Hedging under Non-Gaussian Processes: a fractional calculus approach

Álvaro Cartea

Birkbeck College, University of London

Judge Institute of Management, University of Cambridge

Overview

- Lévy processes
- Fractional calculus
- Fractional-Black-Scholes
- Hedging strategies

Lévy process

Definition 1. ***Lévy process.*** *Let $X(t)$ be a random variable dependent on time t. Then the stochastic process*

$$X(t),\ \text{for } 0 < t < \infty \text{ and } X(0) = 0,$$

is a Lévy process iff it has independent and stationary increments.

Lévy-Khintchine representation

Theorem 1. *Let X(t) be a Lévy process. Then* $\ln \mathbb{E}[e^{i\theta X(t)}] =$

$$ait\theta - \frac{1}{2}\sigma^2 t\theta^2 + t\int \left(e^{i\theta x} - 1 - i\theta x \mathbf{I}_{|x|<1}\right) W(dx),$$

where $a \in \mathbb{R}$, $\sigma \geq 0$, $\mathbf{I}$ *is the indicator function and the Lévy measure* W *must satisfy*

$$\int_{\mathbb{R}/0} \min\{1, x^2\} W(dx) < \infty. \tag{1}$$

Which Lévy process? Why?

- Brownian motion; Bachelier.
- α-Stable or Lévy Stable; Mandelbrot.
- Jump Diffusion; Merton.
- GIG and Generalised Hyperbolic Distribution; Barndorff-Nielsen.
- Variance Gamma; Madan et al.
- CGMY, Carr et al.
- KoBoL, Tempered Stable; Koponen.
- FMLS; Carr and Wu.

Which Lévy process? Why?

When specifying a particular Lévy process we are basically asking how do we want to specify the ‘behaviour’ of the jumps, in other words how is the Lévy density $w(x)$ (ie $W(dx) = w(x)dx$) chosen. For example

- Size and sign of jumps
- Frequency of jumps
- Existence of moments
- **Simplicity**

The Lévy-Stable process

Is a pure jump process with Lévy density

$$w_{LS}(x) = \begin{cases} Cq\,|x|^{-1-\alpha} & \text{for } x < 0, \\ Cpx^{-1-\alpha} & \text{for } x > 0. \end{cases}$$

The log of the characteristic function is $t\Psi(\theta) =$

$$\begin{cases} -t\kappa^{\alpha}|\theta|^{\alpha}\{1 - i\beta\,\mathrm{sign}(\theta)\tan(\alpha\pi/2)\} & \text{for } \alpha \neq 1, \\ -t\kappa|\theta|\left\{1 + \frac{2i\beta}{\pi}\,\mathrm{sign}(\theta)\ln|\theta|\right\} & \text{for } \alpha = 1, \end{cases}$$

here $C > 0$, $p \geq 0$, $q \geq 0$, with $p + q = 1$, $\beta = p - q$.

The CGMY process

A simple answer is then to consider a Lévy density of the form

$$w_{CGMY}(x) = \begin{cases} C|x|^{-1-Y}e^{-G|x|} & \text{for } x < 0, \\ Cx^{-1-Y}e^{-Mx} & \text{for } x > 0, \end{cases}$$

and the log of the characteristic function is given by

$$t\Psi_{CGMY}(\theta) = tC\Gamma(Y)\{(M-i\theta)^Y - M^Y + (G+i\theta)^Y - G^Y\}.$$

Here $C > 0$, $G \geq 0$, $M \geq 0$ and $Y < 2$.

The Damped-Lévy process

$$w_{DL}(x) = \begin{cases} Cq\,|x|^{-1-\alpha}\,e^{-\lambda|x|} & \text{for } x < 0, \\ Cpx^{-1-\alpha}e^{-\lambda x} & \text{for } x > 0, \end{cases}$$

and the natural logarithm of the characteristic equation is given by $t\Psi_{DL}(\theta) =$

$$t\kappa^{\alpha}\left\{p(\lambda - i\theta)^{\alpha} + q(\lambda + i\theta)^{\alpha} - \lambda^{\alpha} - i\theta\alpha\lambda^{\alpha-1}(q-p)\right\},$$

for $1 < \alpha \leq 2$ and $p + q = 1$.

Fractional Integrals

For an n-fold integral there is the well known formula

$$\int_a^x \int_a^x \cdots \int_a^x f(x)dx = \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1} f(t)dt.$$

Note that since $(n-1)! = \Gamma(n)$ the expression above may have a meaning for non-integer values of n.

The Riemann-Liouville Frac Integral

The fractional integral of order $\gamma > 0$ of a function $f(x)$ is given by

$$_aD_x^{-\gamma} f(x) = \frac{1}{\Gamma(\gamma)} \int_a^x (x - \xi)^{\gamma - 1} f(\xi) d\xi,$$

and

$$_xD_b^{-\gamma} f(x) = \frac{1}{\Gamma(\gamma)} \int_x^b (\xi - x)^{\gamma - 1} f(\xi) d\xi.$$

The Riemann-Liouville Frac Derivative

$$ {}_aD_x^\gamma f(x) = \frac{1}{\Gamma(n-\gamma)} \frac{d^n}{dx^n} \int_a^x (x-\xi)^{n-\gamma-1} f(\xi) d\xi, $$

and

$$ {}_xD_b^\gamma f(x) = \frac{(-1)^n}{\Gamma(n-\gamma)} \frac{d^n}{dx^n} \int_x^b (\xi-x)^{n-\gamma-1} f(\xi) d\xi. $$

The Fourier Transform View

Note that if we let $a = -\infty$ and $b = \infty$ we have

$$\mathcal{F}\{D_+^\gamma f(x)\} = (-i\xi)^\gamma \hat{f}(\xi)$$

and

$$\mathcal{F}\{D_-^\gamma f(x)\} = (i\xi)^\gamma \hat{f}(\xi),$$

where $D_+^\gamma = {}_{-\infty}D_x^\gamma$ and $D_-^\gamma = {}_xD_\infty^\gamma$.

The Lévy-Stable FBS

Under the physical measure the price process follows a geometric LS process

$$d(\ln S) = \mu dt + \sigma dL_{LS},$$

where $L \sim S_\alpha(dt^{1/\alpha}, \beta, 0)$ with $1 < \alpha < 2$, $-1 \leq \beta \leq 1$ and $\sigma > 0$.
And under the risk-neutral measure (McCulloch) it follows

$$d(\ln S) = (r - \beta\sigma^\alpha \sec(\alpha\pi/2))dt + dL^\star_{LS} + dL^\star_{DL}$$

where $dL^\star_{LS}$ and $dL^\star_{DL}$ are independent.

The Lévy-Stable FBS

$$
\begin{aligned}
rV(x,t) &= \frac{\partial V(x,t)}{\partial t} + (r - \beta\sigma^\alpha \sec(\alpha\pi/2))\frac{\partial V(x,t)}{\partial x} \\
&\quad -\kappa_2^\alpha \sec(\alpha\pi/2) D_+^\alpha V(x,t) \\
&\quad +\kappa_1^\alpha \sec(\alpha\pi/2)\left(V(x,t) - e^x D_-^\alpha e^{-x} V(x,t)\right)
\end{aligned}
$$

where

$$
\kappa_2^\alpha = \frac{1-\beta}{2}\sigma^\alpha \quad \text{and} \quad \kappa_1^\alpha = \frac{1+\beta}{2}\sigma^\alpha.
$$

Classical BS and the fractional FMLS

Case $\alpha = 2$, Black-Scholes

$$rV(x,t) = \frac{\partial V(x,t)}{\partial t} + (r - \sigma^2)\frac{\partial V(x,t)}{\partial x} + \sigma^2 \frac{\partial^2 V(x,t)}{\partial x^2}.$$

Case $\alpha > 1$ and $\beta = -1$, FMLS

$$\begin{aligned} rV(x,t) &= \frac{\partial V(x,t)}{\partial t} + (r + \sigma^\alpha \sec(\alpha\pi/2))\frac{\partial V(x,t)}{\partial x} \\ &\quad - \sigma^\alpha \sec(\alpha\pi/2) D_+^\alpha V(x,t). \end{aligned}$$

CGMY FBS equation

Let the risk-neutral log-stock price dynamics follow a CGMY process

$$d(\ln S) = (r - w)dt + dL_{CGMY}. \tag{2}$$

The value of a European-style option satisfies

$$\begin{aligned} rV &= \frac{\partial V(x,t)}{\partial t} + (r-w)\frac{\partial V(x,t)}{\partial x} \\ &+\sigma(M^Y + G^Y)V(x,t) + \sigma e^{Mx} D_-^Y\left(e^{-Mx}V(x,t)\right) \\ &+\sigma e^{-Gx} D_+^Y\left(e^{Gx}V(x,t)\right), \end{aligned}$$

where $\sigma = C\Gamma(-Y)$.

Proof

$$V(x,t) = e^{-r(T-t)}\mathbb{E}_t[\Pi(x_T,T)],$$

$$V(x,t) = \frac{e^{-r(T-t)}}{2\pi}\mathbb{E}_t\left[\int_{-\infty+i\nu}^{\infty+i\nu} e^{-ix_T\xi}\hat{\Pi}(\xi)d\xi\right],$$

$$\hat{V}(\xi,t) \;=\; e^{-r(T-t)}e^{-i\xi\mu(T-t)}e^{(T-t)\Psi(-\xi)}\hat{\Pi}(\xi),$$

$$\frac{\partial\hat{V}(\xi,t)}{\partial t} = (r + i\xi\mu - \Psi(-\xi))\hat{V}(\xi,t),$$

with $\hat{V}(\xi,T) = \hat{\Pi}(\xi,T)$.

Dynamic Hedging

Delta hedging, Delta-Gamma hedging, etc.
The Taylor Expansion View

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}dS^2 + \cdots .$$

Portfolio $P(S,t) = V_1(S,t) - \Delta S - bV_2(S,t)$

$$\Delta = \frac{\partial V_1(S,t)}{\partial S} - \frac{\partial V_2(S,t)}{\partial S}b(S,t),$$

$$b(S,t) = \frac{\partial^2 V_1(S,t)/\partial S^2}{\partial^2 V_2(S,t)/\partial S^2}.$$

Fractional Taylor's, Samko et al 1993

Dzherbashyan and Nersesyan

$$f(x) = \sum_{k=0}^{m-1} \frac{D^{(\gamma_k)} f(0)}{\Gamma(1+\gamma_k)} x^{\gamma_k} + \frac{\int_0^x (x-t)^{\gamma_m - 1} D^{(\gamma_m)} f(t) dt}{\Gamma(1+\gamma_m)}$$

for functions having all continuous derivatives.

$$\begin{aligned} dV(S,t) &= \frac{\partial V(S,t)}{\partial t} dt + \frac{\partial V(S,t)}{\partial S} dS \\ &\quad + \frac{1}{\Gamma(2-\gamma)} \, {}_0D_S^{\gamma} V(S,t)(dS)^{\gamma} + \cdots . \end{aligned}$$

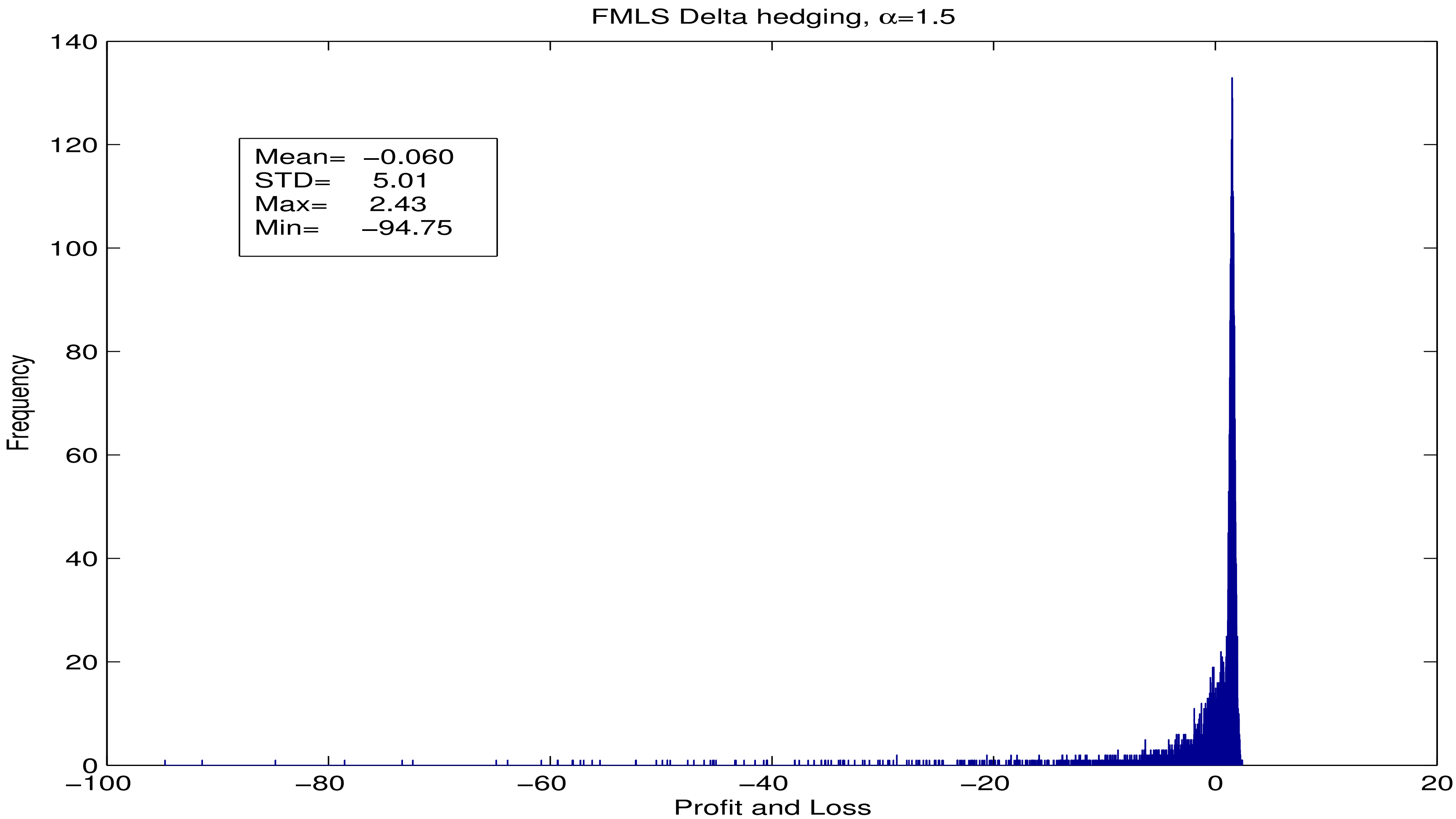
Therefore it seems ‘natural’ to delta and fractional-gamma hedge the portfolio $P(S,t) = V_1(S,t) - a(S,t)S - b(S,t)V_2(S,t)$, hence

$$a(S,t) = \frac{\partial V_1(S,t)}{\partial S} - \frac{\partial V_2(S,t)}{\partial S} b(S,t)$$

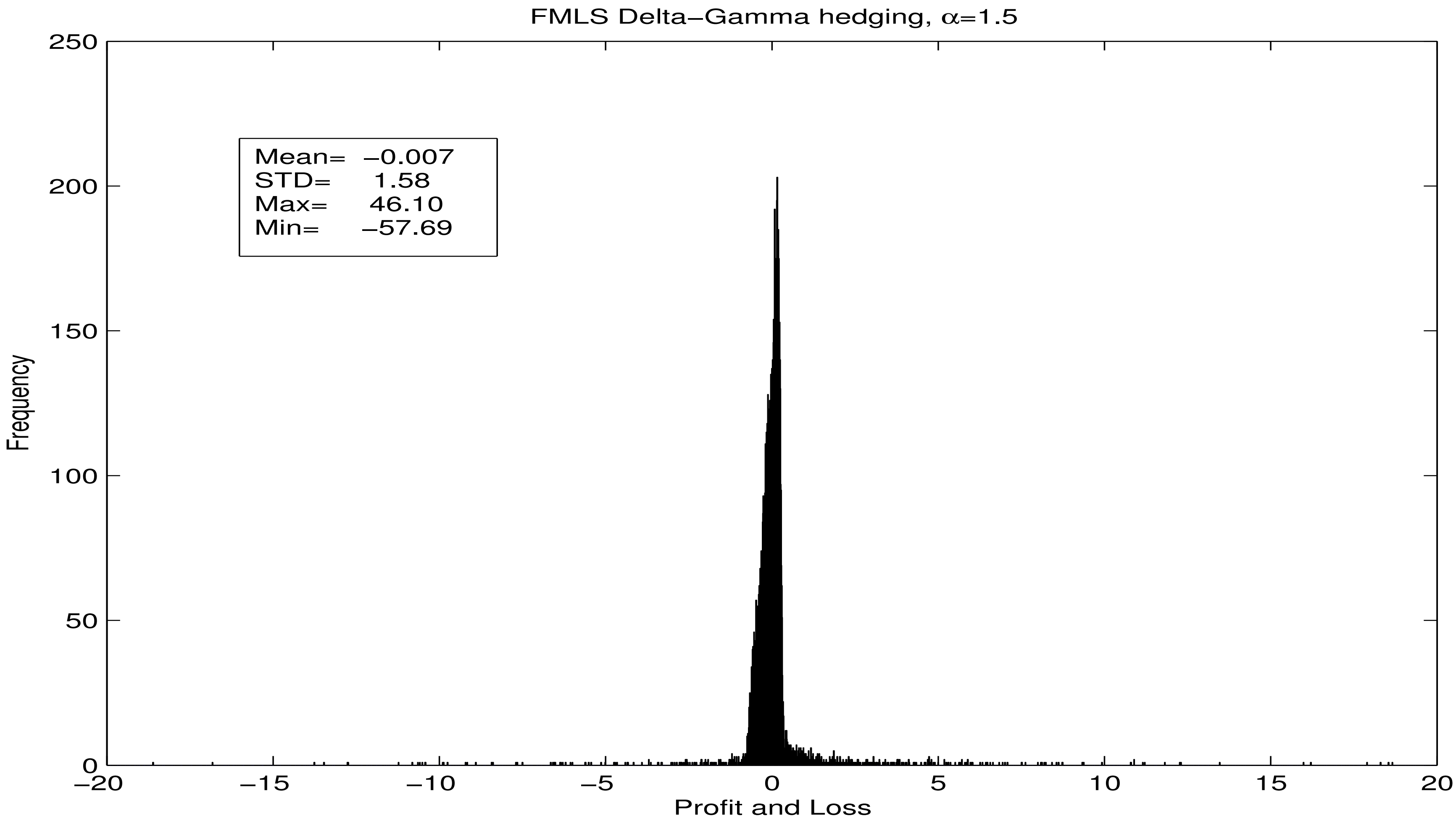
and

$$b(S,t) = \frac{{}_0D_S^\gamma V_1(S,t) - \partial V_1(S,t)/\partial S \; {}_0D_S^\gamma S_t}{{}_0D_S^\gamma V_2(S,t) - \partial V_2(S,t)/\partial S \; {}_0D_S^\gamma S_t}.$$

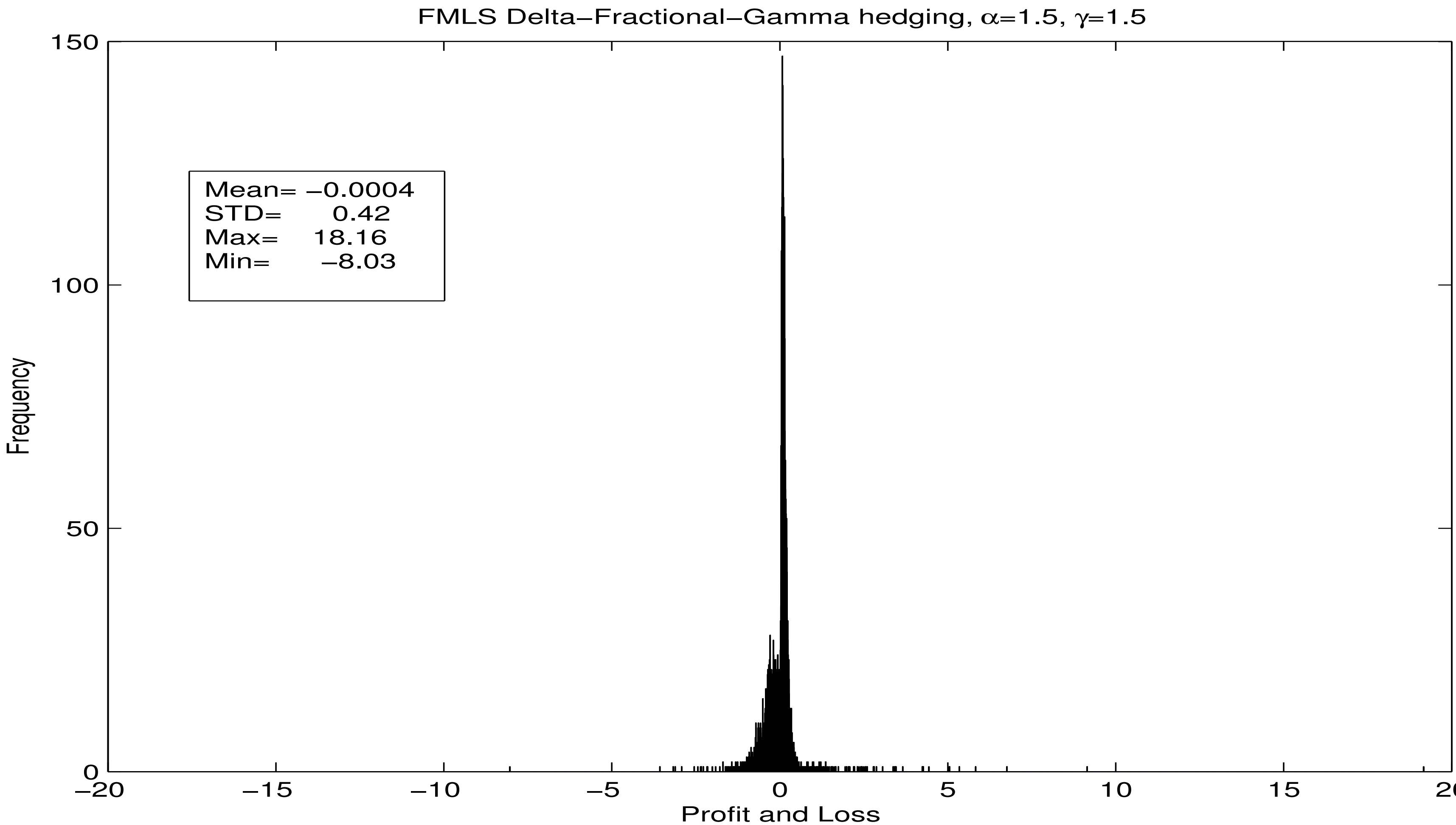
Delta Hedging in the FMLS



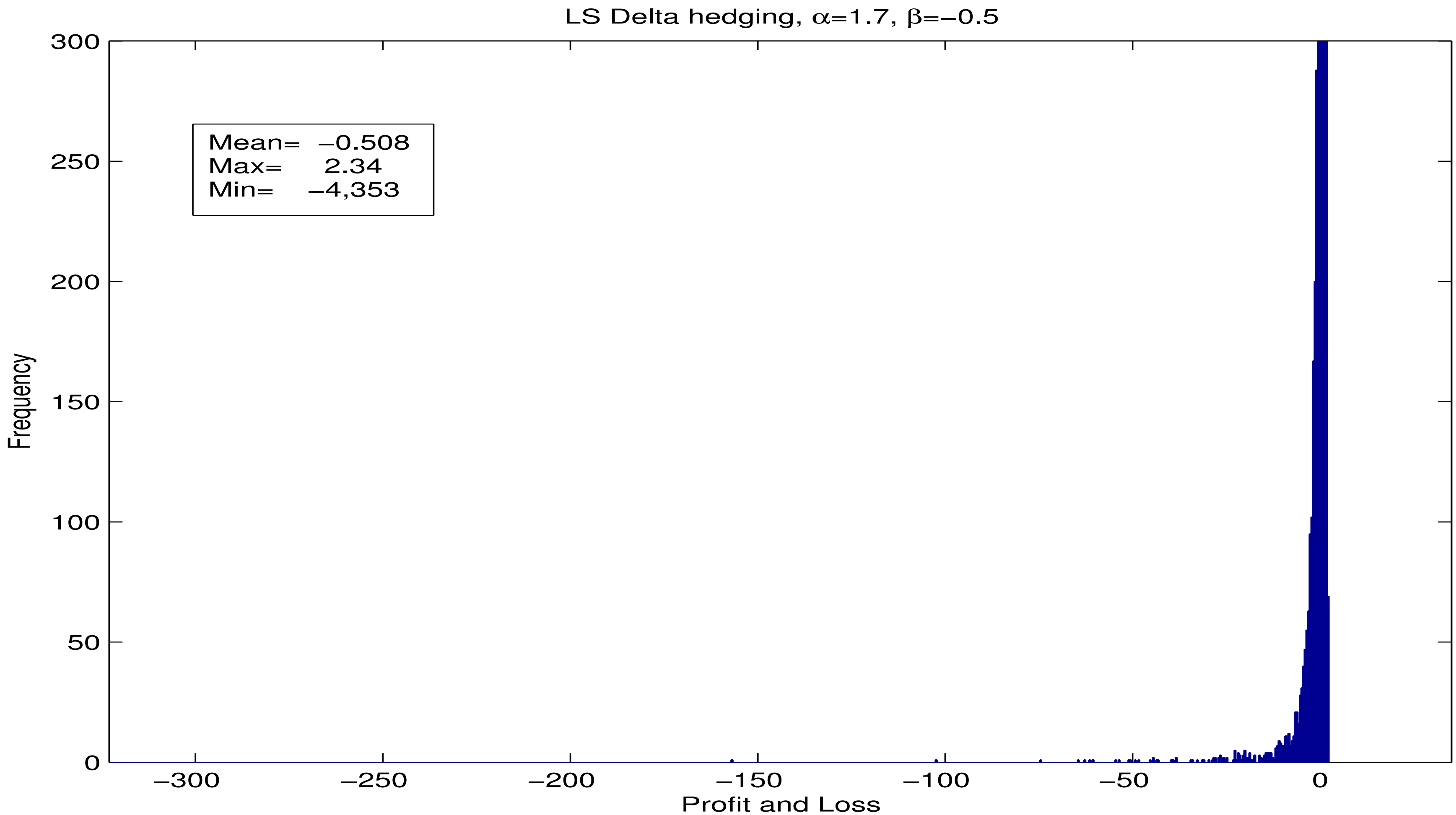
Delta-Gamma Hedging in the FMLS



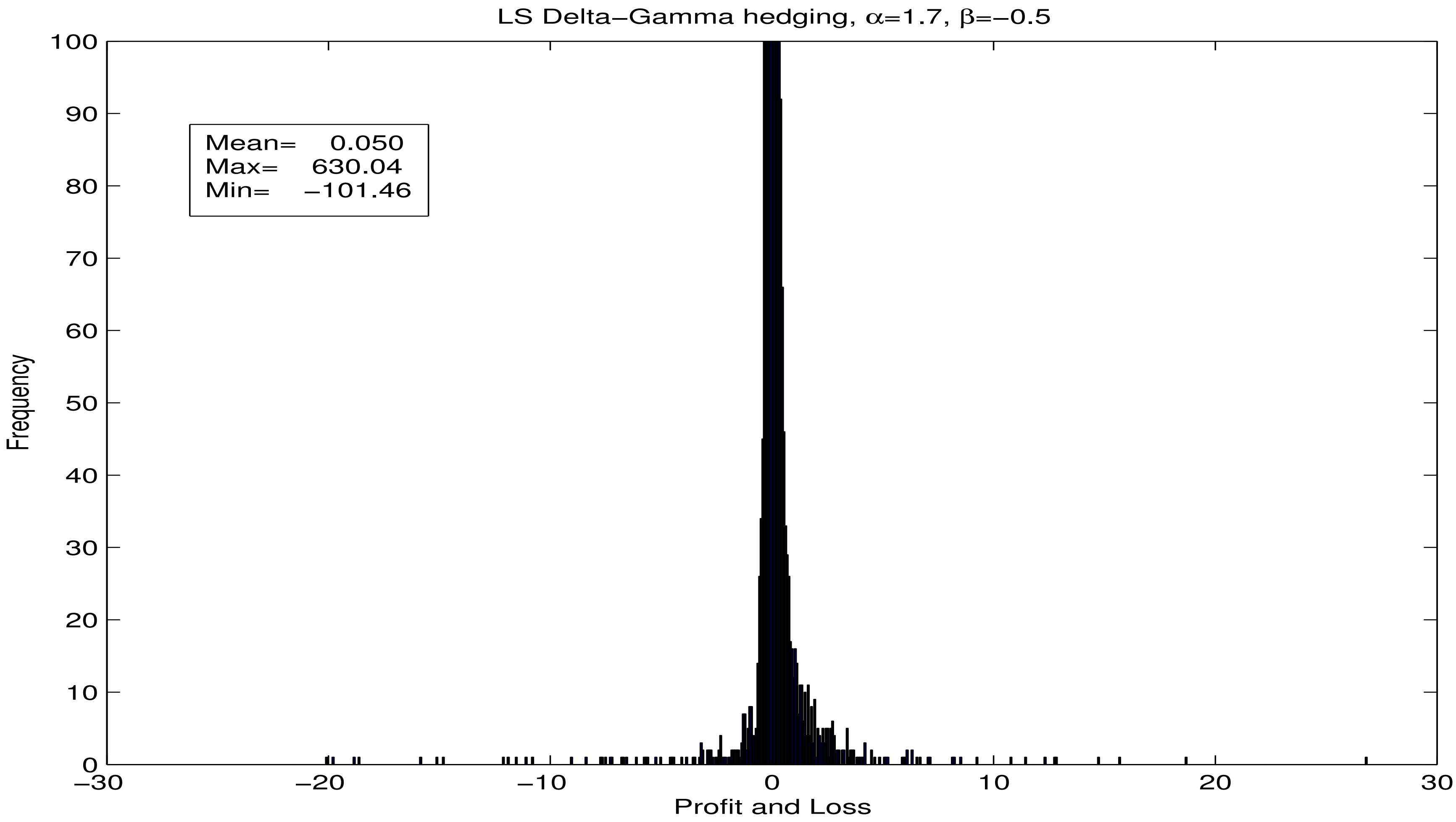
Delta-Frac-Gamma Hedg. in the FMLS



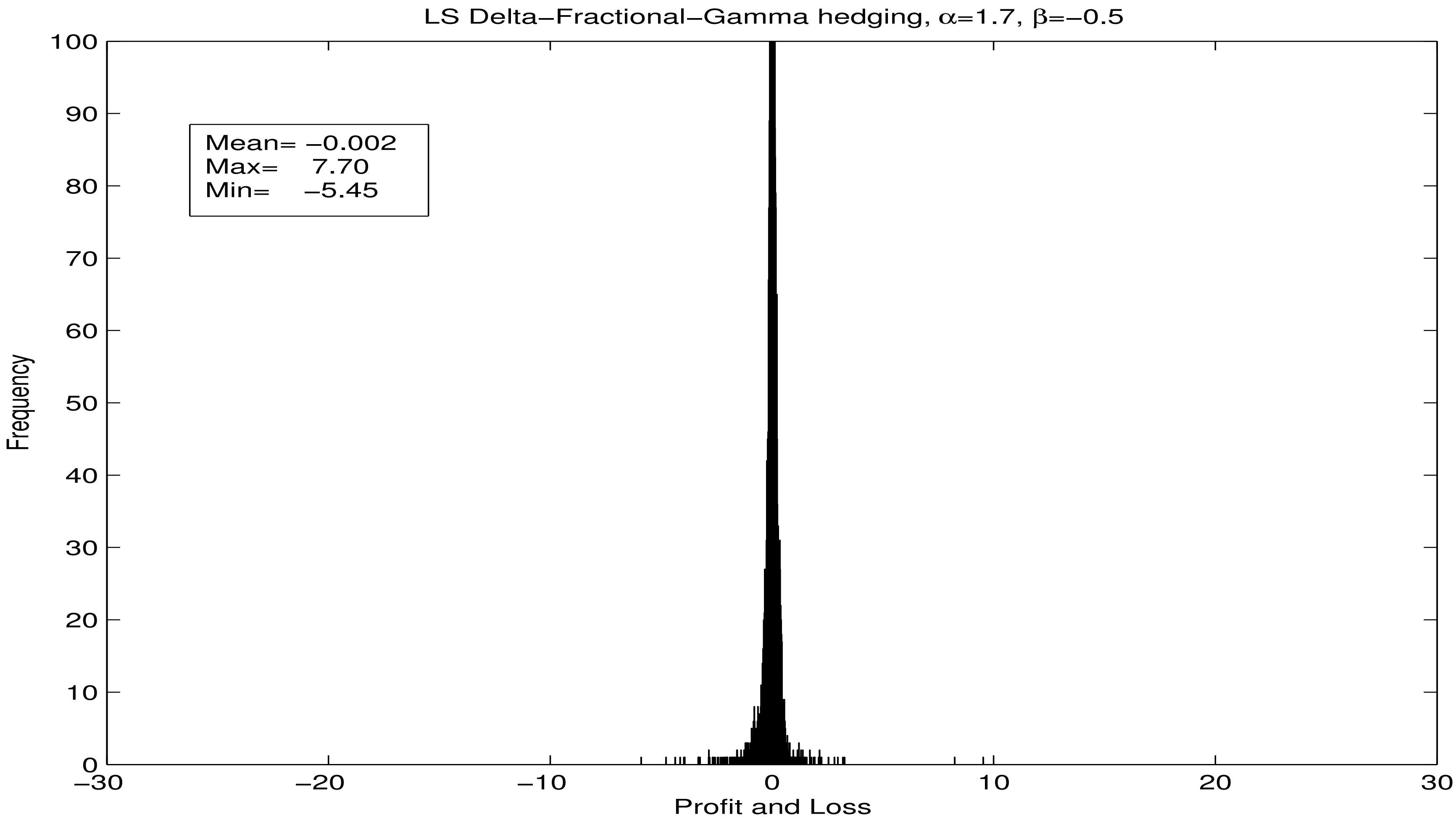
Delta Hedging in the LS



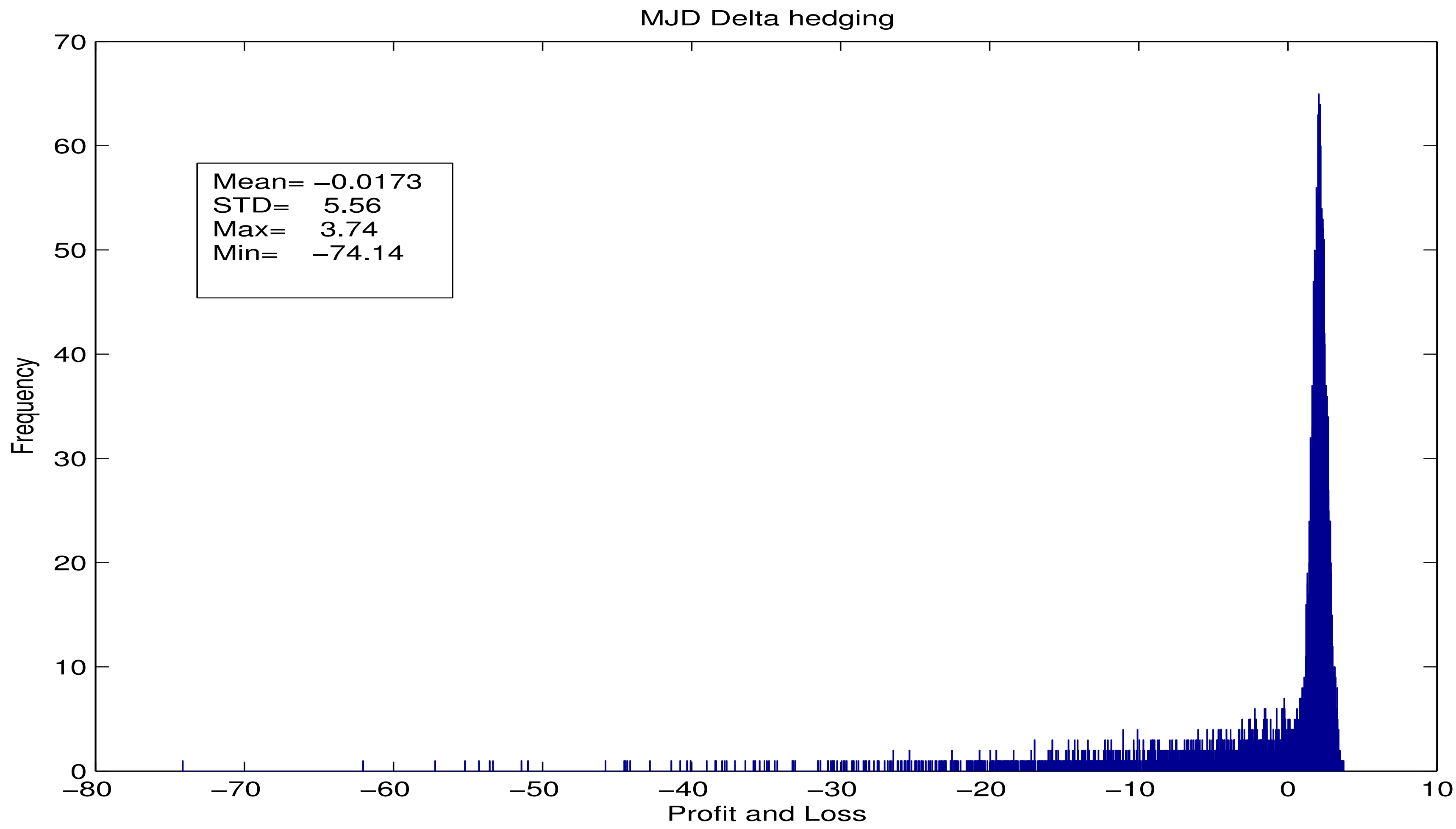
Delta-Gamma Hedging in the LS



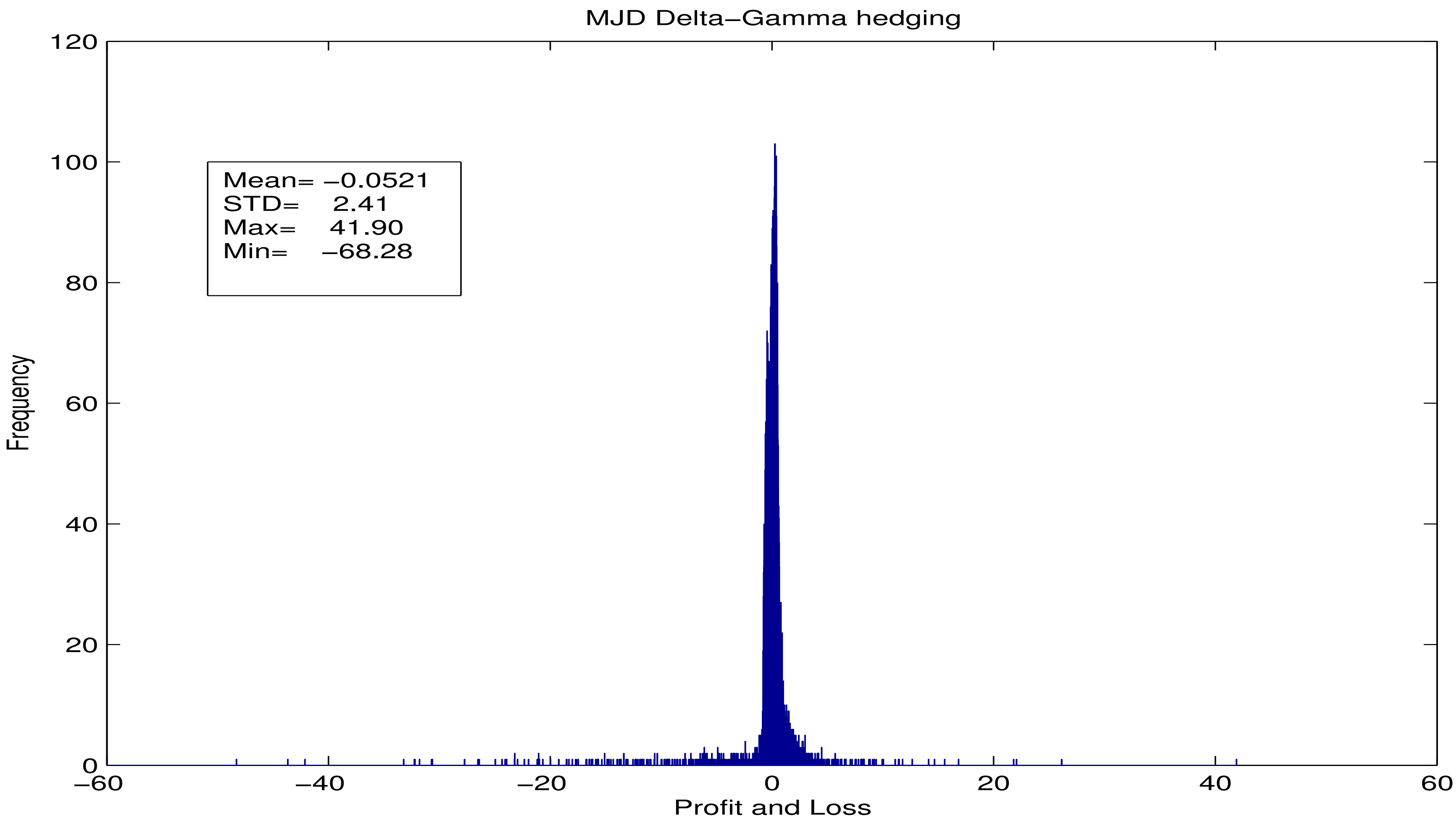
Delta-Frac-Gamma Hedging in the LS



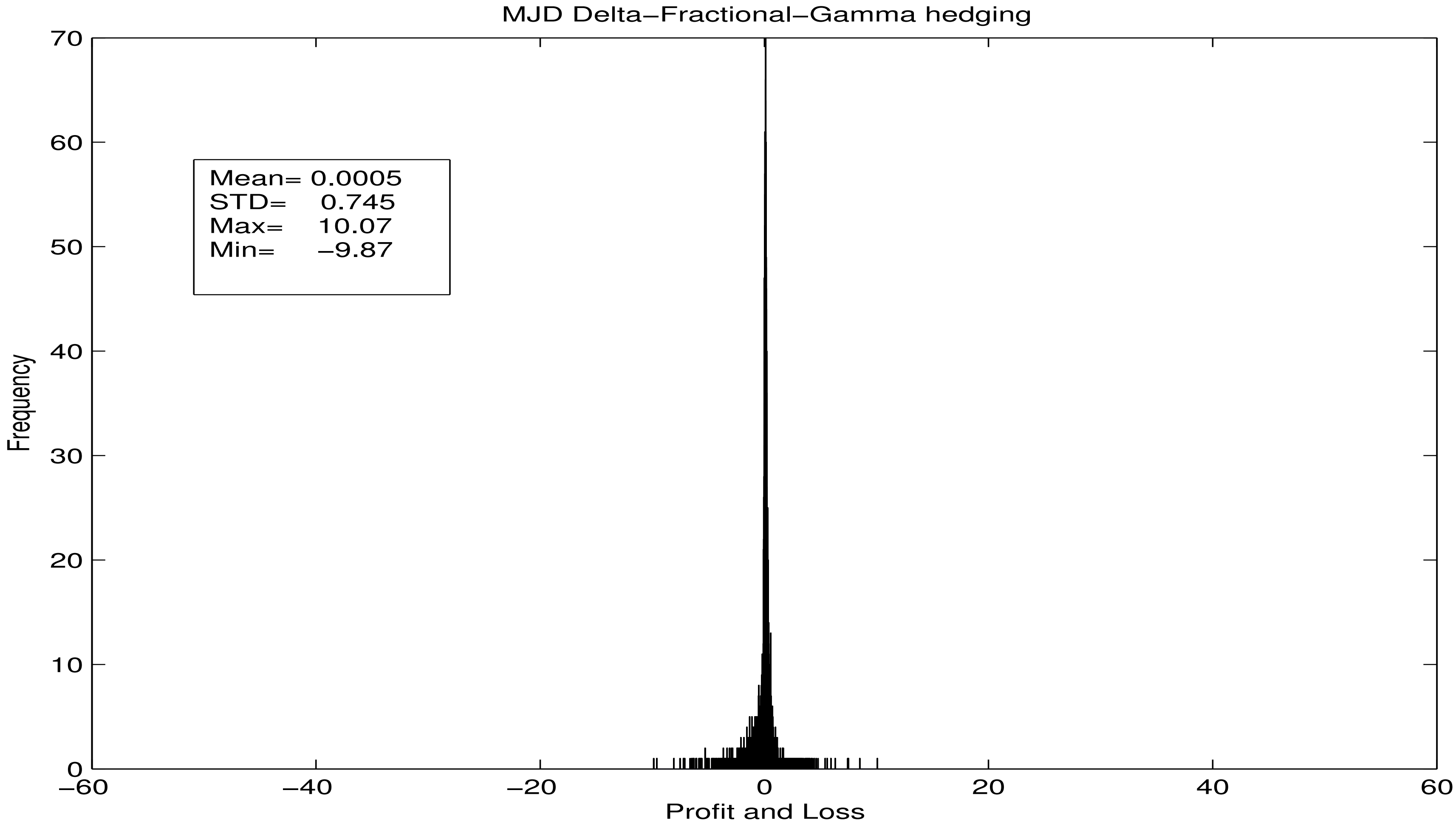
Delta Hedging in the MJD



Delta-Gamma Hedging in the MJD



Delta-Frac-Gamma Hedg in the MJD



Conclusions

For Lévy processes with Lévy densities that have a polynomial singularity at the origin and exponential decay at the tails we can recast the pricing equation in terms of Fractional derivatives.

The non-local property of the fractional operators can be useful when dynamically hedging (any) options.