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A Queueing Network Approach to Portfolio Credit Risk

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Agenda

- Portfolio credit risk.
- Vasicek large portfolio model.
- Rating change models in random environments.
- Preliminary analysis: the ‘leaky bucket’ approach.
- A general model: fluid and diffusion limits.
- Application to credit portfolios.
- Some computational results.

1 Portfolio Credit Risk

The requirements in modelling the credit risk of a portfolio are

- Credible modelling of the interaction effects.
- Efficient computational methods
- Ease of calibration

For large portfolios it makes sense to consider large-sample approximations (law of large numbers, central limit theorem). The approach taken here is based on an analogy with queueing networks.

2 Related Work

- Vasicek – homogeneous large portfolio model (see below)
- Hull and White – extension to non-homogeneous portfolios.
- CreditRisk+ – saddle point approximations (Wilde, Gordy)
- Giesecke and Weber – gaussian approximations in a voter model.

All these models work in a "static" form, this is, they model a fixed point in time.

- Frey and Backhaus – similar model to the one presented here using an arbitrary continuous space state latent process and 'mean field' interaction. Strong convergence results are derived.

3 Vasicek Large Portfolio Model

Obligor i defaults if $X_i < K_i$ where $X_i \sim N(0,1)$ so $K_i = N^{-1}(p_i)$ where p_i is the marginal default probability. Represent X_i as

$$X_i = \rho X + \sqrt{1-\rho^2}\epsilon_i,$$

where $X, \epsilon_1, \epsilon_2, \ldots$ are independent $N(0,1)$. In homogeneous case $p_i = p_1$ for all i and

$$P[\text{Obligor } i \text{ defaults}|X] = N\left(\frac{K_1 - \rho X}{\sqrt{1-\rho^2}}\right) \equiv p(X).$$

Conditional distribution of *proportion* π of obligors defaulting is then binomial with mean $p(X)$ and standard deviation $\sqrt{p(X)(1-p(X)/n}$ where n is the portfolio size.

For large n the standard deviation is small and we have approximately

$$(\pi > \alpha) \Leftrightarrow p(X) > \alpha$$

giving the *unconditional* distribution

$$P[\pi > \alpha] \sim N\left(\frac{K_1 - \sqrt{1-\rho^2}N^{-1}(\alpha)}{\rho}\right).$$

Our objective: do something similar in a *dynamic* context.

4 Models related to Queueing Networks

- Empirical evidence (Crowder, Giampieri & Davis 2003) suggests that the pattern of realized defaults is well represented by a latent variable model where the latent process X_t is a 2-state (good times/bad times) economic variable.

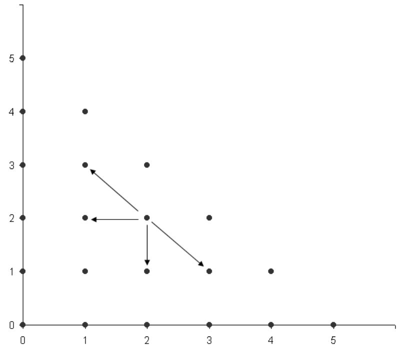
- Obligors move around rating categories at a faster time scale than the economic cycle.

- These facts suggest a model in which obligors move around the rating categories at rates depending on the latent process and occasionally default.

- There is an obvious analogy with queueing networks in which ‘jobs’ move around ‘service stations’ for processing.

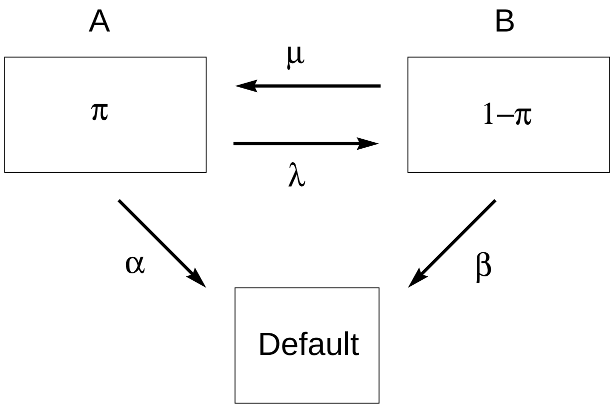
- Recent work by Choudhury, Mandelbaum et al. studies fluid and diffusion limits for queueing networks under random environments.

5 Rating transitions, no latent variable



For a portfolio with n obligors and K possible ratings $k = 1, ..., K$ we define a vector process $Q^n(t)$ taking values in $\mathbf{N}^k$, with each component $Q^n_k(t)$ containing the number of elements in each rating category at time t. Then, $\sum Q^n_k(t)$ is the number of non defaulted obligors at time t. When $n = 5, K = 2$, the state space of $Q^n(t)$ is as shown above. The figure shows all the possible movements given the current credit ratings: transitions (move along the diagonal) and defaults (move to the next diagonal).

6 Leaky bucket analysis



Here π is the proportion of obligors in rating category A.

Assuming $\alpha, \beta \ll \mu, \lambda$ we have the mass balance equation

$$\mu(1-\pi) + \lambda\pi,$$

giving $\pi = \mu/(\mu+\lambda)$ and a default rate

$$d = \alpha\pi + \beta(1-\pi) = \frac{\alpha\mu + \beta\lambda}{\mu+\lambda}$$

Since the obligors are independent in this model, the standard deviation of cumulative defaults in $[0, t]$ is just

$$\sqrt{\frac{d\,t}{n}}.$$

This simple analysis is surprisingly successful in predicting the mean and variance of the exact default distribution obtained by solving the forward equation for the finite-state Markov process, if an adjustment for mean defaults is made.

However, the leaky bucket analysis doesn't depend on the initial distribution of the obligors' ratings and therefore doesn't capture the *short-term* default behaviour.

Parameters

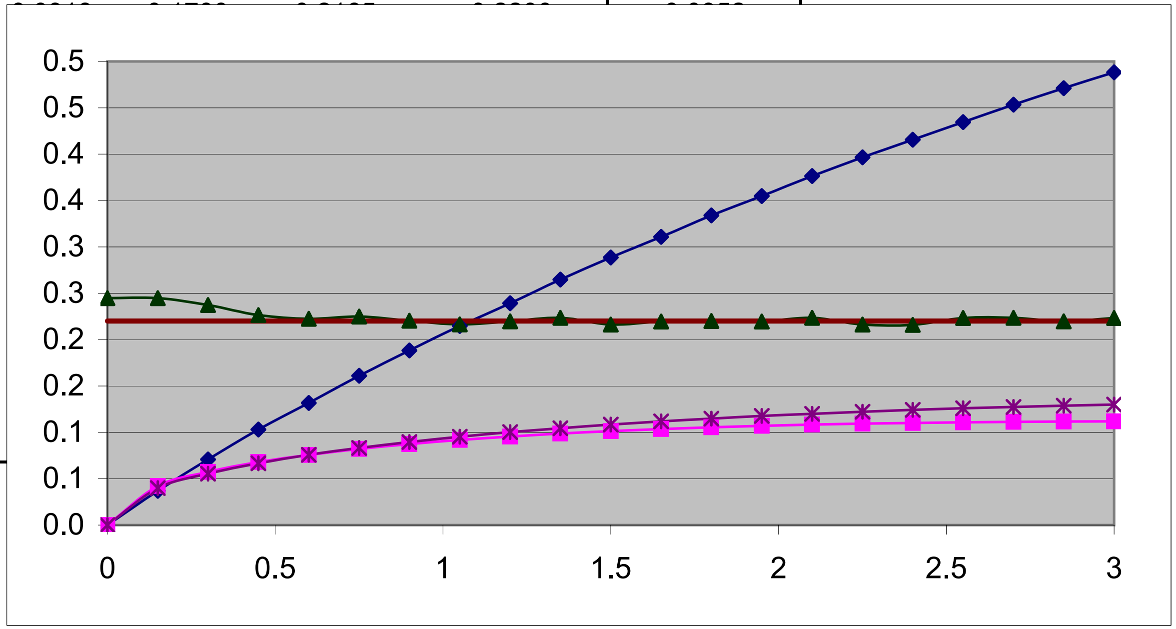
mu	4
lam	1
alpha	0.3
beta	0.2
sut	200
k	15
time	1
elem	20

Leaky Bucket prediction
0.22

0.104881

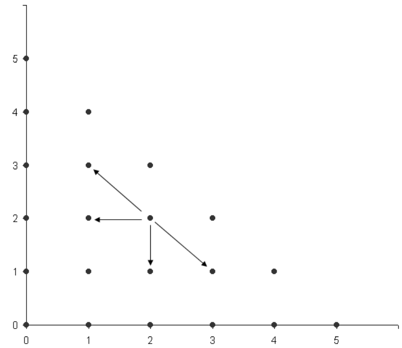
2.104881

Time	Default Statistics Mean	Default Statistics Std Dev	Default Rate	Normalized Def Rate	Leaky Bucket Prediction	Leaky Bucket Std Dev Pred
0	0.0000	0.0000	0.2356	0.2447	0.2200	0.0000
0.15	0.0369	0.0421	0.2356	0.2447	0.2200	0.0399
0.3	0.0707	0.0573	0.2205	0.2373	0.2200	0.0554
0.45	0.1030	0.0680	0.2032	0.2265	0.2200	0.0666
0.6	0.1316	0.0756	0.1932	0.2225	0.2200	0.0757
0.75	0.1610	0.0822	0.1887	0.2250	0.2200	0.0832
0.9	0.1883	0.0874	0.1789	0.2204	0.2200	0.0896
1.05	0.2146					
1.2	0.2393					
1.35	0.2648					
1.5	0.2886					
1.65	0.3109					
1.8	0.3340					
1.95	0.3549					
2.1	0.3765					
2.25	0.3967					
2.4	0.4156					
2.55	0.4345					
2.7	0.4535					
2.85	0.4712					
3	0.4883					



7 Fluid and Diffusion Limits

The finite state environment process defines different 'layers' in which the transition parameters are different.



Conditional in the realisation of the random environment, the process $Q^n(t)$ may be approximated by two processes

$$Q^n(t) \simeq nQ^{(0)}(t) + \sqrt{n}Q^{(1)}(t) \tag{1}$$

where $Q^{(0)}(t)$ is a deterministic process called the *fluid limit* and $Q^{(1)}(t)$ is a diffusion called the *diffusion limit* of the sequence $Q^n(t)$. This is, conditional on the random environment, the distribution of the process $Q(t)$ may be approximated by a normal distribution.

8 A General Model

The random environment process $X(t)$ is a finite state process in continuous time, having at most a finite number of jumps in any bounded interval of $[0, \infty)$.

To construct the process $Q(t)$ we consider a collection of mutually independent Poisson processes $\{A_i\}_{i \in I = \{1,\dots,n\}}$ and a collection of vectors $\{\mathbf{v}_i\}$ in R^K, $K \in \mathrm{N}$, and a collection of non-negative functions of the form $\alpha_i(\cdot, \cdot, x) : [0, \infty) \times \mathrm{R}^K \to [0, \infty)$ for all $i \in I$ and $x \in \mathcal{X}$. We assume each $\alpha_i(t, \cdot, x)$ is Lipschitz bounded with respect to the second argument, this is, exist a locally integrable function $\beta_t : [0, \infty) \to [0, \infty)$ such that

$$\|\alpha_i(t, \cdot, x)\| \leq \beta_t \tag{2}$$

where $\|\cdot\|$ is the Lipschitz norm defined as

$$\|f\| = \sup_{\substack{x \neq y \\ x, y \in D_1}} \frac{|f(x) - f(y)|_{D_2}}{|x - y|_{D_1}} \vee |f(0)|_{D_2}$$

We define a mapping $\mathcal{Q}$ from Ω into $D([0, \infty), \mathrm{R}^K)$ by $(\omega_1, \omega_2) \mapsto Q$, where Q is the process solution to the equation

$$Q(t) = Q(0) + \sum_{i=1}^{n} A_i \left(\int_0^t \alpha_i(s, Q(s), X(s))ds \right) \mathbf{v}_i \tag{3}$$

The law of $Q(t)$ is $P_Q(\cdot) = P(\mathcal{Q}^{-1}(\cdot))$. The probability conditional on the environment process $X(t)$ denoted by $P_Q^{\omega_1}$ is then defined by the conditional probability under the inverse mapping.

We are concerned with the convergence of sequences $Q^\eta : (\omega^1, \omega^2) \to \mathrm{R}^K$ of the form

$$\frac{Q^\eta(t)}{\eta} = \frac{Q^\eta(0)}{\eta} + \frac{1}{\eta}\sum_{i=1}^{n} A_i \left(\eta \int_0^t \alpha_i(s, Q^\eta(s), X(s))ds \right) \mathbf{v}_i$$

for $\eta > 0$.

Theorem 1 *Assume a collection of functions of the form* $\alpha_i : [0, \infty) \times \mathrm{R}^K \times \mathcal{X} \to [0, \infty)$ *for* $i \in I$ *and such that*

$$|\alpha_i(t, y, x)| \leq C(1 + |y|) \tag{4}$$

for some constant $C < \infty$, $s \geq 0$ *and* $y \in \mathcal{R}^K$. *Assume the processes* $X(t)$ *and* $A_i(t)$ *defined as above and* $\mathbf{v}_i$ *a collection of vectors in* R^K.

Then the process $Q(t)$ *defined as the solution of the equation*

$$Q(t) = Q(0) + \sum_{i=1}^{n} A_i \left(\int_0^t \alpha_i(s, Q(s), X(s))ds \right) \mathbf{v}_i \tag{5}$$

has unique solution for a.e. ω.

We want to infer the behaviour of the state of the network when the number of arrivals and the number of serves per node increases while the rate of service remains unchanged. We will present some *quenched* convergence results for the process (X_t, Q_t). The results obtained are two: Firstly, a quenched strong approximation limit for *accelerated sequences* of the form $(X(t), Q(\eta t)/\eta)$, $\eta \to \infty$. Secondly, a quenched weak convergence result states the convergence of accelerated sequences of the form $(X(t), Q(\eta t)/\sqrt{\eta})$ to a diffusion for a given realisation of the random process.

9 Strong Approximations

We define a sequence of network processes $\{(X(t), Q^\eta(t)/\eta); \eta > 0\}$ associated to $(X(t), Q(t))$ as the set of network processes where $Q^\eta(t)$ is the solution to the system

$$Q^\eta(t) = Q^\eta(0) + \sum_{i=1}^{N} A_i \left(\int_0^t \alpha_i^\eta \left(s, Q^\eta(s), X(s)\right)\right) ds \Bigg) \mathbf{v}_i \tag{6}$$

where $\{\alpha_i^\eta(s,\cdot,x)\}$, with $x \in \mathcal{X}$ and $i \in I$, is a collection of functions satisfying

$$\|\alpha_i^\eta(t,\cdot,x)\| \leq \eta\beta_t \tag{7}$$

with β_t a locally integrable function.

The interpretation of the pair $(X(t), \frac{1}{\eta}Q^\eta(t))$ for some $\eta > 0$ is a process with the same characteristics of $Q(t)$ under the same environment $X(t)$ but where the number of servers and rates of arrivals have increased η times.

A quenched law of large numbers for the sequence $(X(t), 1/\eta Q^\eta(t))$:

Theorem 2 *If $\{\alpha_i^\eta | i \in I, \eta > 0\}$ are Lipschitz bounded and*

$$\lim_{\eta\to\infty} \sum_{i=0}^{N} \int_0^t \left\| \frac{\alpha_i^\eta(s,\cdot,x)}{\eta} - \alpha_i^{(0)}(s,\cdot,x) \right\| ds = 0 \tag{8}$$

For $\omega_1 \in \Omega^1$ where the process $Q^\eta(t)$ is that

$$\frac{Q^\eta(0)}{\eta} \to Q^{(0)}(0) \text{ as } \eta \to \infty \tag{9}$$

then

$$\frac{Q^\eta(t)}{\eta} \to Q^{(0)}(t) \text{ as } \eta \to \infty \tag{10}$$

a.s. in $P_Q^{\omega_1}$, where $Q^{(0)}(t)$ defined in Ω^2 is the solution to the equation

$$Q^{(0)}(t) = Q^{(0)}(0) + \int_0^t \alpha^{(0)}(s, Q^{(0)}(s), X(s))ds \tag{11}$$

where

$$\alpha^{(0)}(t,\cdot,x) = \sum_{i=1}^{N} \alpha_i^{(0)}(t,\cdot,x)\mathbf{v}_i \tag{12}$$

and the integral equation is deterministic conditional on the random environment process $X(t)$.

The process $Q^{(0)}(t)$ is referred as the *fluid limit* of the sequence $\{Q^{\eta}(t)\}_{\eta\geq 0}$.

10 Weak convergence

It is possible to derive a quenched functional version of the central limit theorem for $(X(t),(1/\eta)Q^{\eta}(t))$ conditional on $\mathcal{F}_t^{\omega_1}$.

To present this result, we require to define the *scalable Lipschitz derivative* of a function $f : D_1 \to D_2$ at $x \in D_1$, D_1 and D_2 two Banach spaces, as the function $\Lambda f_x(y) : D_1 \to D_2$ such that

$$\lim_{y\to 0} \frac{|f(x+y)-f(x)-\Lambda f_x(y)|_{D_2}}{|y|_{D_1}} = 0$$

whenever such function exists and it is Lipschitz bounded and homogeneous, this is,

$$\|\Lambda f_x(\cdot)\| < \infty,$$

and for all $\lambda \geq 0$

$$\lambda \Lambda f_x(y) = \Lambda f_x(\lambda y).$$

We note that in the case of $f : \mathrm{R}^{d_1} \to \mathrm{R}^{d_2}$, $d_1, d_2 \in \mathrm{N}$, the operator $\Lambda f_x(y)$ generalises the notion of directional derivative, and whenever the differential operator exists we have

$$\Lambda f_x(y) = Df(x)y$$

where $Df(x)$ is the Jacobian matrix valuated at x. When the Jacobian matrix is not defined $\Lambda f_x(y)$ may be not unique.

For a sequence of r.v.'s defined in a measurable space $(\Omega, \mathcal{F})$ we will say that a sequence of r.v.'s $\{Y_n\}$ converges in distribution of Y, denoted by $\lim_{n\to\infty} Y_n =^d$

Y, w.r.t. $\mathbf{P}$ some probability measure on $(\Omega, \mathcal{F})$ if for all $f \in \overline{C}(\Omega)$, the set of bounded continuous real functions defined in Ω, $\lim_{n\to\infty} E_P\{Y_n\} = E_P\{Y\}$.

Some weak convergence results in queuing theory assume a heavy traffic condition, this is, the arrival and total service rates are nearly the same. Here we have no arrivals and eventually $Q^\eta(t) = \mathbf{0}$, so no stationarity may be assumed.

Theorem 3 *Assume*

$$\sum_{i\in I} \int_0^t \overline{\lim_{\eta\to\infty}} \sqrt{\eta} \left\| \frac{\alpha_i^\eta(s,\cdot,x)}{\eta} - \alpha_i^{(0)}(s,\cdot,x) \right\| ds < \infty \tag{13}$$

and

$$\lim_{\eta\to\infty} \sum_{i\in I} \int_0^t \left\| \sqrt{\eta} \left[\frac{\alpha_i^\eta(s,\cdot,x)}{\eta} - \alpha_i^{(0)}(s,\cdot,x) \right] - \alpha_i^{(1)}(s,\cdot,x) \right\| ds = 0 \tag{14}$$

and for all $x \in \mathcal{X}$ and $i \in I$. Assume the function $\alpha^{(0)}(s,\cdot,x)$ has scalable Lipschitz derivative for any values $X(t)$ and $Q^{(0)}(t)$.

For P^1 a.e. $\omega_1 \in \Omega^1$, if

$$\lim_{\eta\to\infty} \sqrt{\eta}\left[\frac{Q^\eta(0)}{\eta} - Q^{(0)}(0)\right] =^d Q^{(1)}(0) \tag{15}$$

w.r.t $P_Q^{\omega_1}$, then

$$\lim_{\eta\to\infty} \sqrt{\eta}\left[\frac{Q^\eta(t)}{\eta} - Q^{(0)}(t)\right] =^d Q^{(1)}(t) \tag{16}$$

w.r.t $P_Q^{\omega_1}$, where the process $Q^{(1)}(t)$ takes values in Ω^2 and it is the solution to the stochastic integral equation

$$\begin{aligned} Q^{(1)}(t) &= Q^{(1)}(0) + \int_0^t \Lambda\alpha_i^{(0)}(s, X_s, Q^{(0)}; Q^{(1)}(s)) + \alpha^{(1)}(s, Q^{(0)}(s), X_s)ds \\ &\quad + \sum_{i\in I} B_i\left(\int_0^t \alpha_i^{(0)}(s, Q^{(0)}(s), X_s)ds\right)\mathbf{v}_i \end{aligned} \tag{17}$$

where $\{B_i(t)\}$ is a collection of independent standard Brownian motions in $(\Omega^2, \mathcal{F}^2, P^{\omega_1})$.

11 Application to Correlated Defaults

We propose a model for the default/rating process of a set of obligors. There is a finite-state random environment process $\{X(t)|t \geq 0\}$ representing some macroeconomic (or sector associated) process that influences the default/transition rates of the obligors. The obligor credit events are independent conditional on the realisation of the environment process and follow a Markov chain with rates being function of the environment process.

Assume a portfolio with n obligors and K possible ratings $1, \ldots, K$. The initial rating composition of the portfolio is represented by the *rating distribution vector* $Q^n(0) \in \mathrm{R}^k$. $Q^n(t)$ will represent the random rating distribution of the portfolio at a later time t.

We define the index set of transition events $I = \{(i,j)|i,j = 1,...,k\}$ and denote the transition rate from rating i to rate j, $i,j = 1,...,k$, by $\mu_{(i,j)}(x) = \mu_{ij}(x)$; default rates are denoted by $\mu_{(i,i)}(x) \equiv \mu_i(x)$. Associated to these credit events we define the set of vectors $\{\mathbf{v}_{(i,j)} \in \mathrm{R}^k | (i,j) \in I\}$ that define the changes in the rating distribution vector in case of a credit event. This is

$$v_{(i,j)} = \begin{cases} \mathbf{e}_j - \mathbf{e}_i & i \neq i \\ -\mathbf{e}_i & i = j \end{cases}$$

where $\mathbf{e}_i$ is the i-th canonical vector in R^k.

Under this assumptions the credit events of each obligors are identically distributed and occur according to the first jump of a Poisson process with rates

$$\hat{\mu}_i(x) = \sum_{j=1}^{k} \mu_{ij}(x)$$

Once a credit event occurs at time t, the obligor defaults with probability $\mu_i(X_t)/\hat{\mu}_i(X_t)$ while a transition to rate $j \neq i$ has probability $\mu_{ij}(X_t)/\hat{\mu}_i(X_t)$.

As stated above, the occurrence of a credit event for an obligor in the credit rate i is given by the first jump of a Poisson process. The rate of occurrence depends on the environment random process but is independent of time. This is, for a set of positive real numbers $\{\mu_{ij}^{x} | x \in \mathcal{X}, (i,j) \in I\}$ we define the transition default rates as

$$\mu_{(i,j)}(t, X_t) = \sum_{x \in \mathcal{X}} \mu_{ij}^{x} \mathrm{I}_{\{X_t = x\}}$$

for $(i,j) \in I$.

12 The Fluid Limit

According to the notation used above, we have the set of rate functions

$$\alpha^n_{(i,j)}(t, \mathbf{y}, x) = \mathbf{y}_i \mu_{(i,j)}(t, x)$$

and since the rate function does not depend on n is obvious that

$$\alpha^{(0)}_{(i,j)}(t, \mathbf{y}, x) = \mathbf{y}_i \mu_{(i,j)}(t, x)$$

satisfy the conditions of theorem 2. Using vector notation

$$\alpha^{(0)}(t, \mathbf{y}, x) = A_t(x)\mathbf{y}$$

where A is the infinitesimal generator of the process.

We can verify that by assuming

$$\lim \frac{1}{n} Q^n(0) = Q^{(0)}(0)$$

for all $n \geq 0$ all conditions of therorem 2 hold and the fluid limit process $Q^{(0)}(t)$ is the solution to the deterministic PDE system

$$\frac{d}{dt}Q^{(0)}(t) = A_t Q^{(0)}(t) \tag{18}$$

and whenever $A = A_t$ is time independent (no environment influence) the solution is given by

$$Q^{(0)}(t) = e^{tA}Q^{(0)}(t)$$

otherwise, since X_t has at most finite jumps in any bounded interval of time, we can define a countable set of jump times of X_t $t_0 = 0, t_1 < ...$ and define $Q^{(0)}(t)$ recursively

$$Q^{(0)}(t) = e^{(t-t_i)A(x_{t_i})}Q^{(0)}(t) \tag{19}$$

for $t_i < t_{i+1}$.

13 The diffusion limit

Since $\alpha^n_{(i,j)} = \alpha^{(0)}_{i,j}$ for all $n \geq 0$ and $(i,j) \in I$ we can verify that by defining $\alpha^{(1)}_{(i,j)} = 0$ for all $(i,j) \in I$ the conditions in theorem 3 hold. Therefore assuming

$$\lim_{n\to\infty} \sqrt{n}\left[\frac{Q^n(0)}{n} - Q^{(0)}(0)\right] =^d Q^{(1)}(0) \tag{20}$$

implies

$$\lim_{n\to\infty} \sqrt{n}\left[\frac{Q^n(t)}{n} - Q^{(0)}(t)\right] =^d Q^{(1)}(t) \tag{21}$$

where $Q^{(1)}(t)$ satisfies a SDE which may be expressed as the following integral equation

$$Q^{(1)}(t) = Q^{(1)}(0) + \int_0^t A_t Q^{(1)}(t) + \sum_{l=1}^{k} \int_0^t \left(Q_l^{(0)}(t)\right)^{1/2} B_l dW_t^{(l)}$$

where W^l is a k-dimensional vector of independent standard Brownian motions for $l = 1, ..., k$. The matrices B_l have components

$$(B_l(t))_{ij} = \begin{cases} -\mu_{(i,j)}^{1/2} & \text{if } i = j \\ \mu_{(i,j)}^{1/2} & \text{if } l = i \neq j \\ 0 & \text{otherwise} \end{cases}$$

It is always possible rewrite (22) as

$$Q^{(1)}(t) = Q^{(1)}(0) + \int_0^t A_t Q^{(1)}(t) + \int_0^t B(s) d\hat{W}_t^{(l)}$$

where $\hat{W}_t$ is a k-dimensional vector of independent standard Brownian motions. In the case $k = 2$ the SDE is

$$\begin{aligned} dQ_1^{(1)}(t) &= -Q_1^{(1)}(t)(\mu_1(t) + \mu_{12}(t))dt + Q_2^{(1)}(t)\mu_{12}(t)dt \\ &\quad -(Q_1^{(0)}(t))^{1/2}(\mu_1^{1/2}(t)dW_{1,t}^{(1)} + \mu_{12}^{1/2}(t)dW_{2,t}^{(1)}) + (Q_2^{(0)}(t)\mu_{21}(t))^{1/2}dW_{1,t}^{(2)} \\ dQ_2^{(1)}(t) &= -Q_2^{(1)}(t)(\mu_2(t) + \mu_{21}(t))dt + Q_1^{(1)}(t)\mu_{21}(t)dt \\ &\quad -(Q_2^{(0)}(t))^{1/2}(\mu_2^{1/2}(t)dW_{2,t}^{(2)} + \mu_{21}^{1/2}(t)dW_{1,t}^{(2)}) + (Q_1^{(0)}(t)\mu_{12}(t))^{1/2}dW_{(2,t}^{(1)} \end{aligned}$$

That is equivalent to the the following SDE system

$$dQ^{(1)}(t) = A_t Q^{(1)} dt + B(t) d\hat{\mathbf{W}}_t$$

where

$$B(t) = \begin{pmatrix} \sigma_1(t) & 0 \\ \rho(t)\sigma_2(t) & \sqrt{1-\rho^2(t)}\sigma_2(t) \end{pmatrix}$$

$$\begin{aligned}
\sigma_1^2(t) &= Q_1^{(0)}(t)(\mu_1(t) + \mu_{12}(t)) + Q_2^{(0)}(t)\mu_{21}(t) \\
\sigma_2^2(t) &= Q_2^{(0)}(t)(\mu_2(t) + \mu_{21}(t)) + Q_1^{(0)}(t)\mu_{12}(t) \\
\rho(t)\sigma_1(t)\sigma_2(t) &= -Q_1^{(0)}(t)\mu_{12}(t) - Q_2^{(0)}(t)\mu_{21}(t)
\end{aligned}$$

and

$$\hat{\mathbf{W}}_t = (\hat{W}_t^{(1)}, \hat{W}_t^{(2)})^t$$

is a bi dimensional standard Brownian motion with independent components.

Assuming $A_t = A$ time independent, the solution of the SDE is given by

$$Q^{(1)}(t) = e^{tA} Q^{(1)}(0) + \int_0^t e^{(t-s)A} B(s) d\hat{\mathbf{W}}_s \tag{22}$$

with $Q^{(1)}(0) = \mathbf{0}$. The process is a stable Gaussian system with covariance matrix given by the integral

$$\mathrm{Cov}[Q^{(1)}(t), Q^{(1)}(t)] = \int_0^t e^{(t-s)A} B(s) B(s)^{tr} (e^{(t-s)A})^{tr} ds \tag{23}$$

that can be calculated numerically by solving a matrix ordinary differential equation (the Lyapunov equation).

In the general case of X_t taking values in $\mathcal{X}$ the process is defined similarly to the case of the fluid limit, this is

$$Q^{(1)}(t) = e^{(t-t_i)A_{t_i}} Q^{(1)}(t_i) + \int_{t_i}^t e^{(t-s)A_{t_i}} B(s) d\hat{W}_t \tag{24}$$

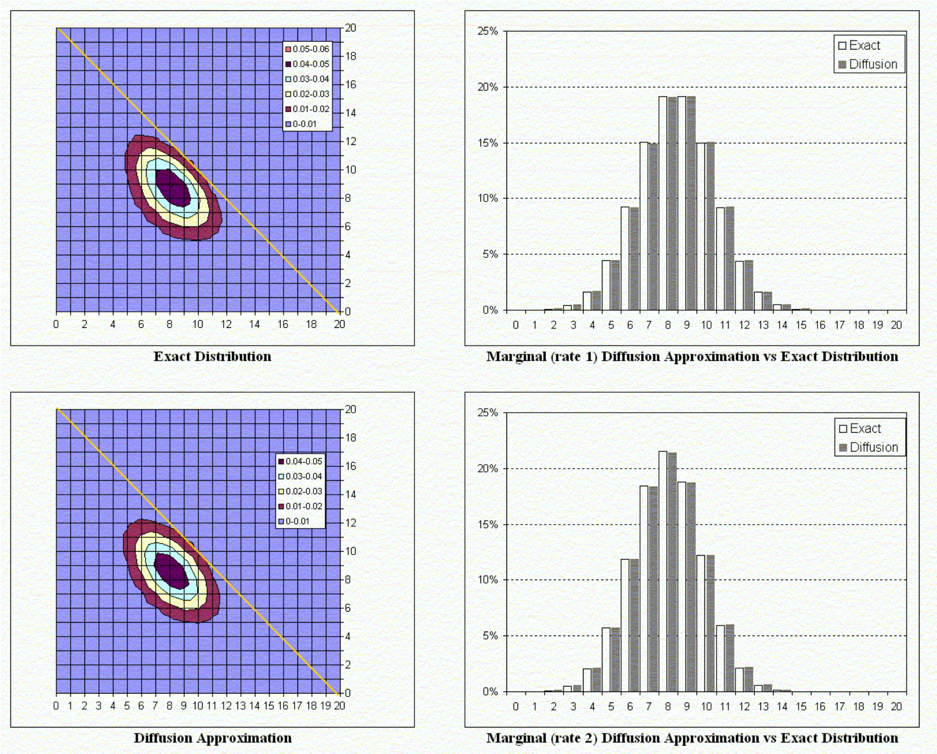
for $t_i < t < t_{i+1}$ where t_i is the time of the i-th jump of X_t.

14 Some numerical results

We assume a two state (two credit rates) system and 20, 50 and 100 elements. We assume a two state external random environment where jumps occur according to a standard Poisson process (rate 1). By sampling 1000 times the random environment we construct both the diffusion approximation and the exact distribution. The latter is obtained by integrating the Kolmogorov forward equation associated to the process using Runge-Kutta. The parameters are shown in the table.

	$X(t) = 0$	$X(t) = 1$
μ_1	0.1	0.2
μ_2	0.2	0.4
μ_{12}	0.3	0.3
μ_{21}	0.2	0.2

The exact and approximated distributions for the case of 20 elements and initial distribution 50/50. We can observe that the fitting of the marginal distributions is outstanding despite the relatively small number of elements.



Exact Distribution

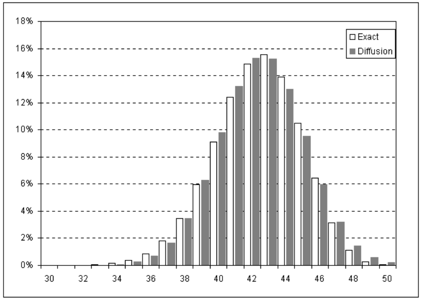
Marginal (rate 1) Diffusion Approximation vs Exact Distribution

Diffusion Approximation

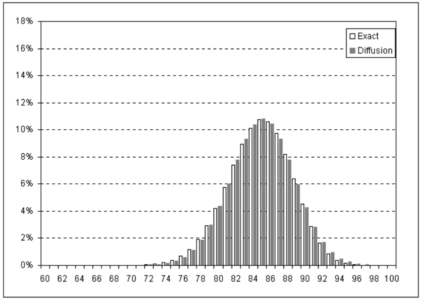
Marginal (rate 2) Diffusion Approximation vs Exact Distribution

We consider the approximation to the number of survivors for different number of elements and initial distribution in Graphs 2.

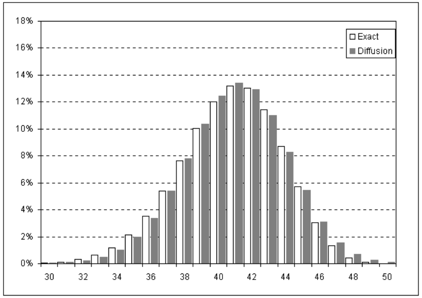
Graphs 2. Approximation to the number of survivors



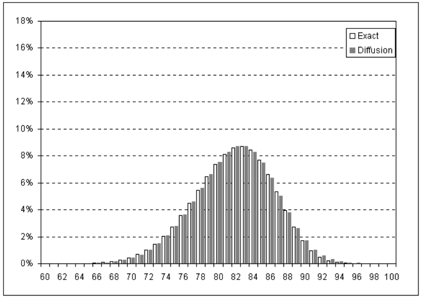
Case 1: 50 elements, 50/50 Initial Distribution



Case 2: 100 elements, 50/50 Initial Distribution



Case 3: 50 elements, 25/75 Initial Distribution



Case 4: 100 elements, 25/75 Initial Distribution

15 Concluding Remarks

- Method seems effective in predicting default performance with low computational cost.
- Investigate pricing applications: calibration to iTraxx tranche quotes.
- Risk management applications using empirical change-of-rating data.