Entropic Calibration Revisited

-from gamma to option pricing-

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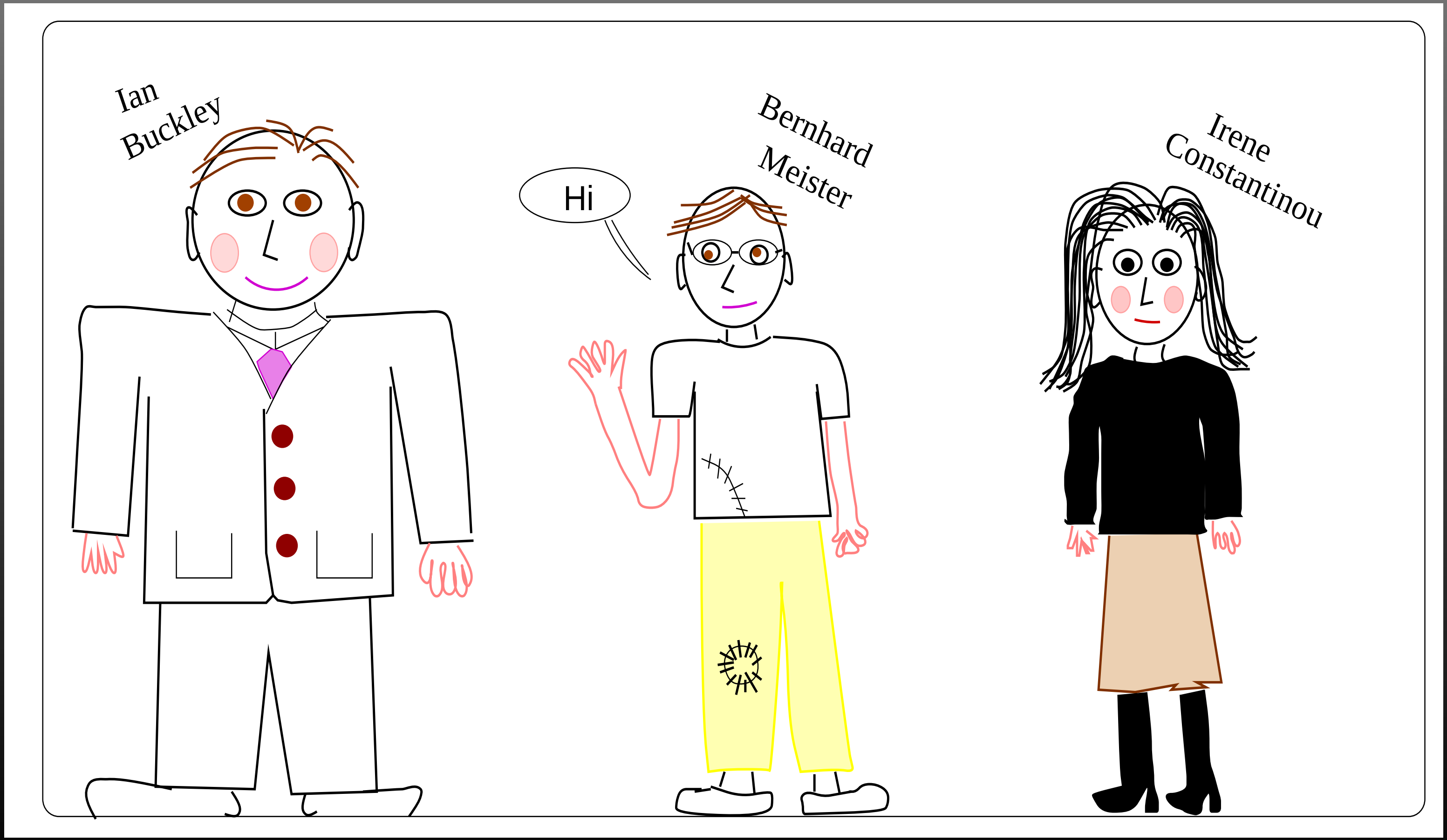
Work based on:

- D. C. Brody & B. K. Meister, “From gamma to option pricing” (Imperial College Preprint 2003)
- D. C. Brody, I. R. C. Buckley & I. C. Constantinou ”Option Price Calibration from Rényi Entropy” (Imperial College Preprint 2004)
- D. C. Brody, I. R. C. Buckley, & B. K. Meister, “Preposterior analysis for option pricing” *Quant. Fin.* **4**, 465-477 (2004)
- D. C. Brody, I. R. C. Buckley, I. C. Constantinou & B. K. Meister, “Entropic calibration revisited” *Phys. Lett.* A**335** (2005)

Downloadable from:

http://www.imperial.ac.uk/research/theory/people/brody/

Collaborators



Option price

We set the terminal asset price to be:

$$S_T = S\,\mathrm{e}^{Z_T + \int_0^T r_s \mathrm{d}s}$$

$Z_T \cdots$ some random variable; $r_t = r \cdots$ short rate.
The price of the option is given by

$$C(S) = K\mathrm{e}^{-rT}\mathbb{E}^*\left[\left((S/K)\mathrm{e}^{Z_T + rT} - 1\right)^+\right].$$

$\mathbb{E}^*[-] \cdots$ expectation in the risk-neutral probability measure.

Option gamma

Differentiating $C(S)$ once in S gives

$$C'(S) = \mathbb{E}^* \left[\mathrm{e}^{Z_T} \Theta \left((S/K) \mathrm{e}^{Z_T + rT} - 1 \right) \right],$$

where $\Theta(x)$ is the Heaviside step function.
Differentiating this once more, we obtain

$$C''(S) = K^{-1} \mathrm{e}^{rT} \mathbb{E}^* \left[\mathrm{e}^{2Z_T} \delta \left((S/K) \mathrm{e}^{Z_T + rT} - 1 \right) \right],$$

where $\delta(x)$ is the Dirac delta-function.

Option gamma

In terms of the risk-neutral density $\rho(z)$ for Z_T we have

$$C''(S) = K^{-1}\mathrm{e}^{rT}\int_{-\infty}^{\infty}\rho(z)\mathrm{e}^{2z}\delta\left((S/K)\mathrm{e}^{z+rT}-1\right)\mathrm{d}z.$$

The integrand survives for $z = \ln(K/S) - rT$ and gives us

$$\gamma(S) = S^{-1}\tfrac{K}{S}\mathrm{e}^{-rT}\rho\left(\ln(\tfrac{K}{S}\mathrm{e}^{-rT})\right).$$

From this we observe at once a simple scaling property

$$\gamma(\xi K, S) = \xi^{-1}\gamma(K, \xi^{-1}S).$$

Gamma as a probability

We have $\gamma(S) \geq 0$.

The integral of gamma gives:

$$\int_0^\infty \gamma(S)\mathrm{d}S = \int_{-\infty}^\infty \mathrm{e}^z \rho(z)\mathrm{d}z = 1.$$

$\Rightarrow \gamma(S)$ defines a probability density function.

The mean

The first moment of gamma is:

$$\mathbb{E}^{\gamma}[S] = K\mathrm{e}^{-rT}.$$

This follows from

$$\int_0^\infty S\gamma(S)\,\mathrm{d}S = \Big(SC'(S) - C(S)\Big)\Big|_0^\infty .$$

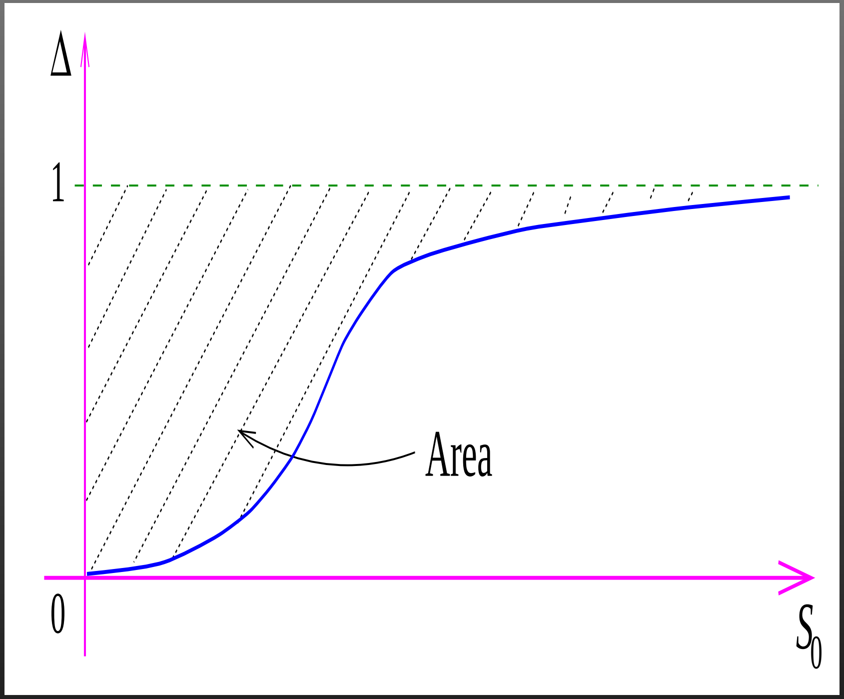
$\Rightarrow$ This is universal.

Mean and the area law

The mean is equivalent to the area above delta:

$$\mathbb{E}^{\gamma}[S] = \int_0^{\infty} (1 - \Delta(S))\, \mathrm{d}S$$

$$\Leftrightarrow \quad K\mathrm{e}^{-rT}$$



Higher moments

For the n^{th} moment of $\gamma(S)$ we have:

$$\mathbb{E}^{\gamma}\left[S^{n}\right] = \left(K\mathrm{e}^{-rT}\right)^{n}\int_{-\infty}^{\infty}\mathrm{e}^{-(n-1)z}\rho(z)\,\mathrm{d}z.$$

It follows that the price of a ‘power payoff’ derivative is:

$$\mathrm{e}^{-rT}\mathbb{E}^{*}\left[\left(\frac{S_0}{S_T}\right)^{n-1}\right] = K^{-n}\mathbb{E}^{\gamma}\left[S^{n}\right].$$

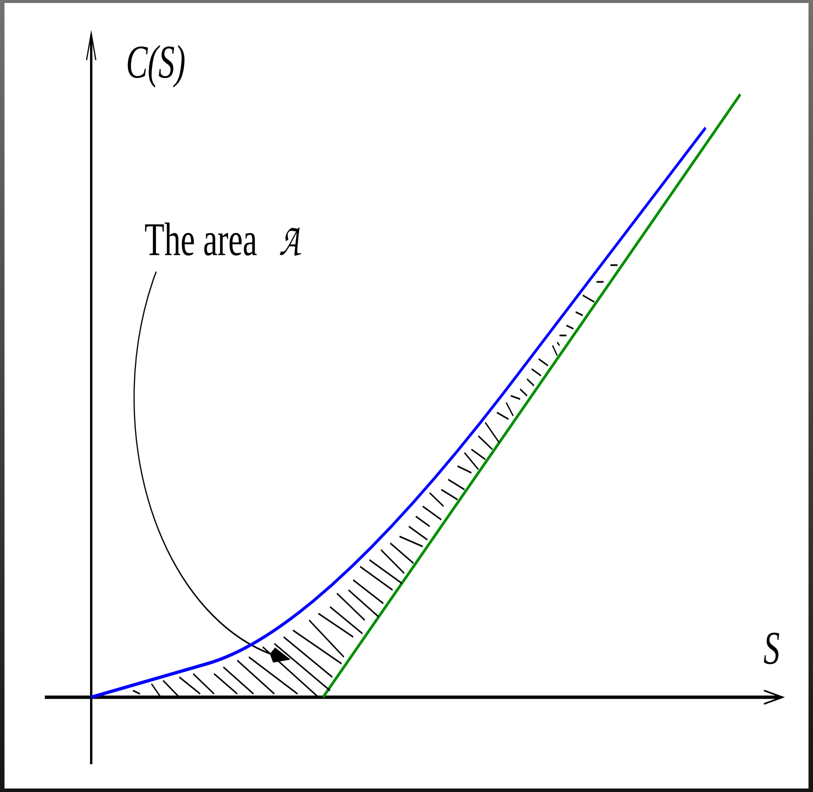
Variance and the area law

The variance, in particular, is given by

$$\mathbb{V}^\gamma[S] = K^2 e^{-2rT} \times \left(\int_{-\infty}^{\infty} e^{-z} \rho(z)\, dz - 1 \right).$$

The area under $C(S)$ is:

$$\mathfrak{A} = \frac{1}{2}\mathbb{V}^\gamma[S]$$



From gamma to option price

There is a one-to-one correspondence between the value function $C(S)$ for a vanilla option and the associated gamma:

$$C(S) = \int_0^S \int_0^u \gamma(x)\,\mathrm{d}x\mathrm{d}u$$

Note that the converse transformation

$$C(S) \to \gamma(S)$$

is also unique.

Put-call reversal and parity

We also have the identities:

$$\int_0^\infty (S-u)^+ \gamma(u)\,\mathrm{d}u = C(S)$$

$$\int_0^\infty (u-S)^+ \gamma(u)\,\mathrm{d}u = K\mathrm{e}^{-rT} - S + C(S)$$

from which we recover the put-call parity:

$$P(S) - C(S) = K\mathrm{e}^{-rT} - S$$

Option price representations

There is a generic representation:

$$C(S) = S\int_0^S \gamma(u)\,\mathrm{d}u - K\mathrm{e}^{-rT}\int_0^S (u/K\mathrm{e}^{-rT})\gamma(u)\,\mathrm{d}u$$

which is equivalent to writing

$$C(S) = S\int_\eta^\infty \mathrm{e}^z\rho(z)\,\mathrm{d}z - K\mathrm{e}^{-rT}\int_\eta^\infty \rho(z)\,\mathrm{d}z$$

where $\eta = \ln(K\mathrm{e}^{-rT}/S)$.

Entropy maximisation

Suppose that we are provided with the information that a set of functions $f_i(S)$ have definite expectation values F_i:

$$\int_0^\infty f_i(S)\gamma(S)\,\mathrm{d}S = F_i.$$

The most plausible choice for $\gamma(S)$ is obtained by maximising the Shannon entropy

$$H[\gamma] = -\int_0^\infty \gamma(S)\ln\gamma(S)\,\mathrm{d}S$$

subject to the given set of constraints.

Entropy maximisation

Another basic constraint is:

$$\int_0^\infty \gamma(S)\,\mathrm{d}S = 1$$

We thus consider the variational relation

$$\frac{\delta}{\delta\gamma}\left(-\gamma\ln\gamma - \gamma\sum_i \beta_i f_i - \beta_0\gamma\right) = 0.$$

Entropy maximisation

The solution reads

$$\gamma(S) = \exp\left(-\sum_i \beta_i f_i(S) - \beta_0 - 1\right),$$

where the Lagrange multipliers β_i are determined implicitly by the constraints.

Relevant constraints in calibration:

Normalisation:

$$\int_0^\infty \gamma(S)\mathrm{d}S = 1,$$

Mean condition:

$$\int_0^\infty S\gamma(S)\mathrm{d}S = K\mathrm{e}^{-rT},$$

Call price datum:

$$\int_0^\infty (S_0 - S)^+\gamma(S)\mathrm{d}S = C_0.$$

Maxent gamma

The result of the calibration gives:

$$\gamma(S) = \exp\left(-\beta S - \mu(S_0 - S)^+ - \phi(\beta, \mu)\right),$$

where

$$\phi(\beta, \mu) = \ln\left(\frac{\beta \mathrm{e}^{-\mu S_0} - \mu \mathrm{e}^{-\beta S_0}}{\beta(\beta - \mu)}\right)$$

is the normalisation.

The constraints:

The two constraints for the Lagrange multipliers are given by the relations:

$$\frac{\mathrm{e}^{-\mu S_0} + \mu S_0 \mathrm{e}^{-\beta S_0}}{\mu \mathrm{e}^{-\beta S_0} - \beta \mathrm{e}^{-\mu S_0}} + \frac{2\beta - \mu}{\beta(\beta - \mu)} = K\mathrm{e}^{-rT},$$

and

$$\frac{\beta S_0 \mathrm{e}^{-\mu S_0} + \mathrm{e}^{-\beta S_0}}{\beta \mathrm{e}^{-\mu S_0} - \mu \mathrm{e}^{-\beta S_0}} - \frac{1}{\beta - \mu} = C_0.$$

$\Rightarrow$ Can be solved numerically.

The call option price

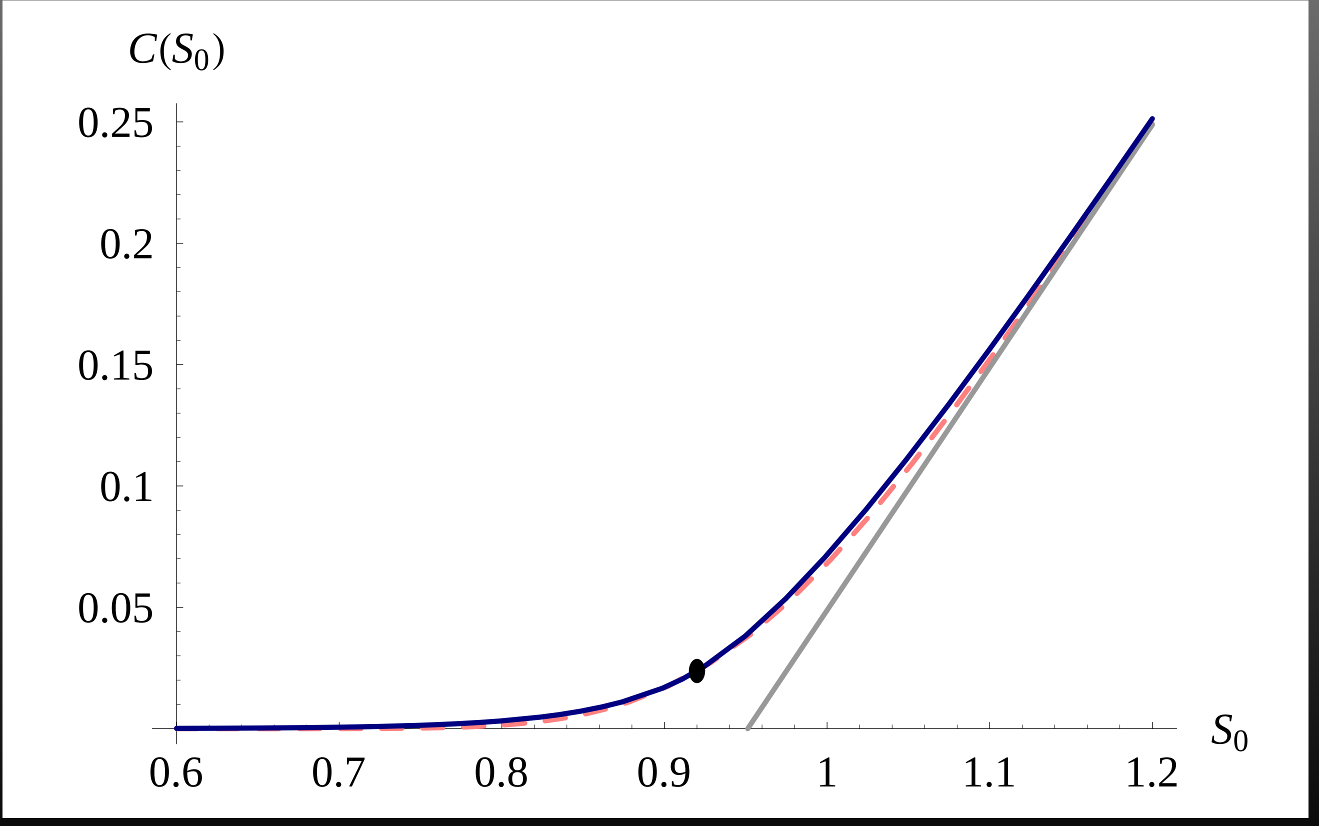
The resulting call option price can be written as

$$C(S) = \begin{cases} \frac{1}{\beta-\mu} \mathrm{e}^{-\mu S_0 - \phi} \left(S - \frac{1}{\beta-\mu} \left(1 - \mathrm{e}^{-(\beta-\mu)S} \right) \right) \\ S - K\mathrm{e}^{-rT} + \left(C_0 + K\mathrm{e}^{-rT} - S_0 \right) \mathrm{e}^{-\beta(S-S_0)} \end{cases}$$

for $S \leq S_0$ and $S > S_0$, respectively.

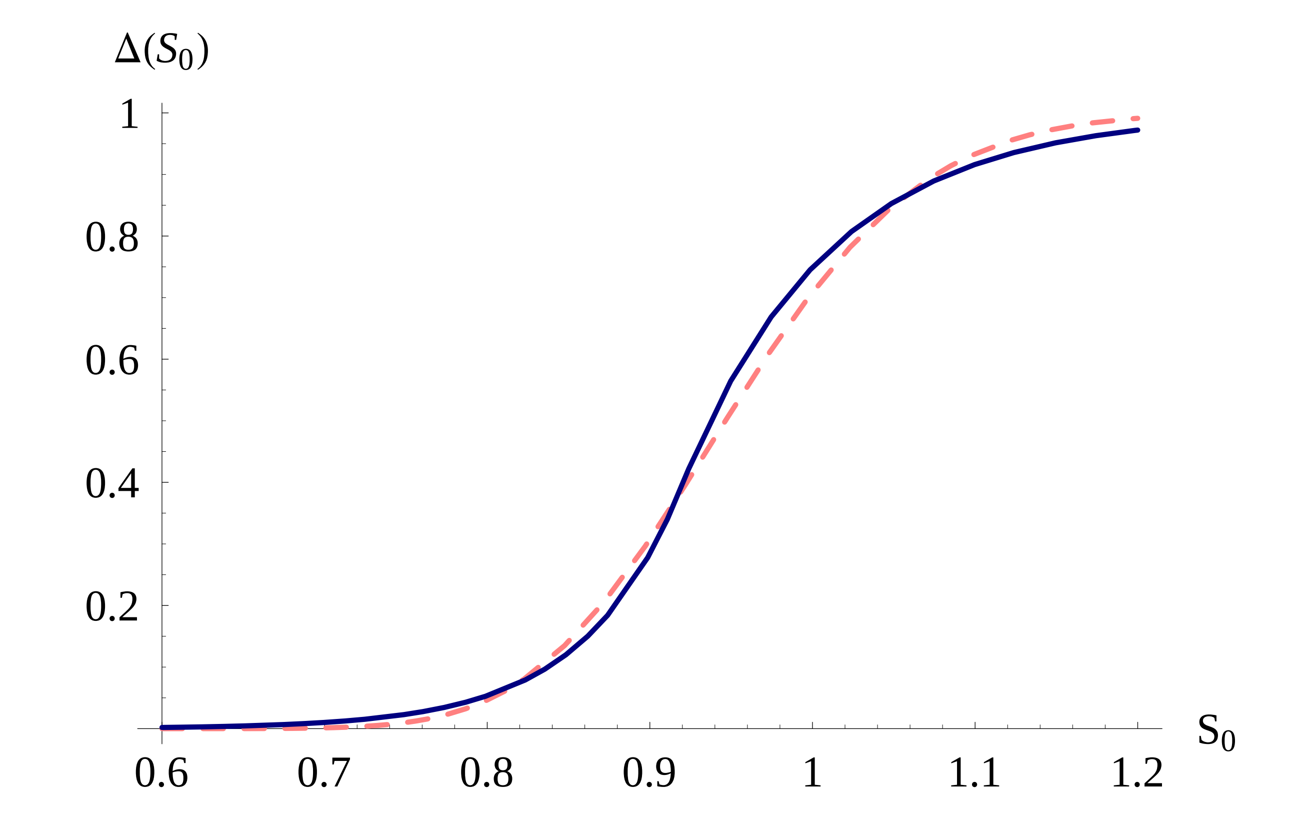
Option price

The resulting call price:



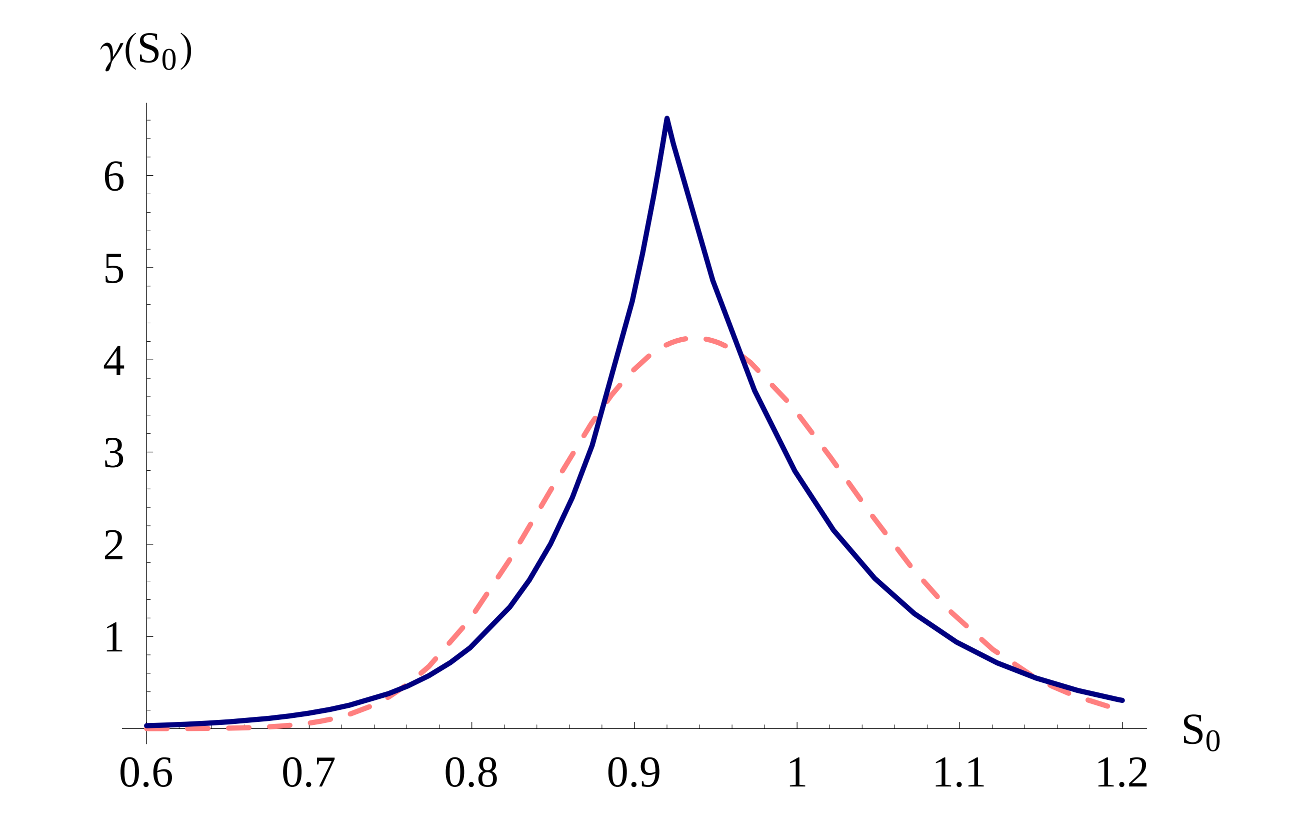
Option delta

The corresponding delta ($S_0 = 0.92K$):

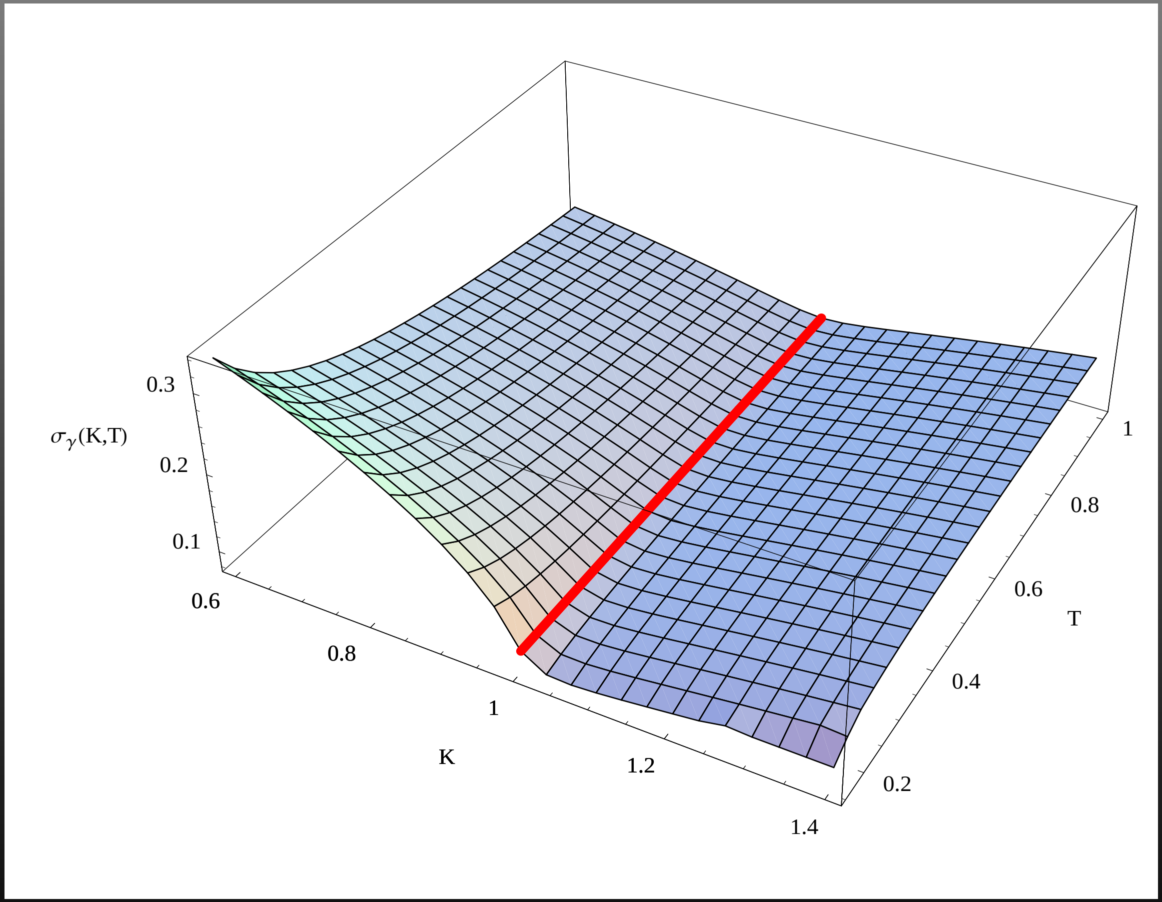


Option gamma

Gamma is calibrated against the B-S price at $S_0 = 0.92K$:

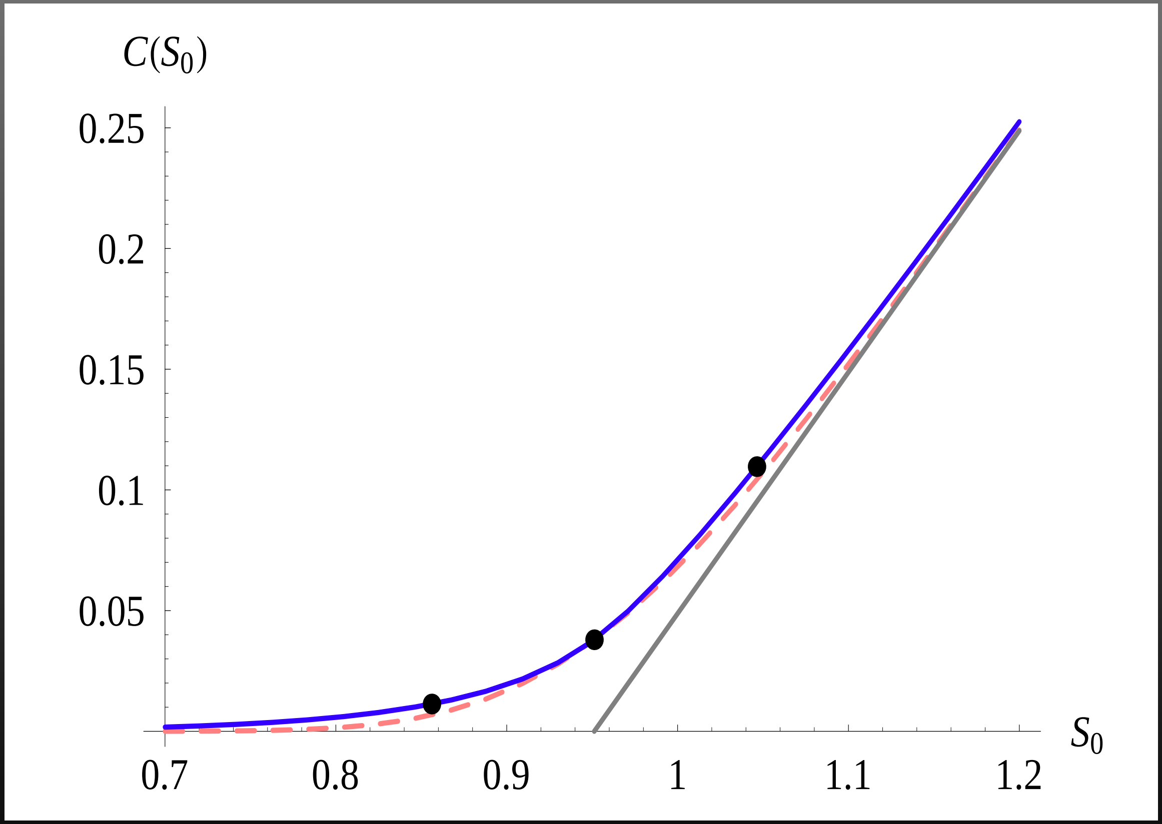


Implied (B-S) volatility smile:



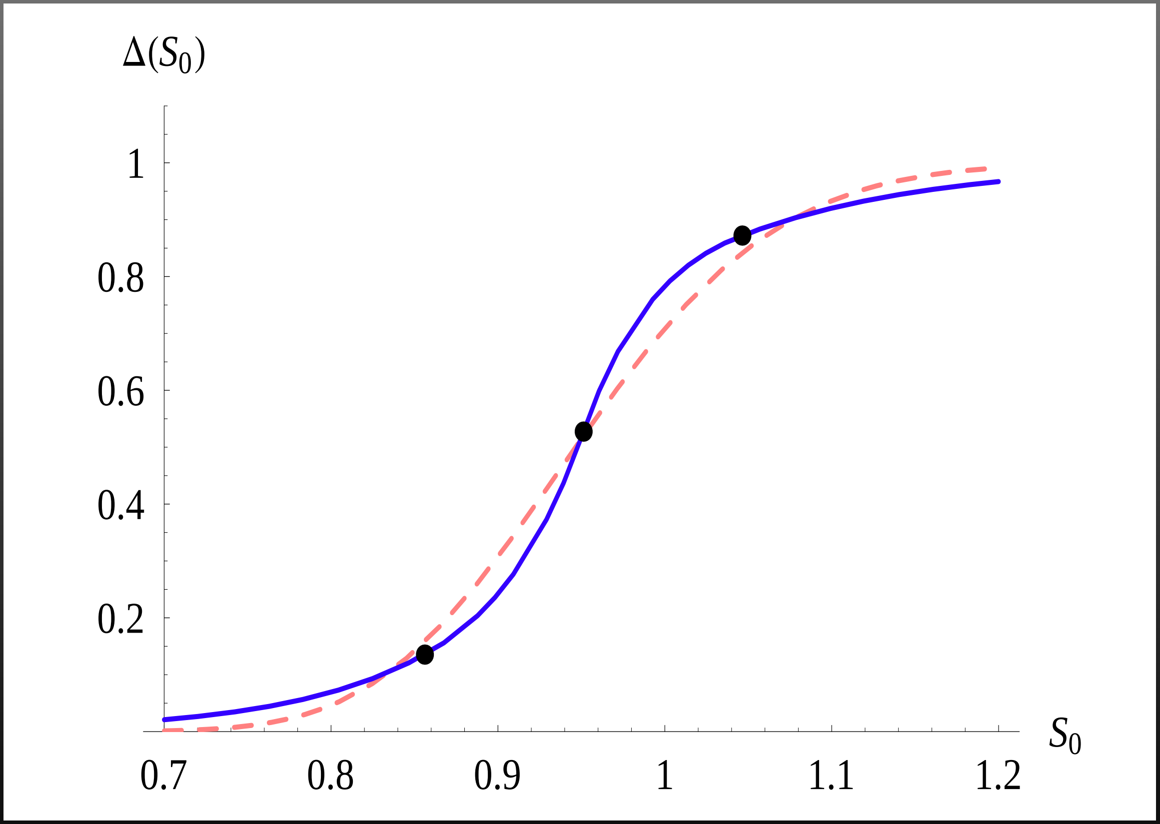
Three input prices:

$C(S)$



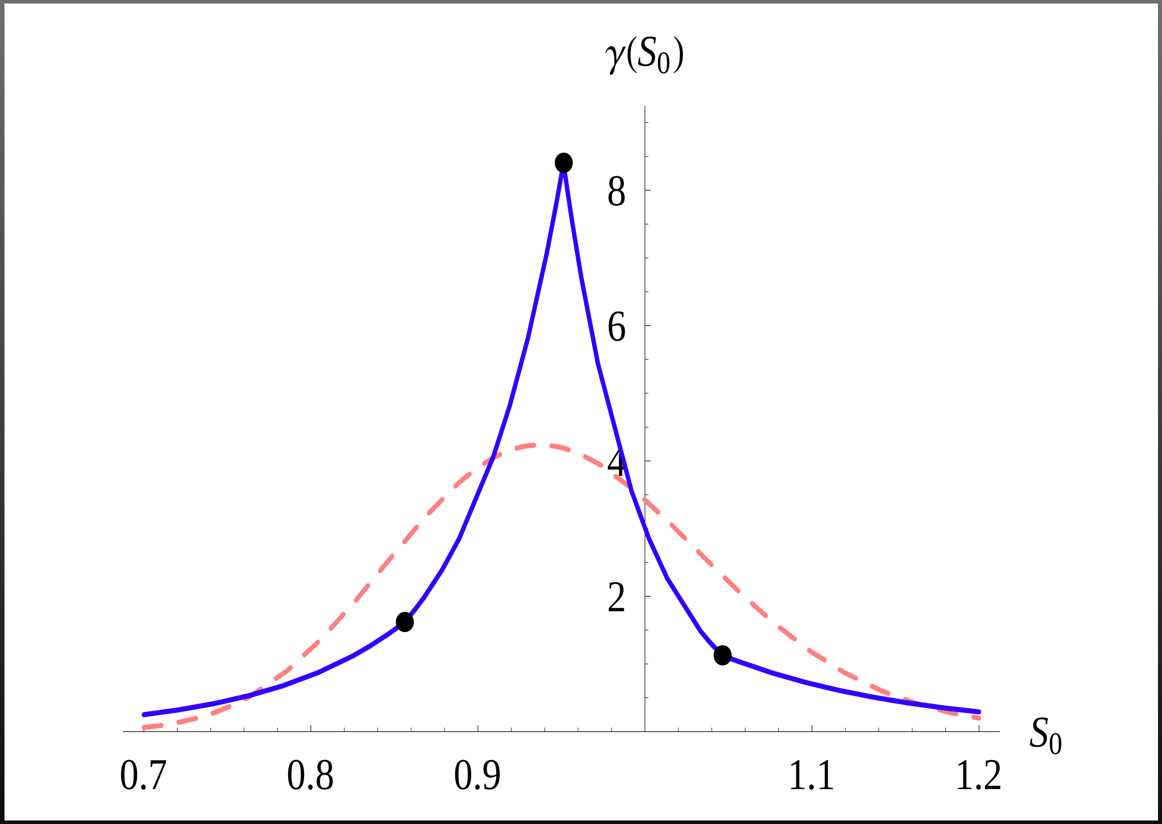
Three input prices:

$\Delta(S)$

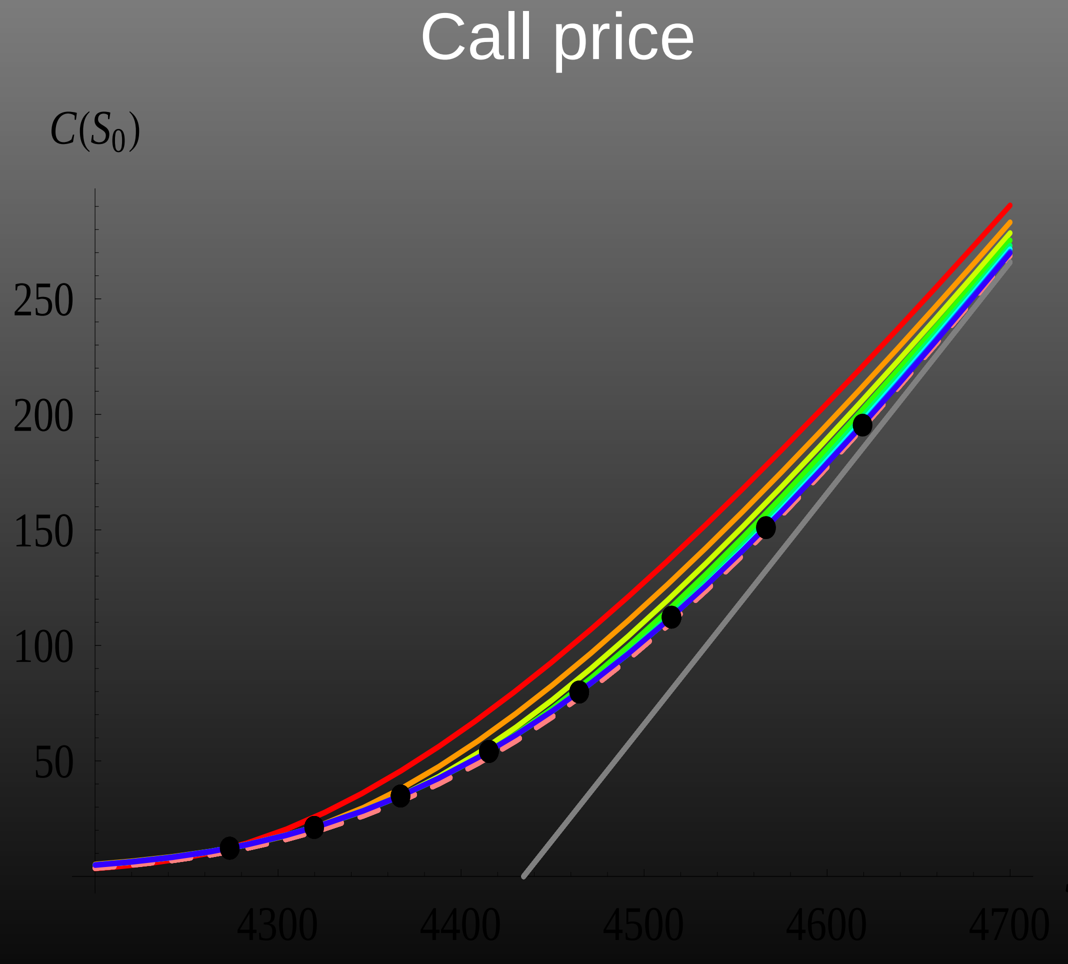


Three input prices:

$\gamma(S)$

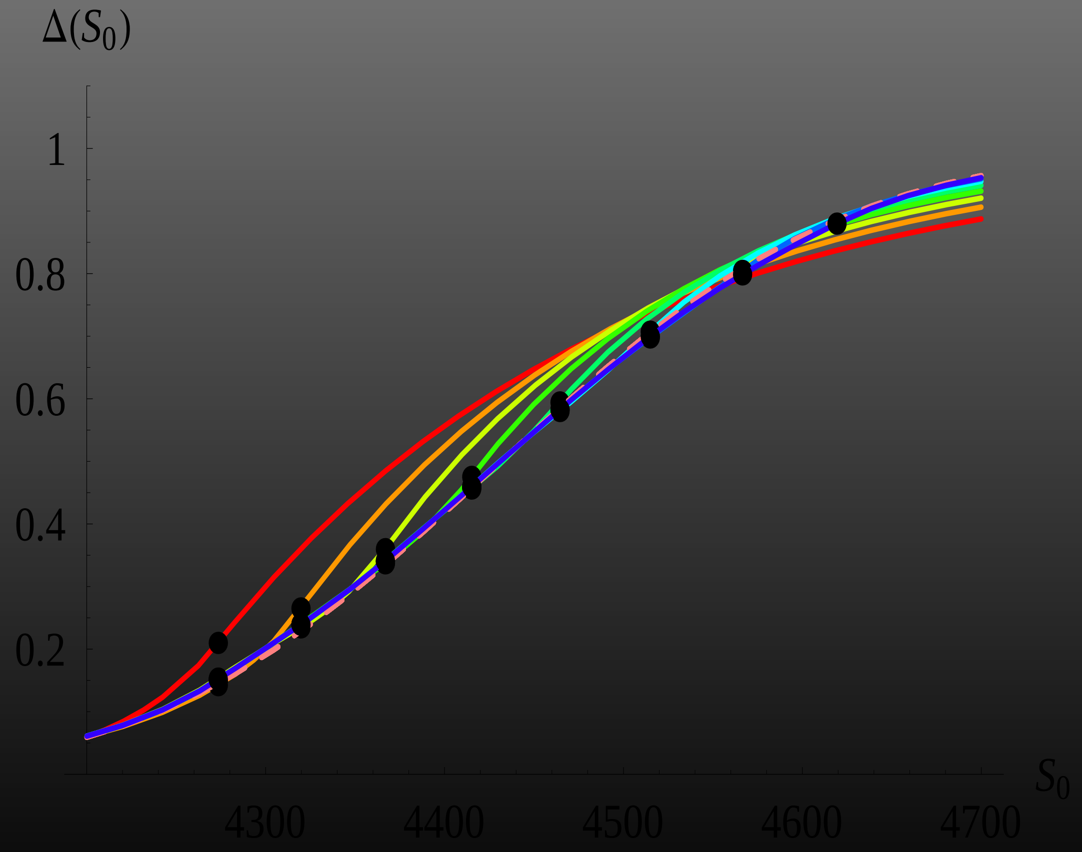


Calibrating options on FTSE 100 futures



Calibrating options on FTSE 100 futures

Option delta



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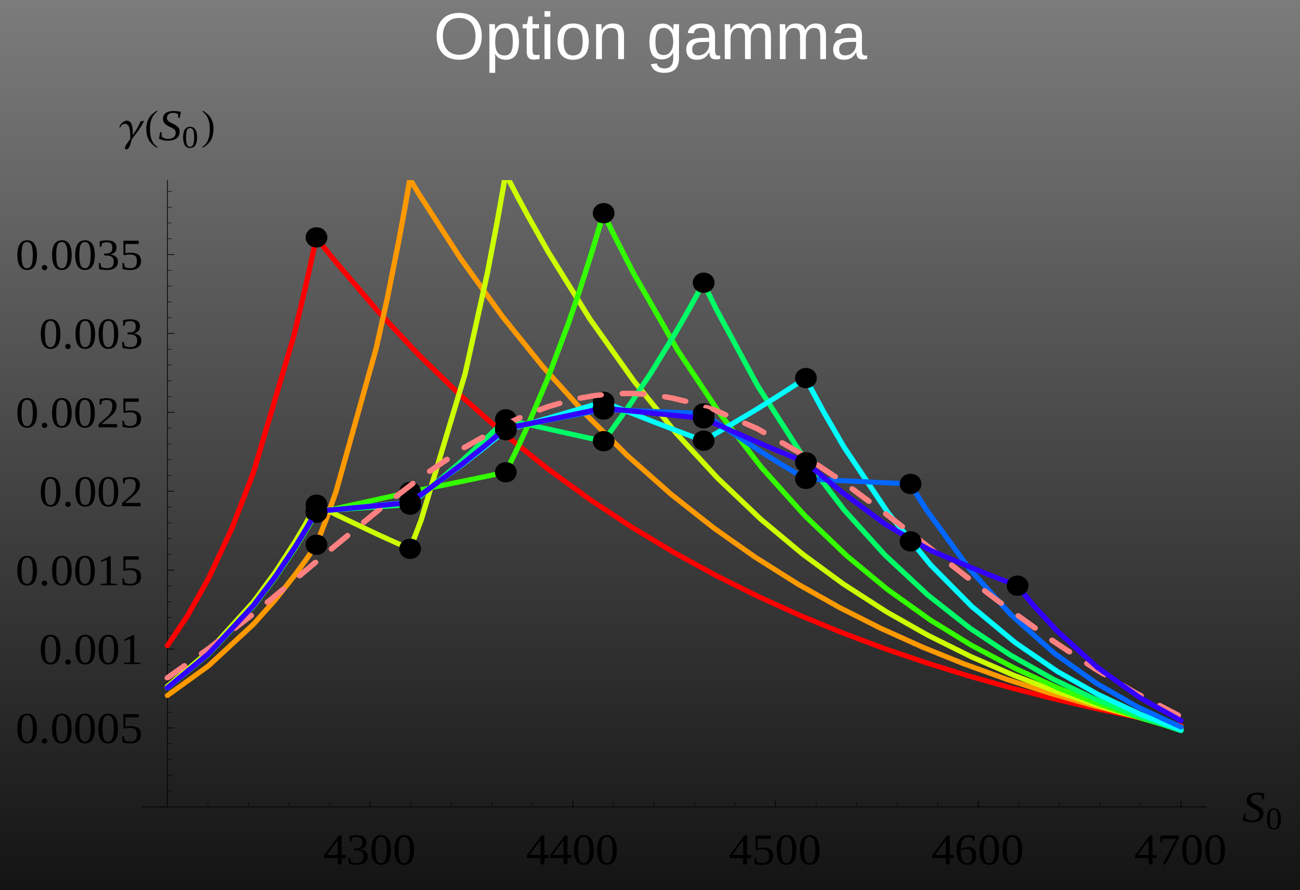
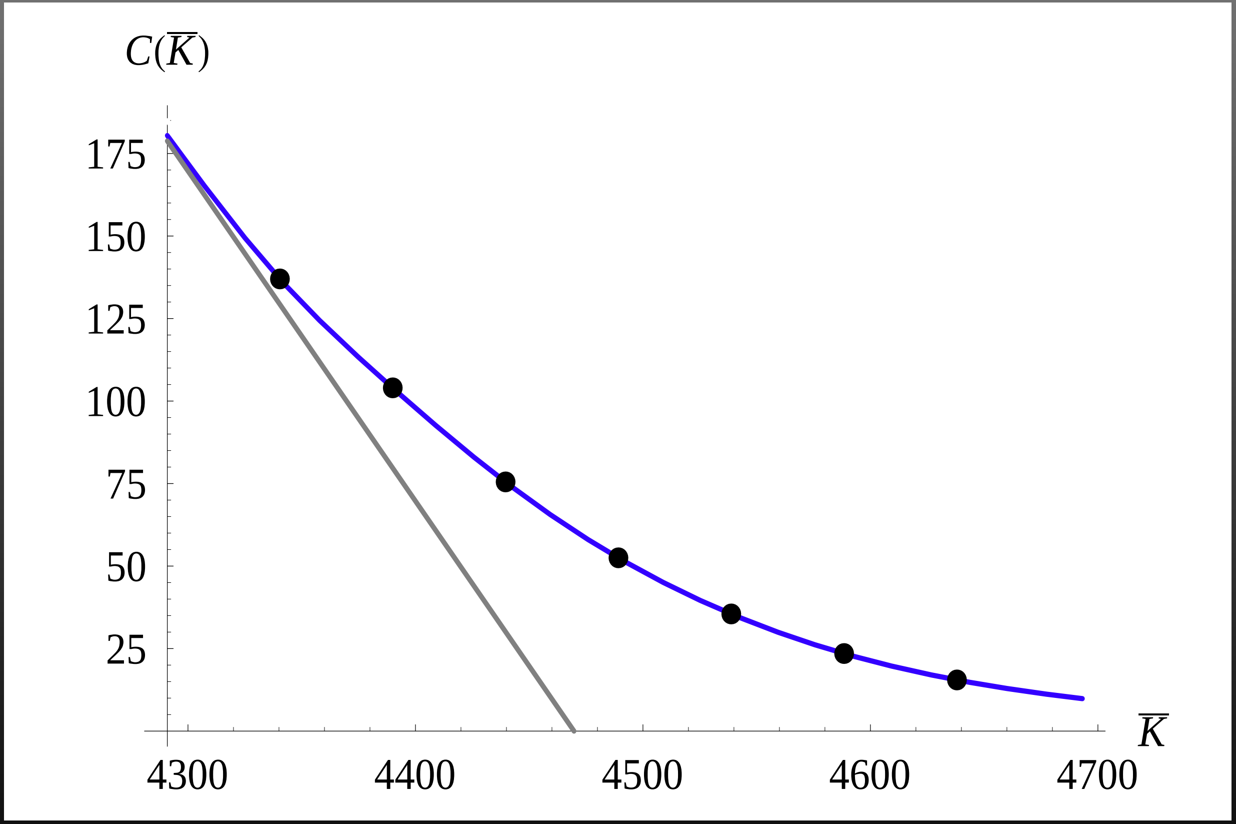


Calibrating options on FTSE 100 futures

Call price against the strike



Hunting arbitrage

- Data addition.
- Varying the lower end.
- Varying the middle.
- Varying the upper end.

From Shannon to Rényi: basic axioms

- symmetry,
- continuity,
- normalisation
- additivity, and
- mean value condition

$$H(p \cup q) = f^{-1}\left(\frac{w(p)f[H(p)] + w(q)f[H(q)]}{w(p) + w(q)}\right).$$

Rényi entropy:

Rényi showed that the only functions $f(x)$ satisfying these axioms are linear function $f_1(x) = ax + b$ and the exponential function $f_\alpha(x) = \mathrm{e}^{-(1-\alpha)x}$ with $\alpha > 0, \neq 1$.
The former leads to the familiar Shannon entropy, while the latter leads to the one-parameter family of entropies:

$$H_\alpha(p) = \frac{1}{1-\alpha} \ln \int_0^\infty p^\alpha(x)\, \mathrm{d}x.$$

Relevant constraints:

Normalisation:

$$\int_0^\infty p(x)\mathrm{d}x = 1,$$

Mean (martingale) condition:

$$\int_0^\infty x p(x)\mathrm{d}x = S_0 \mathrm{e}^{rT},$$

Call price datum:

$$\int_0^\infty (x - K_0)^+ p(x)\mathrm{d}x = C_0 \mathrm{e}^{rT}.$$

Maxent distribution:

Maximisation procedure yields:

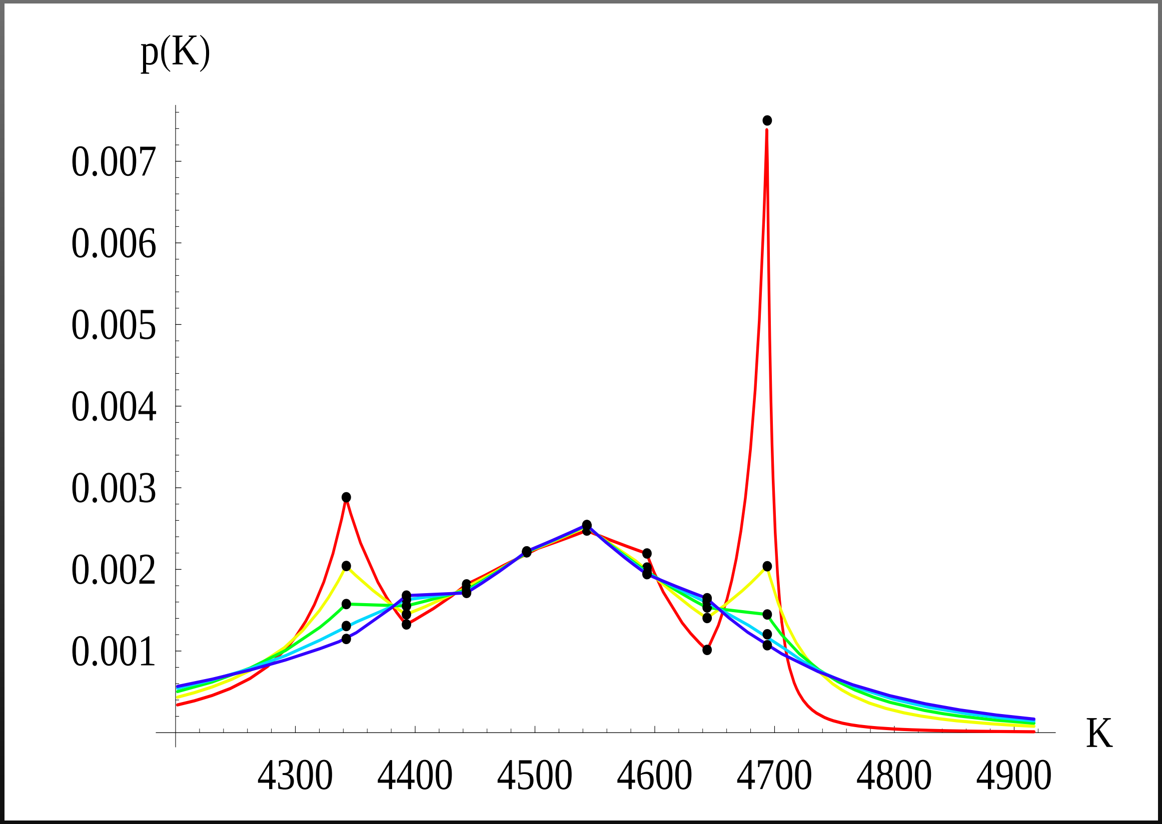
$$p(x) = \frac{1}{Z}\Big(\lambda + \beta_0 x + \beta_1 (x - K_0)^+\Big)^{\frac{1}{\alpha-1}},$$

where

$$Z = \frac{1-\alpha}{\alpha}\left(\frac{1}{\beta_0}\lambda^{\frac{\alpha}{\alpha-1}} - \frac{(\lambda + \beta_0 K_0)^{\frac{\alpha}{\alpha-1}}\beta_1}{\beta_0(\beta_0+\beta_1)}\right).$$

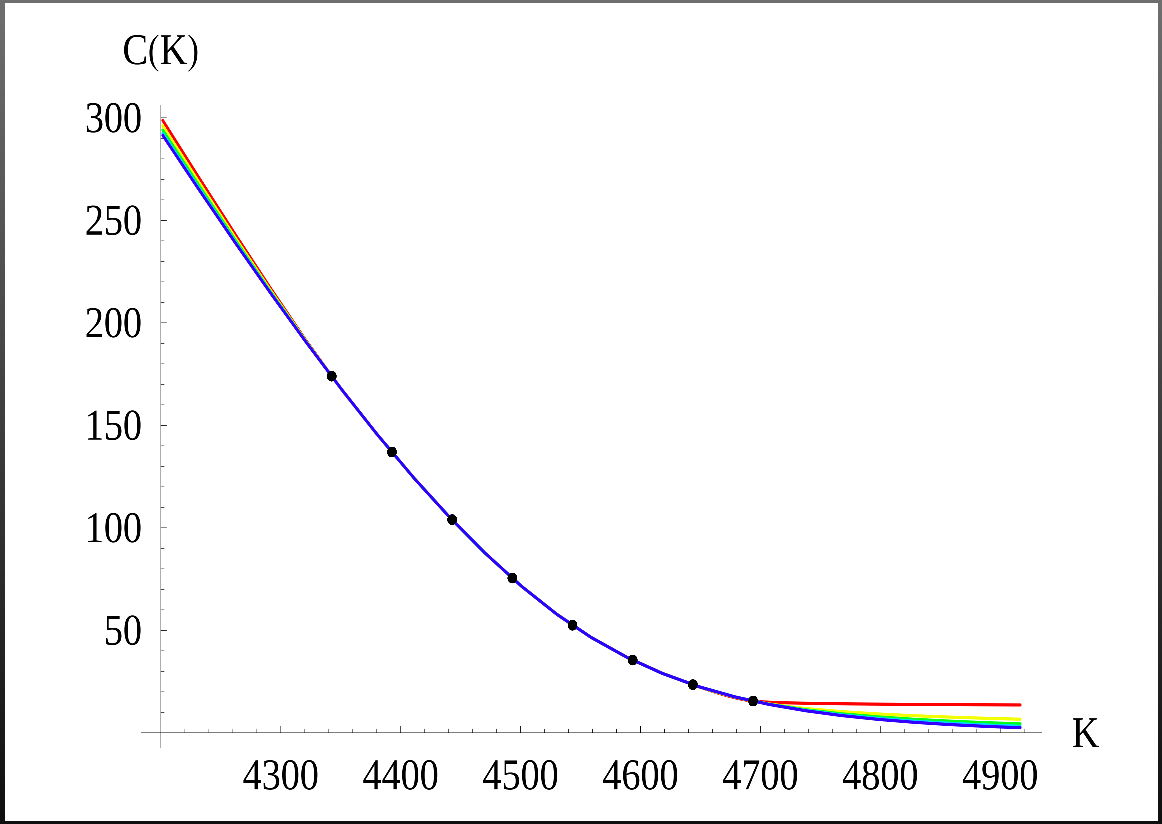
Maxent Rényi distributions:

$p(K)$



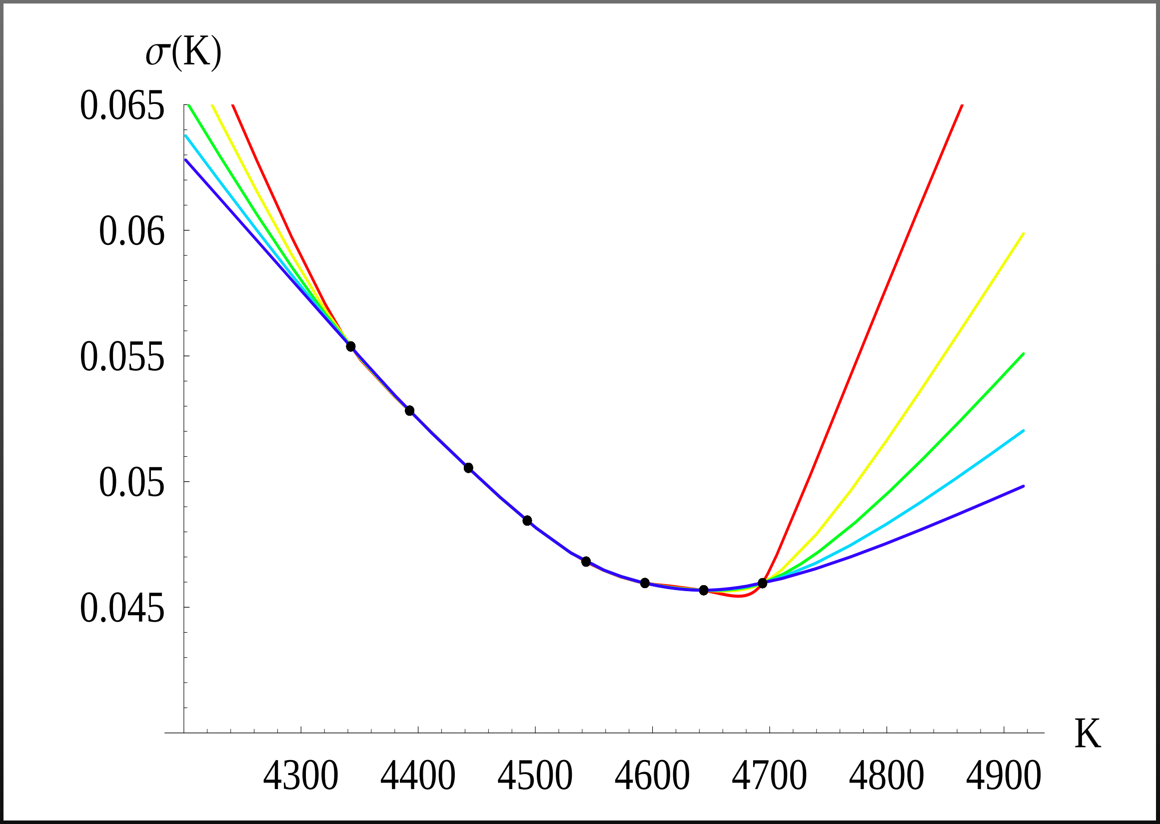
Call prices for different tales:

$C(K)$



Implied volatilities:

$\sigma(K)$



Concluding remark:

“Every science has its pipe dreams, pursuing them without ever catching them; but along the way useful insights can be seized”

Bernard LE BOUYER de FONTENELLE,
Dialogues of the Dead