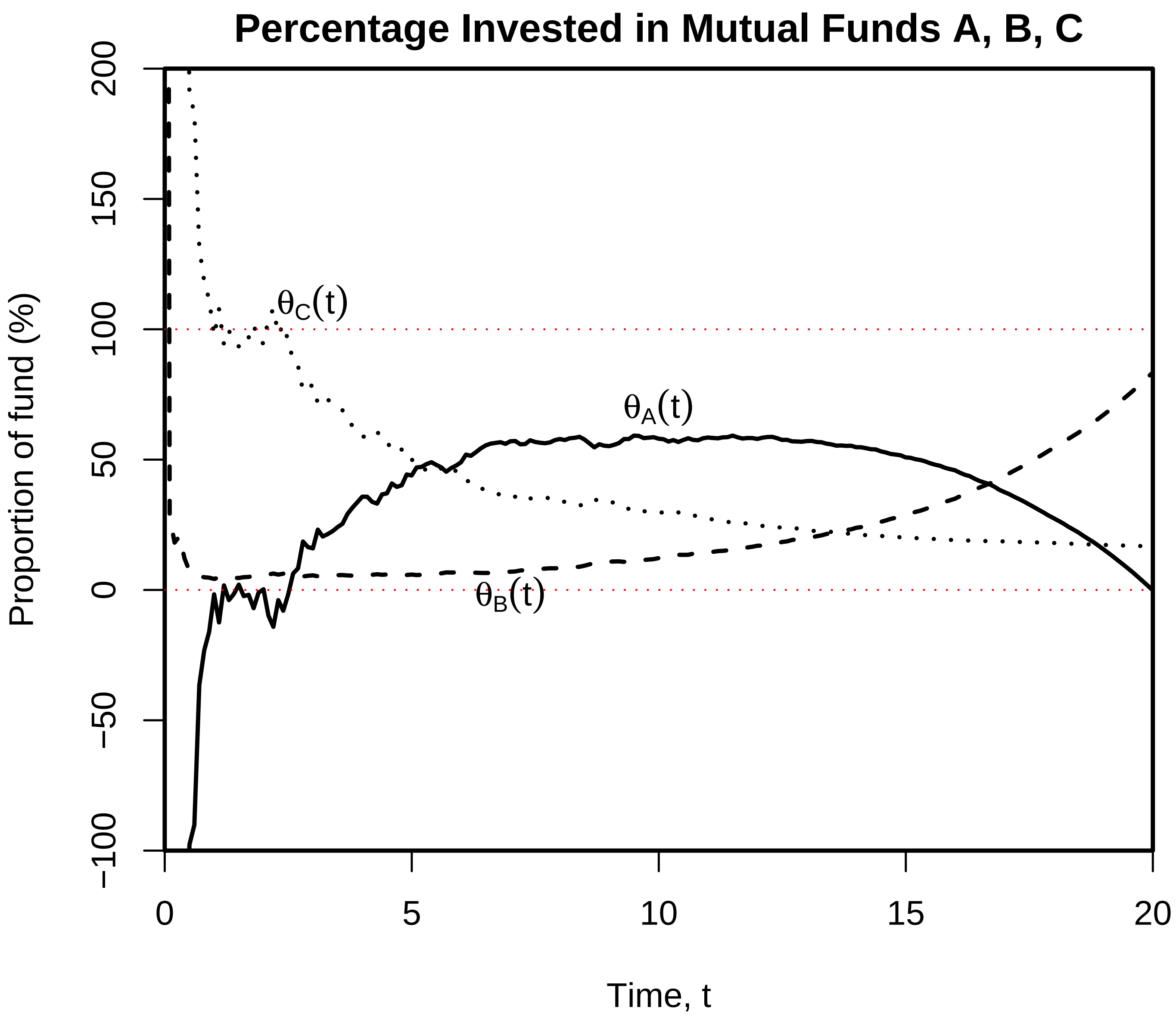
Example 1:

- Relative risk aversion: $RRA = 6$ (moderate)
- Duration of contract: $T = 20$ years
- Contribution rate: 10% of salary

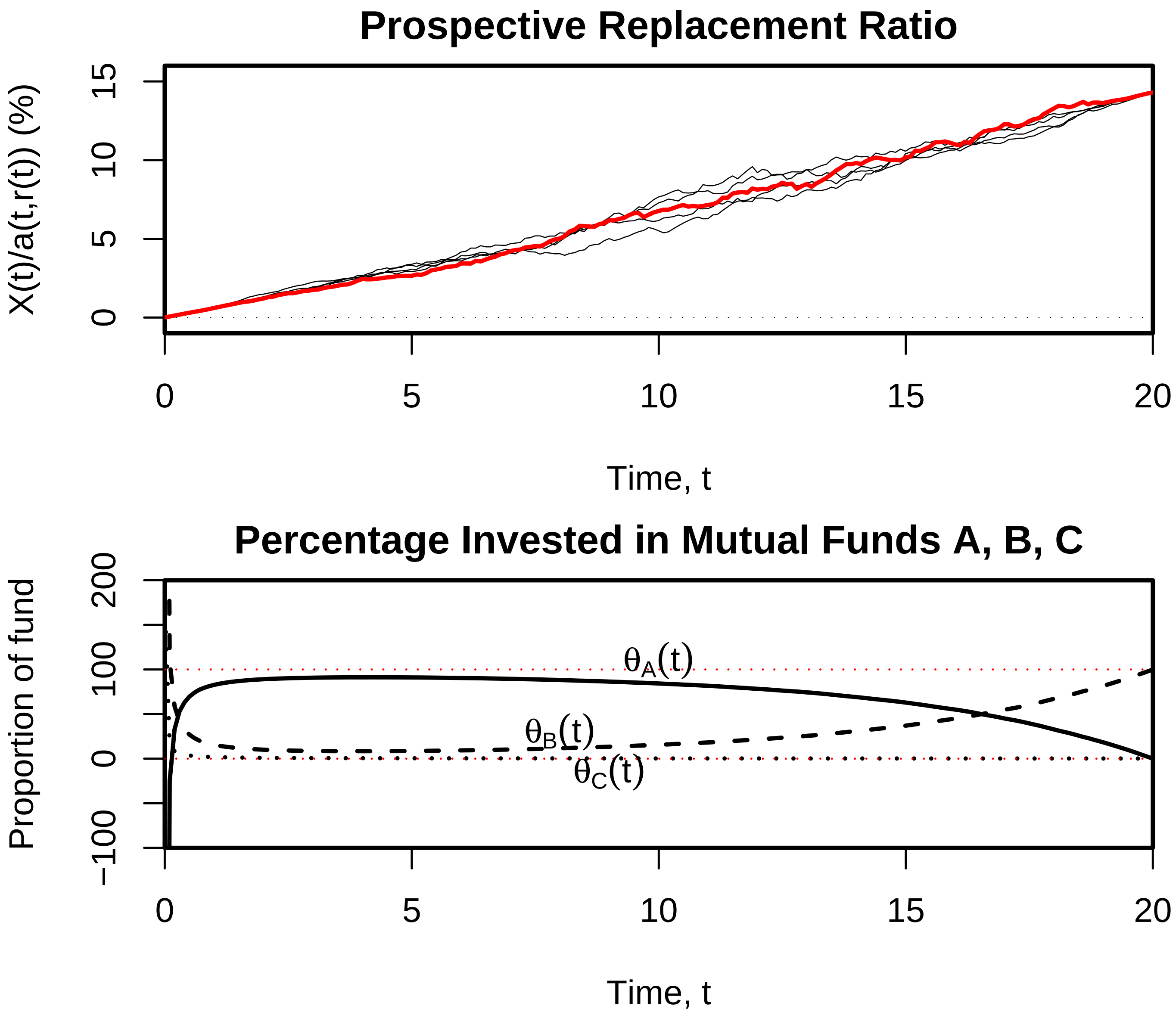
Example 1: $RRA = 6, T = 20$



Example 1: $RRA = 6, T = 20$



Example 2: Very high RRA, $T = 20$



Comparison with other strategies

Optimal strategy versus:

- Salary-hedged static strategy (S)
- Merton-static strategy (M)
- Deterministic lifestyle strategies:
  - initially 100% in equities
  - gradual switch over last 10 years into 100% bonds (B-10) or 100% cash (C-10)

Tables show:

- Expected terminal utility, $V(0,0)$ (normalised):

  starting at time 0

  with $W(0) = 0$

- Cost:
  - Benchmark: 10% cont. rate with optimal strategy
  - Other strategies: % contribution rate to match optimal utility

(c)	$RRA = 6, \quad T = 20$				
Strategy:	Optimal	Static		Deterministic lifestyle	
	stochastic	S	M	B-10	C-10
$V(0,0)$	-100	-134.58	-205.42	-141.00	-191.47
Cost	10.00%	10.61%	11.55%	10.71%	11.39%

(c)	$RRA = 6, \quad T = 20$						
Strategy:	Optimal	Static		Deterministic lifestyle			
	stochastic	S	M	B-10	B-5	A-10	A-5
Cost	10.00%	10.61%	11.55%	10.71%	11.42%	11.39%	11.88%

(d)	$RRA = 6, \quad T = 40$						
Strategy:	Optimal	Static		Deterministic lifestyle			
	stochastic	S	M	B-10	B-5	A-10	A-5
Cost	10.00%	11.52%	12.58%	12.86%	14.04%	13.67%	14.68%

(a)	$RRA = 1, \quad T = 20$						
Strategy:	Optimal	Static		Deterministic lifestyle			
	stochastic	S	M	B-10	B-5	A-10	A-5
Cost	10.00%	13.79%	13.78%	20.18%	18.67%	21.39%	19.23%

(c)	$RRA = 6, \quad T = 20$						
Strategy:	Optimal	Static		Deterministic lifestyle			
	stochastic	S	M	B-10	B-5	A-10	A-5
Cost	10.00%	10.61%	11.55%	10.71%	11.42%	11.39%	11.88%

(e)	$RRA = 12, \quad T = 20$						
Strategy:	Optimal	Static		Deterministic lifestyle			
	stochastic	S	M	B-10	B-5	A-10	A-5
Cost	10.00%	10.61%	12.08%	11.70%	13.77%	12.65%	14.40%

(b)	$RRA = 1, \quad T = 40$						
Strategy:	Optimal	Static		Deterministic lifestyle			
	stochastic	S	M	B-10	B-5	A-10	A-5
Cost	10.00%	17.37%	17.36%	32.21%	29.67%	34.33%	30.64%

(d)	$RRA = 6, \quad T = 40$						
Strategy:	Optimal	Static		Deterministic lifestyle			
	stochastic	S	M	B-10	B-5	A-10	A-5
Cost	10.00%	11.52%	12.58%	12.86%	14.04%	13.67%	14.68%

(f)	$RRA = 12, \quad T = 40$						
Strategy:	Optimal	Static		Deterministic lifestyle			
	stochastic	S	M	B-10	B-5	A-10	A-5
Cost	10.00%	12.38%	13.17%	16.57%	19.72%	17.82%	20.77%

Summary

- Commercial strategies can be costly
- Optimal strategy has some drawbacks:
  - regular rebalancing $\Rightarrow$ difficult to implement??
  - short selling

$\Rightarrow$ we need to find a compromise

$\Rightarrow$ future work to find a robust dynamic strategy that takes account of plan member's risk aversion

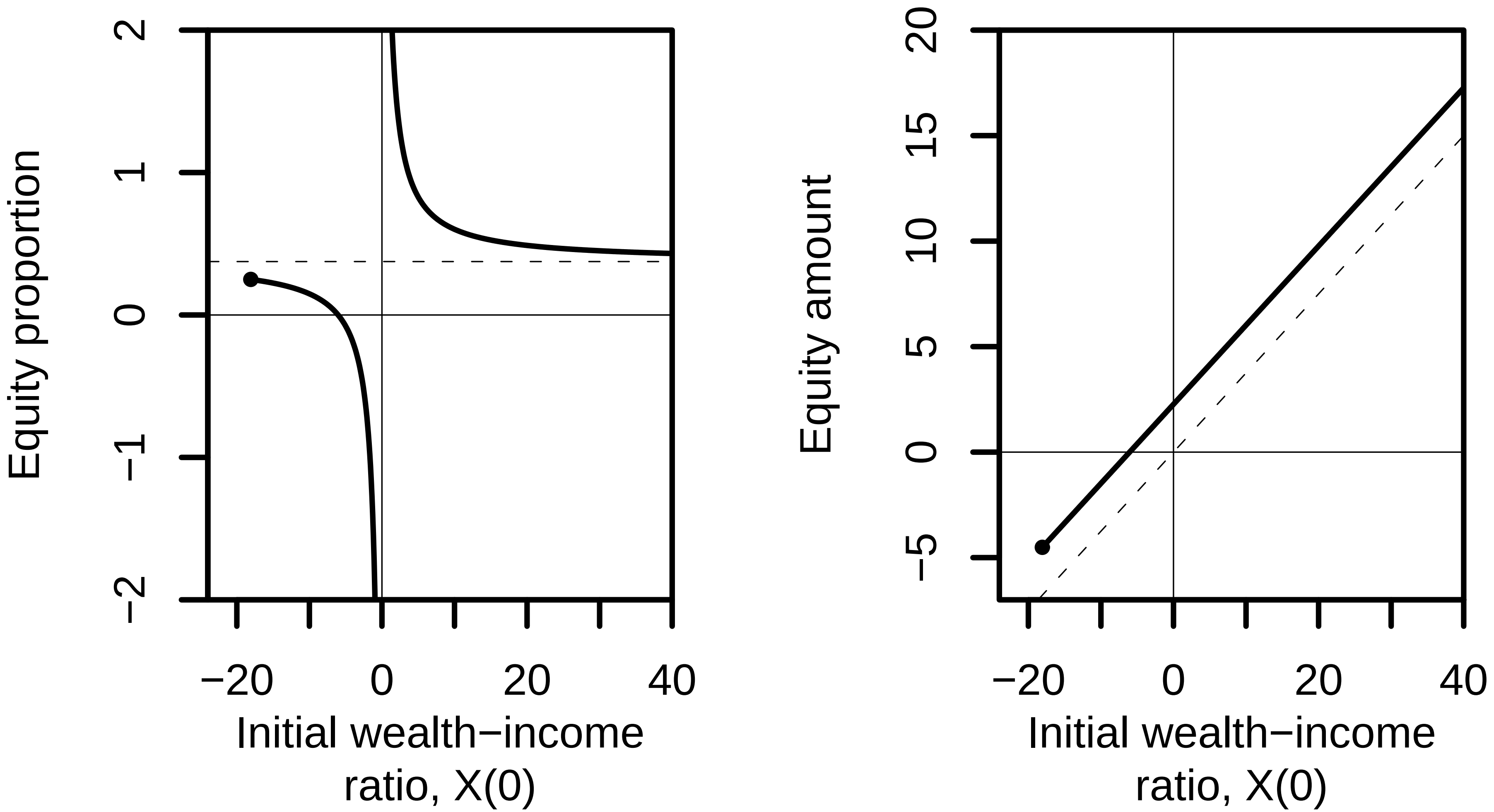
$r(t) =$ constant, r

- Case 1: $\pi = 0$, $\sigma_{Y0} = 0$.
- Case 2: $\pi = 0$, $\sigma_{Y0} \neq 0$.
- Case 3: $\pi > 0$, $\sigma_{Y0} = 0$.

* Case 4: $\pi > 0$, $\sigma_{Y0} \neq 0$.

Cases 1, 2, 3 have analytical solutions.

Case 4 $\Rightarrow$ numerical solution.



Case 3: $\pi > 0$, $\sigma_{Y0} = 0$

Case 4: $\pi > 0$, $\sigma_{s0} \neq 0$, 1 risky asset.

Solution by HJB equation.

- No analytical solution

  $\Rightarrow$ numerical solution required

- $V(t, x)$ has a singularity at $x = 0$

Result: Misery! (for a while).

Static optimisation problem:

$$\Rightarrow p^*(t,x) = p^*(t,x;V) = \frac{1}{\sigma_1}\left(\sigma_{Y1} - \frac{V_x}{xV_{xx}}(\xi_1 - \sigma_{Y1})\right)$$

$\Rightarrow p^*(t,x)$ only depends upon σ_{Y0} through $V(t,x)$.

Solve the non-linear PDE:

$$V_t + \mu_X^{p^*} V_x + \frac{1}{2} \sigma_X^{p^*\,2} V_{xx} \;=\; 0$$

subject to $V(T, x) \;=\; \frac{1}{\gamma} x^\gamma$

$$\mu_X^{p^*} \;=\; \pi \;+\; x \left(-\mu_s + \sigma_{Y0}^2 + \sigma_{Y1}^2 \right) \\ +\; x p^*(t, x; V) \sigma_1 (\xi_1 - \sigma_{Y1})$$

$$\sigma_X^{p^*\,2} \;=\; x^2 \left(\sigma_{Y0}^2 + \{ p^*(t, x; V) \sigma_1 - \sigma_{Y1} \}^2 \right).$$

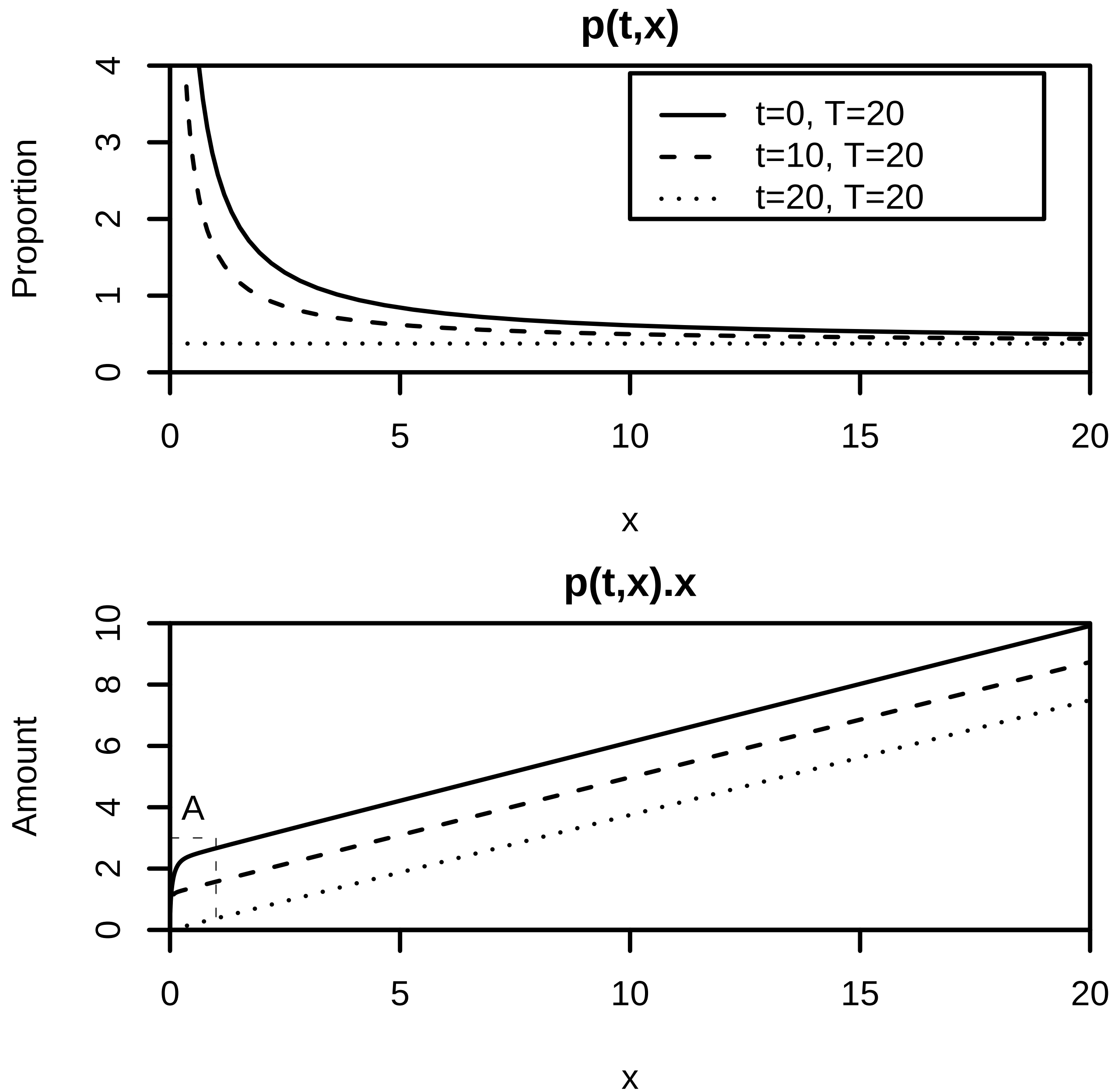
Numerical solution: Finite Difference Method

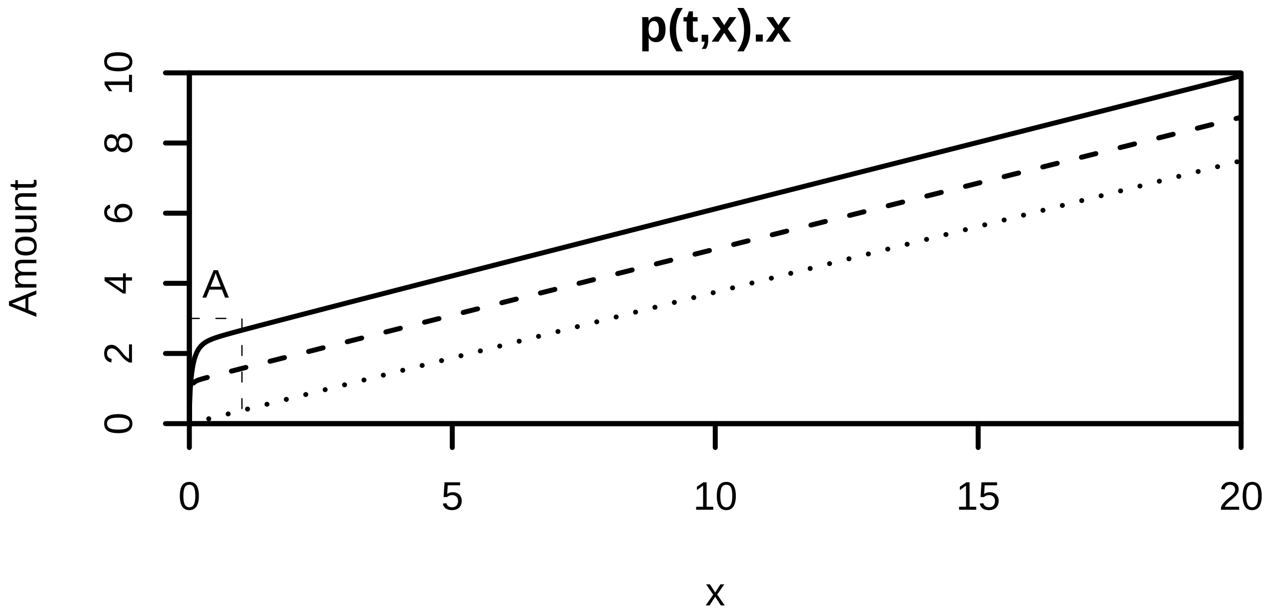
Problem (e.g. $\gamma < 0$) as $x \to 0$:

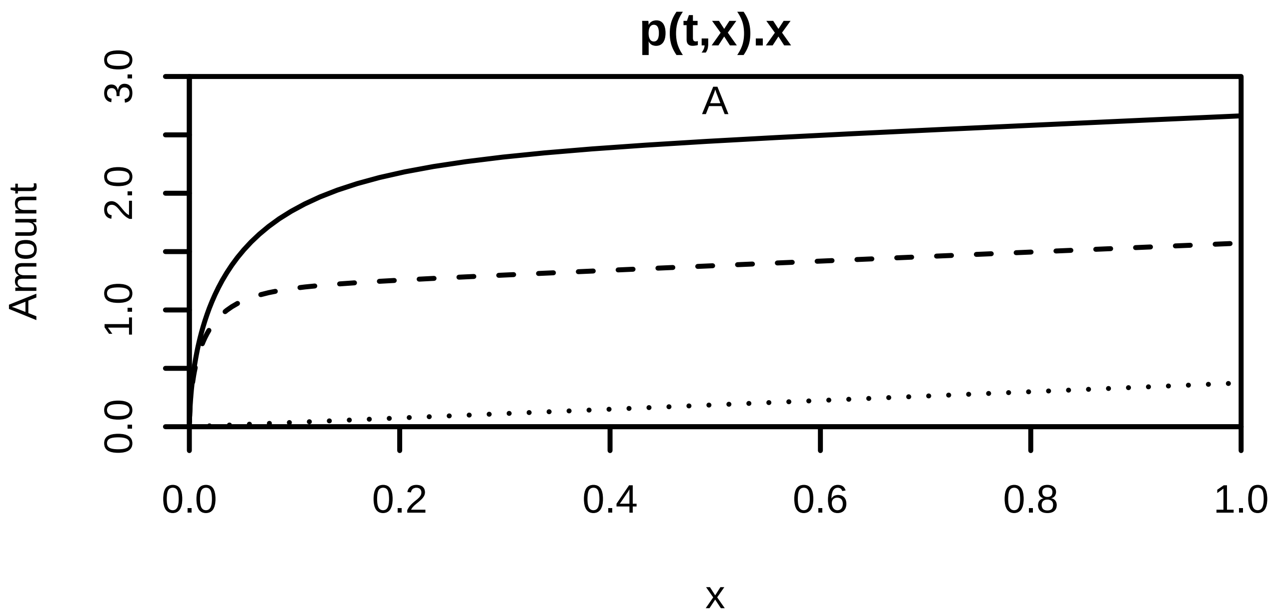
$$V(t,x) \;\to\; \begin{cases} -\infty, & \text{if } t = T \\ l(t), & -\infty < l(t) < 0,\; t < T \end{cases}$$

$$\frac{\partial V}{\partial t}(t,0) \;\to\; -\infty \quad \text{as } t \to T$$

$\Rightarrow$ numerical solution: unstable near $x = 0$??


p(t,x)
Proportion
0
1
2
3
4
t=0, T=20
t=10, T=20
t=20, T=20
0
5
10
15
20
x
p(t,x).x
Amount
0
2
4
6
8
10
A
0
5
10
15
20
x


p(t,x).x
Amount
10
8
6
4
2
0
A
0
5
10
15
20
x


p(t,x).x
Amount
3.0
2.0
1.0
0.0
A
0.0
0.2
0.4
0.6
0.8
1.0
x

Numerical results $\Rightarrow$ for $t < T$

$$p^*(t,x)\sqrt{x} \to \phi \text{ as } x \to 0$$

Value of ϕ is critical!

- $\phi = \infty \quad \Rightarrow \quad X(t)$ might hit 0

** $\phi = 0 \quad \Rightarrow \quad X(t)$ never hits 0

- $0 < \phi < \infty \quad \Rightarrow \quad X(t)$ might or might not hit zero

Numerical solutions suggest (**).

(... but see Duffie et al., 1997)

Case 4: upper bound

Introduce an extra asset, $R_2(t)$, to complete the market.

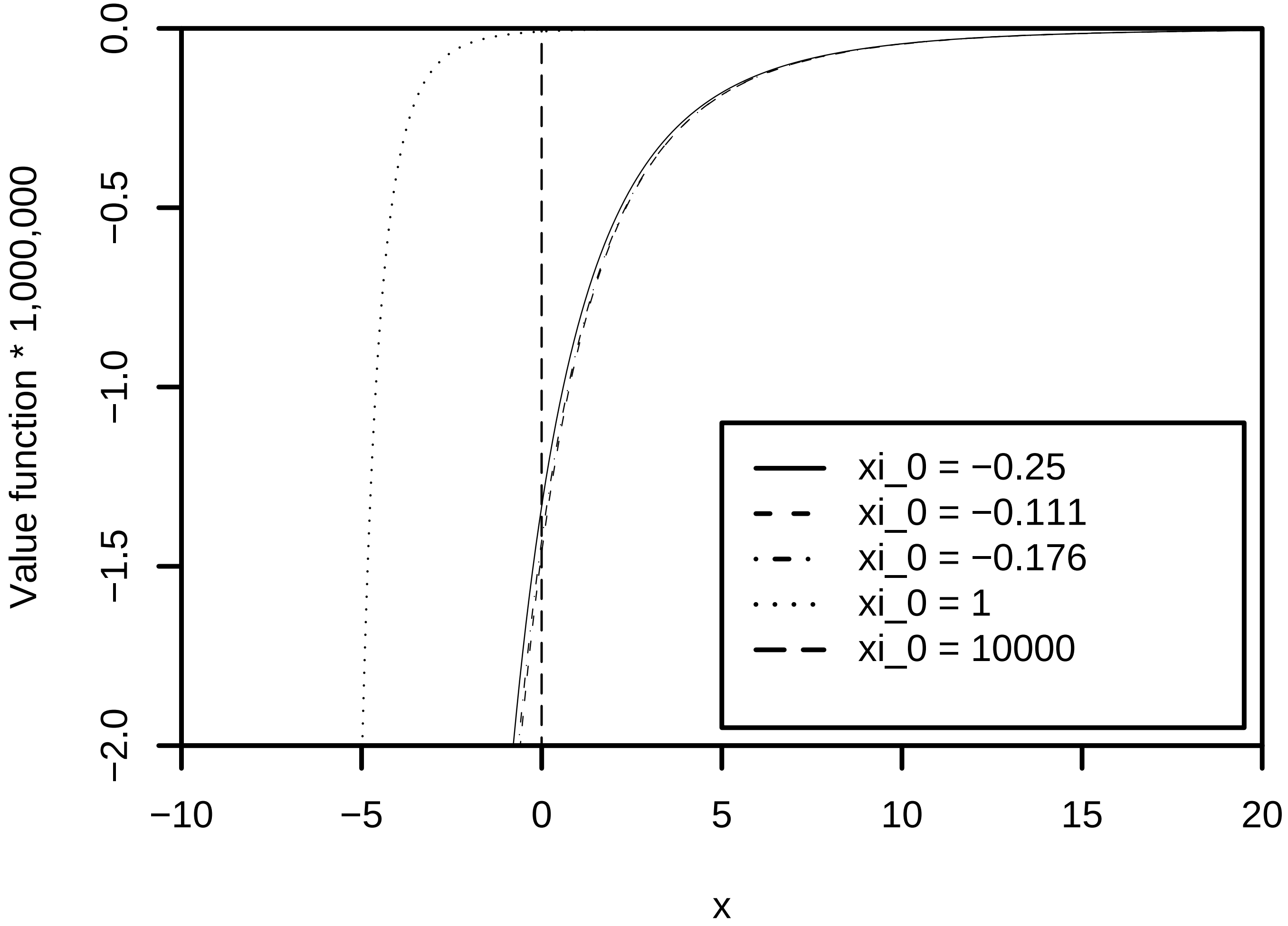
$$dR_2(t) \;=\; R_2(t)\Big[\,(r + \xi_0\sigma_{Y0})\,dt + \sigma_{Y0}dZ_0(t)\Big].$$

$\xi_0 =$ arbitrary market price of risk: to be specified

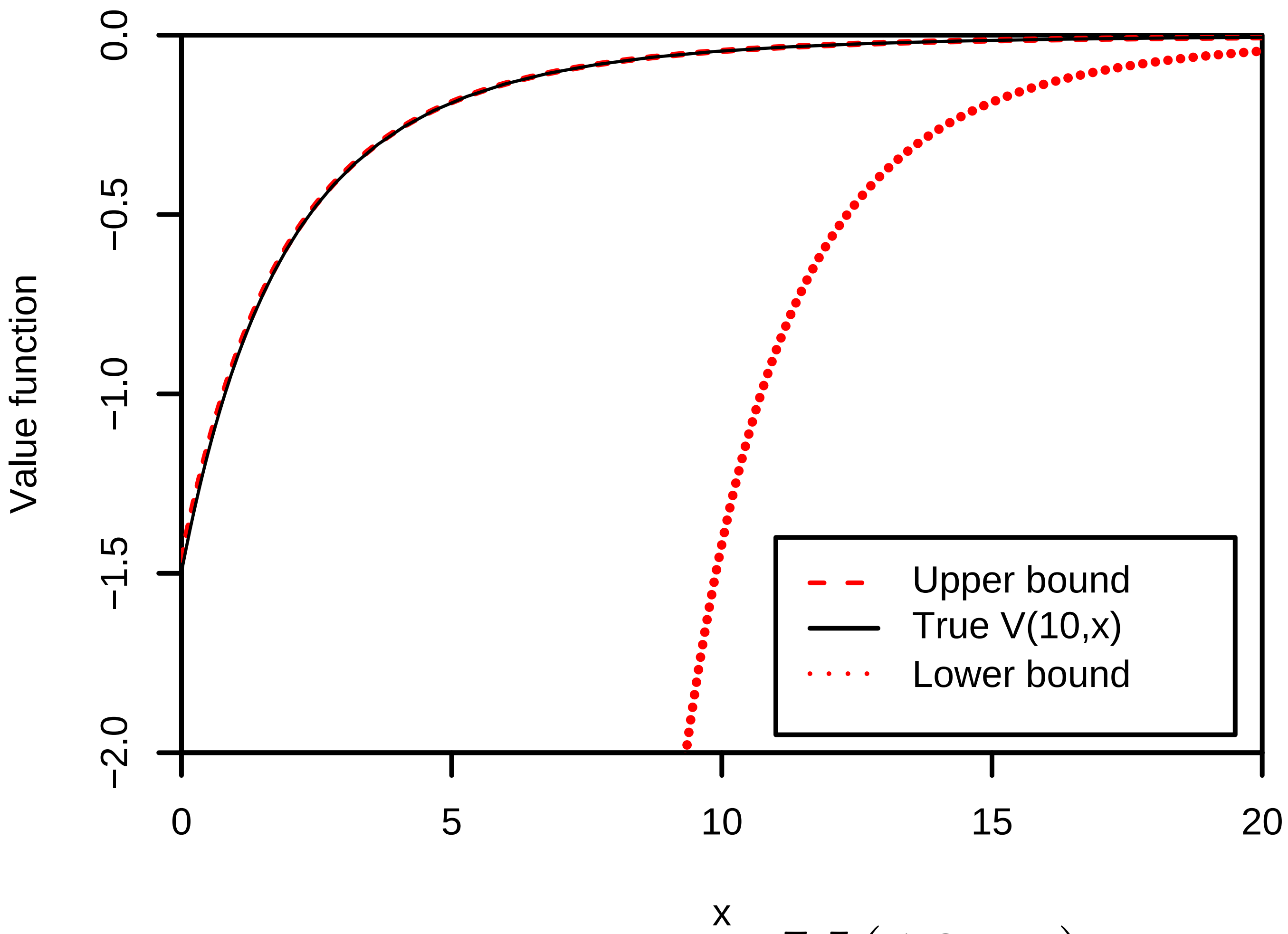
More choice $\Rightarrow$ increased $E[u(W(T), Y(T))]$

$\Rightarrow$ analytical upper bounds (like Case 3), $V^u(t, x; \xi_0)$.

$$\text{Then} \;\; V(t,x) \le V^u(t,x) = \inf_{\xi_0 \in R} V^u(t, x; \xi_0)$$



Construction of the upper bound for $V(t, x)$



The true optimal value function $V(10, x)$ when $\gamma = -5$ and $T - t = 10$ versus its upper and lower bounds.