University Finance Seminar

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Managing Correlation Risk with Spread Option Models

M A H Dempster

Centre for Financial Research

Judge Institute of Management, University of Cambridge

&

Cambridge Systems Associates Limited

mahd2@cam.ac.uk http://www-cfr.jims.cam.ac.uk

Co-worker: S S G Hong, UBS, London

Contents

1. Introduction

What is a Spread Option ?

- Two Underlying Assets: S_1, S_2
- Spread (basis): $S_1 - S_2$
- Payoff: $(S_1(T) - S_2(T) - K)_+$
- Price: $V_T(K) = E_Q[\, e^{-rT} (S_1(T) - S_2(T) - K)_+ \,]$

Why are they important?

- Invaluable tools for hedging and speculating...

- ... in almost all markets!

Energy Crack spread, Spark spread

Commodity Crush spread, Cotton calendar spread

Equity Index spread

Bond NOB spread, TED spread

Credit Derivatives Credit spread

- Indispensable for managing "correlation risks"

Hedging Using Spread Options

An oil refinery firm can short a call on the spread of oil future prices: $F_l - F_s$

- F_l : long output = Refined product
- F_s : short input = Brent crude
- K : strike = marginal conversion cost
- $(F_l(T) - F_s(T) - K)_+$: payoff of the crack spread

Hedging Using Spread Options

- If the spread is greater than the cost the option is exercised by the holder and the firm meets its obligation by producing
- If the spread is less than the cost the option expires worthless and the firm will not produce
- Either way the firm earns the option premium

  i.e. a call on the spread replicates the payoff structure of a firm's production schedule
- Also used to bridge delivery locations

Speculating Using Spread Options

A speculator can trade the correlation between two prices, indices or bond yields (LTCM):

- If we speculate on a correlation drop, we long a call on spread
- If we speculate on a correlation rise, we short a call on spread

The reasoning is similar to going long on a vanilla call on a single asset if we think volatility will rise, with the variance of the spread replacing the volatility of the single asset

Speculating Using Spread Options

The spread variance depends on:

- volatility of the long leg
- volatility of the short leg
- correlation between the two

The first two can be traded by options on individual prices

We need a spread option to trade the third (Mbanefo 1997)

The Problem

- Set up good models for the dynamics of the factors which accommodate stochasticities in interest rates, volatility...
- Compute the price of a spread option under such models
- Study how the price depends on the model specification in particular the volatility and correlation structure
- Design appropriate calibration procedures

2. Spread Option Pricing Review

Existing Approaches: I

- Model the spread as a geometric Brownian motion:

$$\boldsymbol{X} := \boldsymbol{S_1} - \boldsymbol{S_2}$$

$$d\boldsymbol{X} = \boldsymbol{X} (\mu \, dt + \sigma \, d\boldsymbol{W})$$

- Apply the Black-Scholes formula:

$$V_T(K) = \mathrm{E}_Q [e^{-rT} [\boldsymbol{S_1}(T) - \boldsymbol{S_2}(T) - K]_+]$$

$$:= \mathrm{E}_Q [e^{-rT} [\boldsymbol{X}(T) - K]_+]$$

- Simple but dangerous!

- spread can go negative

- a multi-factor problem by nature

Existing Approaches: II

- Model $\boldsymbol{S_1}, \boldsymbol{S_2}$ as geometric Brownian motions:

$$d\boldsymbol{S_1} = \boldsymbol{S_1} (\mu_1 \; dt + \sigma_1 \; d\boldsymbol{W_1})$$
$$d\boldsymbol{S_2} = \boldsymbol{S_2} (\mu_2 \; dt + \sigma_2 \, d\boldsymbol{W_2})$$

where $E_Q [d\boldsymbol{W_1} \, d\boldsymbol{W_2}] = \rho \, dt$

- ρ is the correlation between the prices

- Apply a conditioning technique to turn the two-dimensional integral into a single one
(K Ravindran 1993, D Shimko 1994)

$$V_T(K) = e^{-rT}\int_0^\infty \int_{S_2+K}^\infty (S_1 - S_2 - K) f_T(S_1, S_2) dS_1 dS_2$$

$$= e^{-rT}\int_0^\infty \left[\int_{S_2+K}^\infty [S_1 - (S_2 + K)]_+ f_{1|2}(S_1 | S_2) dS_1 \right] f_2(S_2) dS_2$$

$$= e^{-rT}\int_0^\infty C(S_2) f_2(S_2) dS_2$$

where

- $f_T(\cdot|\cdot)$: joint p.d.f. of $\boldsymbol{S_1}(T)$, $\boldsymbol{S_2}(T)$ ... bivariate log-normal
- $f_{1|2}(\cdot|\cdot)$: conditional density of $\boldsymbol{S_1}(T)$ given $\boldsymbol{S_2}(T)$ ... log-normal
- $f_2(\cdot)$: marginal density of $\boldsymbol{S_2}(T)$ .... log-normal
- $C(\cdot)$: an integral similar to the Black-Scholes call price

Existing Approaches: II

Simple, two-factor, but...

- Only works when distributions are normal
- Prices are the only sources of randomness...
- No stochastic interest rate or convenience yield
- Constant (deterministic) volatility
- Trivial correlation structure

Existing Approaches: III

Variants on the previous approach

- Approximation by piecewise linear payoff function (N D Pearson 1995)
- Edgeworth series expansion (D Pilipovic & J Wengler 1998)
- Lattice and PDE methods (Brooks 1995)
- A GARCH model with co-integration is also proposed and the spread option is valued using a Monte Carlo method (J C Duan, S R Pliska 1999)
- Gaussian mixture (C Alexander 2003)
- Survey (R Carmona & V Durrleman 2003)

3. Fourier Transform Techniques for Vanilla Options

What is a Fourier Transform ?

$$f(x) \mapsto \phi(\upsilon) = \int_{-\infty}^{\infty} f(x) \cdot e^{i\upsilon x} dx$$

$$\phi(\upsilon) \mapsto f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(\upsilon) \cdot e^{-i\upsilon x} d\upsilon$$

- probability density functions $\rightarrow$ characteristic functions
- differentiation w.r.t. x $\rightarrow$ multiplication by $-i\upsilon$ and inverting

$$f'(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (-i\upsilon \cdot \phi(\upsilon)) \cdot e^{-i\upsilon x} d\upsilon$$

- option pricing = integration of p.d.f. times payoff

… and a Fast Fourier Transform?

- An efficient algorithm for computing the sum

$$Y_k = \sum_{j=0}^{N-1} X_j \cdot e^{-\frac{2\pi i}{N} jk} \quad \text{for } k = 1, \ldots, N$$

for a complex array $X=(X_j)$ of size N

- Reduces the number of multiplications from an order of N^2 to $N \log_2 N$ Strassen (1967)

- Crucial for approximating the Fourier integral as a function of υ

$$\int_{-\infty}^{\infty} f(x) \cdot e^{i\upsilon x} dx \approx \sum_{j=0}^{N-1} f(x_j) e^{i\upsilon x_j} \Delta x$$

Black-Scholes via Fourier Transform

- S Heston (1993), G Bakshi & D B Madan(1999), P Carr & D B Madan (1999)

To price a European call under Black-Scholes we need:

- $s_T := \log(S_T)$: log-price of the underlying at maturity
- $q_T(.)$: risk-neutral density of the log-price s_T
- $k := \log(K)$: log of the strike price
- $C_T(k)$: price of a T-maturity call with strike e^k
- $f_T(.)$: characteristic function of the risk-neutral density q_T

Characteristic function under Black-Scholes

$$d \ln S = (r - \frac{1}{2}\sigma^2)dt + \sigma dW$$

$$\Rightarrow \quad s_T \sim N\left(s_0 + (r - \tfrac{1}{2}\sigma^2)T, \sigma^2 T\right)$$

$$\Rightarrow \quad \phi_{\mathrm{T}}(u) := E_Q\left[e^{iu \cdot s_T}\right]$$

$$= \int_{-\infty}^{\infty} e^{iu \cdot s} q_T(s) ds$$

$$= \exp\left[\left[s_0 + (r - \tfrac{1}{2}\sigma^2)T\right] - \tfrac{1}{2}\sigma^2 T \cdot u^2\right]$$

Fourier transform of the (modified) call

(P Carr & D B Madan 1999)

$$C_T(k) \equiv E_Q\left[e^{-rT}\left(S_T - K\right)_+\right]$$

$$\equiv \int_k^{\infty} e^{-rT}\left(e^s - e^k\right) q_T(s)\,ds$$

- The call price is not square-integrable since

$$C_T(k) \to S_0 \quad , \quad k \to -\infty$$

- Define the modified call price for some $\alpha > 0$

$$c_T(k) := \exp(\alpha k)\, C_T(k)$$

- The Fourier transform of the modified call price c_T is given by

$$
\begin{aligned}
\psi_T(\upsilon) &:= \int_{-\infty}^{\infty} e^{i\upsilon k} c_T(k)\,dk \\
&= \int_{-\infty}^{\infty} \int_{k}^{\infty} e^{-rT} e^{(\alpha+i\upsilon)k} (e^s - e^k) q_T(s)\,ds\,dk \\
&= \int_{-\infty}^{\infty} e^{-rT} q_T(s) \int_{-\infty}^{s} e^{(\alpha+i\upsilon)k} (e^s - e^k)\,dk\,ds \\
&= \int_{-\infty}^{\infty} e^{-rT} q_T(s) \left[\frac{e^{(\alpha+1+i\upsilon)s}}{(\alpha+i\upsilon)(\alpha+1+i\upsilon)} \right] ds \\
&= \frac{e^{-rT}\phi_T\left(\upsilon-(\alpha+1)i\right)}{(\alpha+i\upsilon)(\alpha+1+i\upsilon)}
\end{aligned}
$$

- But ϕ_T is also known in closed form!

- Inverting thus yields the call price:

$$C_T(k) = \frac{e^{-\alpha k}}{2\pi} \int_{-\infty}^{\infty} e^{-i\upsilon k} \psi_T(\upsilon)\, d\upsilon$$

- Approximate this, using trapezoid or Simpson's rule, with a finite sum and then apply the Fast Fourier Transform

$$C_T(k_m) \approx \frac{e^{-\alpha k_m}}{2\pi} \sum_{j=0}^{N-1} e^{-i\upsilon_j k_m} \psi_T(\upsilon_j)\eta$$

$$= \frac{e^{-\alpha k_m}}{2\pi} \sum_{j=0}^{N-1} \left[(-1)^{j+m} \psi_T(\upsilon_j)\eta \right] \cdot e^{-\frac{2\pi i}{N} jm}$$

for $m=0,\ldots,N-1$, where

$$\upsilon_j = (j - N/2)\eta \qquad k_m = (m - N/2)\lambda \qquad \lambda \cdot \eta = \frac{2\pi}{N}$$

Note: With an N grid for the Fourier sum this gives option prices with N equally spaced strikes

Extending the payoff

By modifying the input function of the inverse transform $\psi_T(\cdot)$ we can handle the following instrument with the same technique:

- $\left[e^{A \cdot s_T + B} - e^{k}\right]_+$: call on bonds (s_T is now the short rate)
- $\left[(A \cdot s_T + B) - k\right]_+$: call on yields
- $P(s_T)$: payoff contingent on polynomial in s_T
- $H(s_T)$: can even do general payoff in C^∞ via Taylor series expansion! (G Bakshi & D B Madan 1999)

Extending the distribution

- Normality can be relaxed...
- Explicit expression of the p.d.f. not needed
- Key: Characteristic functions!
- The underlying can evolve as
  - O. U. or C. I. R. processes
  - Affine diffusion with jumps
  - VG (Variance Gamma) process...
- Many of the above have no analytic density but their characteristic functions are known
- Needed for spreads on prices of pseudo-commodities such as kwH

Extending numbers of factors

- Stochastic volatility, stochastic interest rate... can be incorporated

$$dS = S\left(r\,dt + \sqrt{v}\,dW_1\right)$$
$$dv = \kappa_v\left(\mu_v - v\right)dt + \sqrt{v}\,dW_2$$
$$dr = \kappa_{rv}\left(\mu_{rv} - r\right)dt + \sigma_r dW_3$$
$$\vdots$$

... as long as the factors have analytic characteristic functions

- This includes pretty much all the diffusion models in the literature:
- Multifactor CIR models (Chen-Scott...)
- General affine diffusion models (Duffie, Kan, Singleton...)
- Gaussian interest rate models (Longstaff-Schwartz...)
- Stochastic volatility models (Heston, Bates, Hull-White...)

Extending the number of assets

- Now consider options whose payoffs are contingent on two assets S_1, S_2
- Example (Bakshi & Madan 1999): a generalisation of European call with the following payoff: $\left(e^{s_1(T)} - e^{k_1}\right)_+ \cdot \left(e^{s_2(T)} - e^{k_2}\right)_+$

We can price it in a similar fashion

$$c_T(k_1, k_2) := \exp(\alpha_1 k_1 + \alpha_2 k_2) \cdot C_T(k_1, k_2)$$

$$\equiv e^{\alpha_1 k_1 + \alpha_2 k_2} \int_{k_1}^{\infty} \int_{k_2}^{\infty} e^{-rT} \left(e^{s_1(T)} - e^{k_1}\right) \cdot \left(e^{s_2(T)} - e^{k_2}\right) q_T(s_1, s_2) ds_2 ds_1$$

- Consider its Fourier transform

$$\psi_T(\upsilon_1,\upsilon_2) := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\upsilon_1 k_1 + i\upsilon_2 k_2} c_T(k_1,k_2)\,dk_2\,dk_1$$

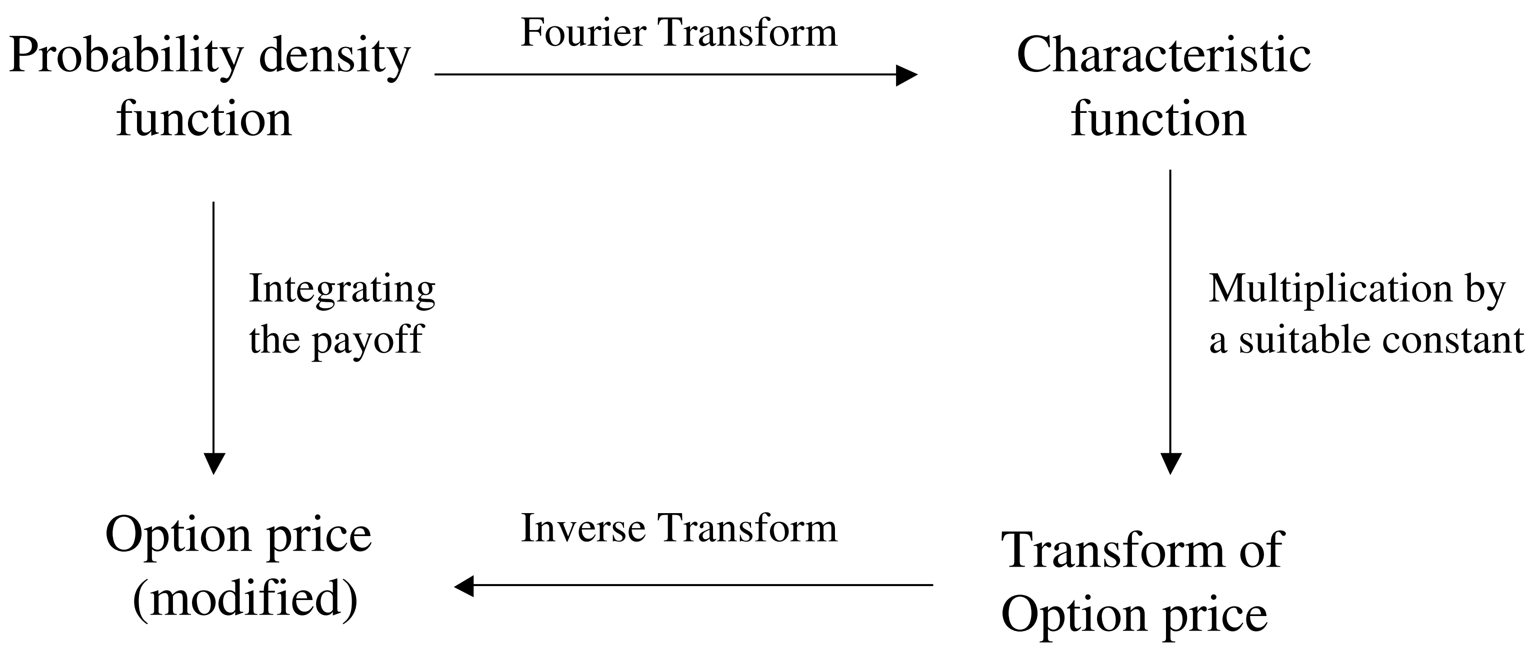
$$= \frac{e^{-rT}\phi_T\left(\upsilon_1-(\alpha_1+1)i, \upsilon_2-(\alpha_2+1)i\right)}{(\alpha_1+i\upsilon_1)(\alpha_1+1+i\upsilon_1)(\alpha_2+i\upsilon_2)(\alpha_2+1+i\upsilon_2)}$$

- Inverting thus yields the option price

$$C_T(k_1,k_2) = \frac{e^{-\alpha_1 k_1 - \alpha_2 k_2}}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i\upsilon_1 k_1 - i\upsilon_2 k_2}\psi_T(\upsilon_1,\upsilon_2)\,d\upsilon_2\,d\upsilon_1$$

- Compute this with a two-dimensional FFT

Moral of the story



4. Pricing Spread Options with the FFT

- Let us now try to price a call on the spread $\boldsymbol{S_1 - S_2}$

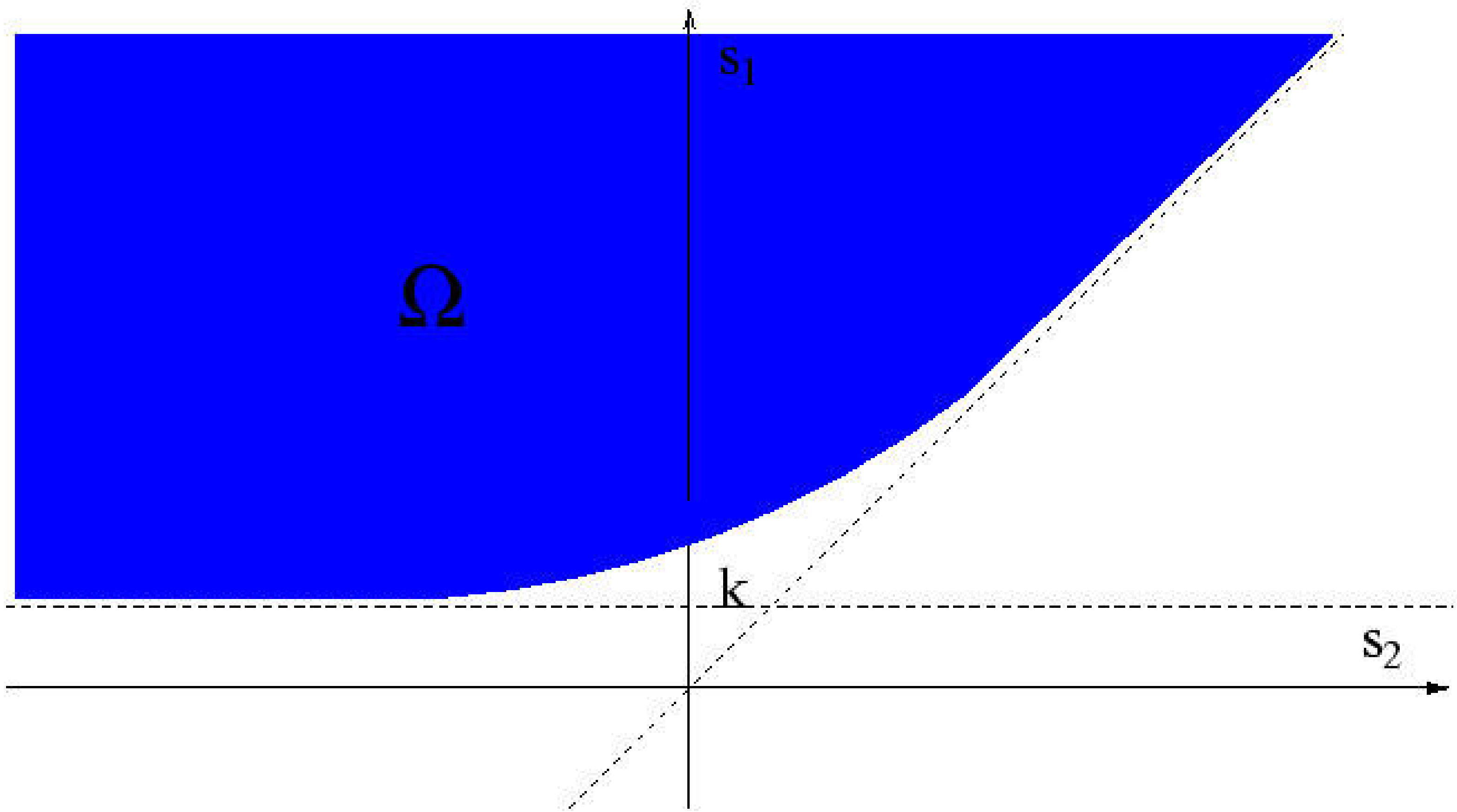
$$V_T(k) = E_Q\left[e^{-rT}(S_1 - S_2 - K)_+\right]$$

$$= e^{-rT}\int_{-\infty}^{\infty}\int_{\log(e^{s_2}+e^k)}^{\infty}(e^{s_1} - e^{s_2} - e^k)q_T(s_1, s_2)ds_1 ds_2$$

$$\equiv e^{-rT}\int\int_{\Omega}(e^{s_1} - e^{s_2} - e^k)q_T(s_1, s_2)ds_1 ds_2$$

- Big problem: the exercise region Ω to be integrated over has a curved boundary

$$\Omega := \left\{(s_1, s_2) \in R^2 \mid e^{s_1} - e^{s_2} - e^k \geq 0\right\}$$



The simple 2-D FFT (Bakshi & Madan 1999) trick will not work here!

Approximating the Exercise Region

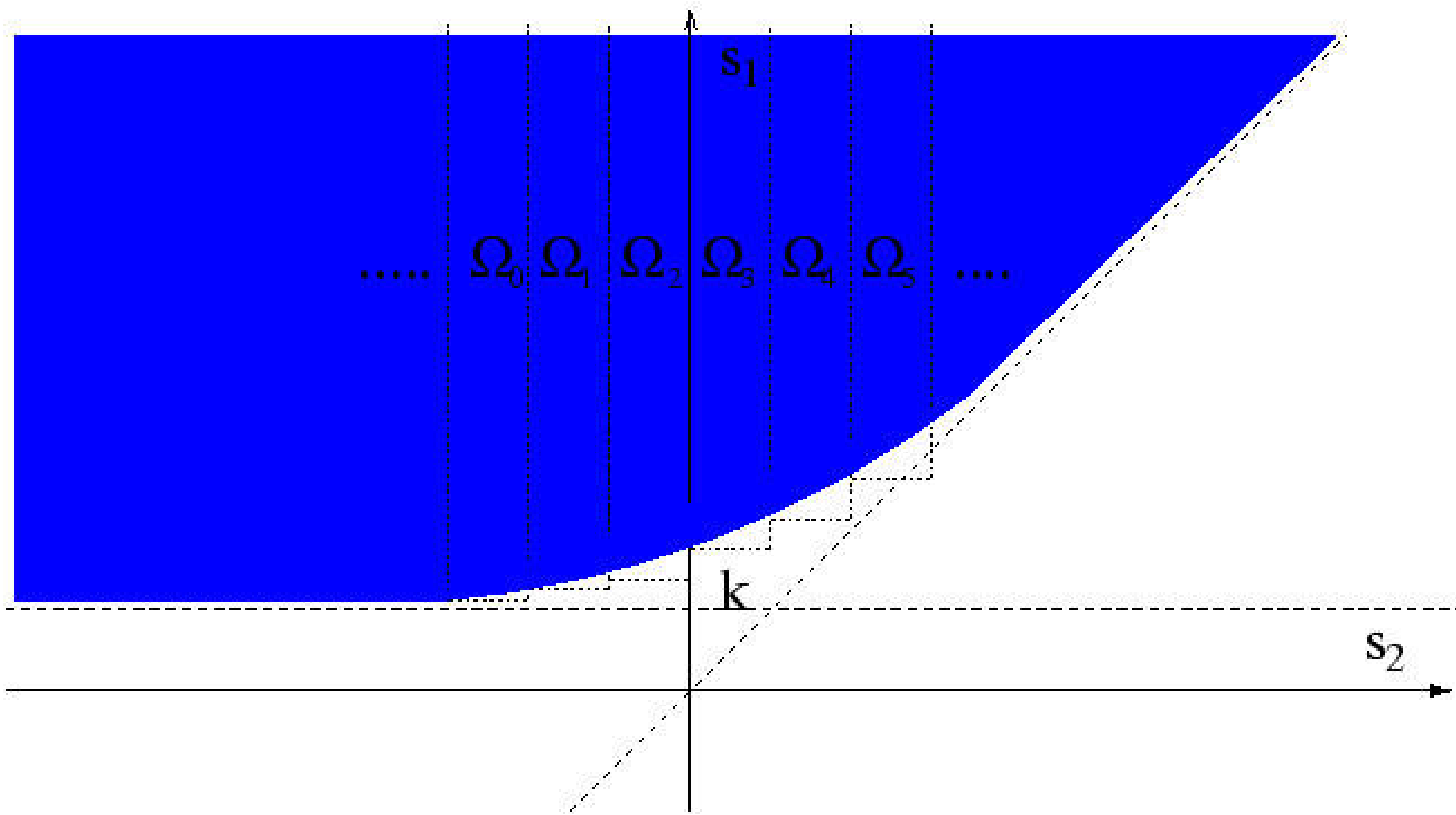
- Approximate it with rectangular strips (Riemann) as

$$V_T(k) = e^{-rT} \int\int_{\Omega} (e^{s_1} - e^{s_2} - e^{k}) q_T(s_1, s_2) ds_1 ds_2$$

$$\approx e^{-rT} \sum_{u=0}^{N-1} \int\int_{\Omega_u} (e^{s_1} - e^{s_2} - e^{k}) q_T(s_1, s_2) ds_1 ds_2$$

- The integral can be computed over each rectangular region Ω_u, $u=0,\ldots,N-1$

Riemann Approximation



Riemann approximation with rectangular strips

- We DON'T have to do N integrals!!!

- A single 2-D transform will produce $N \times N$ of

$$\int_{k_1(m)}^{\infty}\int_{k_2(n)}^{\infty}(e^{s_1}-e^{s_2}-e^{k})q_T(s_1,s_2)ds_1ds_2$$

for $m,n = 0, \ldots, N\text{-}1$

- These are sufficient for the N components we require since for different strikes k of the spread option we only need to pick different components to sum and no additional transform is needed

Why the FFT?

- Consider the following model :

$$dS_1 = S_1\left(r\,dt + \sqrt{v_1}\,dW_1\right)$$

$$dS_2 = S_2\left(r\,dt + \sqrt{v_2}\,dW_2\right)$$

$$dv_1 = \kappa_1(\mu_1 - v_1)dt + \sqrt{v_1}\,dW_3$$

$$dv_2 = \kappa_2(\mu_2 - v_2)dt + \sqrt{v_2}\,dW_4$$

with $E_Q\,[d\mathbf{W}_i\,d\mathbf{W}_j] = \rho_{ij}\,dt$

- Direct generalisation of 1-D stochastic volatility models with non-trivial correlation!

- No existing method can handle this!
  - conditioning trick won't work
  - lattice obviously fails...
  - a PDE in 4 space variables
  - slow convergence for Monte Carlo

- But easy (relatively) with the Fourier transform approach!
  - as the number of factors go up the payoff structure based on the price differences remains the same
  - the characteristic function involves more parameters and complicated expressions (naturally) but is still known in closed form
  - the transform will still be two dimensional

5. Computational Results

- Athlon 650 MHz with 512 MB RAM running Linux
- Code in C++
- Invoke Simpson's rule for approximation of the Fourier integral
- Use the award winning FFTW code ("Fastest Fourier Transform in the West") written by M Frigo and S Johnson from MIT (1999)

Pricing Spread Options under Two-factor GBM

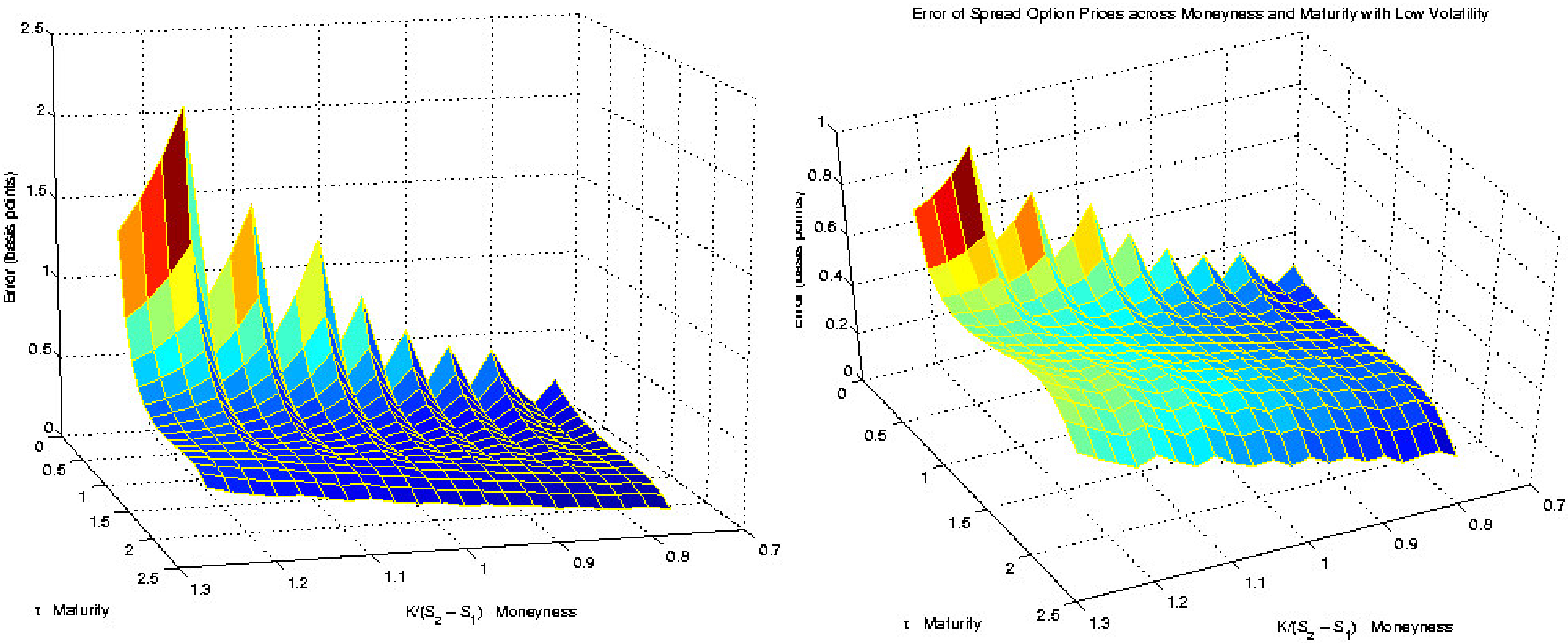
- First we compute spread option prices with the model (Existing Approach II):

$$dS_1 = S_1 (\mu_1 \, dt + \sigma_1 \, dW_1)$$
$$dS_2 = S_2 (\mu_2 \, dt + \sigma_2 \, dW_2)$$

where $E_Q [dW_1 \, dW_2] = \rho \, dt$

- We compare prices to those obtained by direct 1-D integration (using conditioning)

Errors in Spread Prices



Errors in Spread Prices across Strikes and Maturities for the FFT Method with High and Low Volatility N=4096

Error Variation With Strikes

Strike	Analytic	FFT	Error (b.p.)
0	6.56469	6.564078	0.932488
0.1	6.52267	6.522448	0.341628
0.2	6.480852	6.480436	0.641932
0.3	6.439226	6.439017	0.324435
0.4	6.397804	6.397531	0.426712
0.5	6.356578	6.356316	0.410849
0.6	6.315548	6.315321	0.359201
0.7	6.27472	6.27449	0.367451
0.8	6.234087	6.233878	0.335393
0.9	6.193652	6.19345	0.325424
1	6.153411	6.153223	0.306302
1.1	6.113369	6.113193	0.288202
1.2	6.07352	6.073361	0.261818
1.3	6.03387	6.033721	0.247201
1.4	5.994414	5.994279	0.2244
1.5	5.955153	5.95503	0.205267
1.6	5.916084	5.915977	0.181615
1.7	5.877211	5.877117	0.161329
1.8	5.838531	5.83845	0.138798
1.9	5.800047	5.79998	0.115989
2	5.761753	5.761697	0.098485

Maturity = 1.0
Interest Rate = 0.1
Initial price of Asset 1 = 100
Initial price of Asset 2 = 100
Dividend of Asset 1 = 0.05
Dividend of Asset 2 = 0.05
Volatility of Asset 1= 0.2
Volatility of Asset 1= 0.1
Correlation = 0.5

Number of Discretisation N = 4096
Integration step η = 1.0
Scaling factor α = 2.5

Table 1. Two-factor spread option prices across strikes

Accuracy of Alternative Methods

(Athlon 650 MHz with 512 MB RAM)

Fast Fourier Transform				Monte Carlo				
Number of Discretisation	Lower	Upper		Number of Simulations	Time Steps 1000		Time Steps 2000	
512	4.44	25.6		10000	129.15	0.051839	70.81	0.050949
1024	1.13	13.9		20000	22.34	0.036225	40.67	0.035899
2048	0.32	7.2		40000	7.44	0.025737	7.63	0.025733
4096	0.1	3.65		80000	18.34	0.018076	4.94	0.018184

Table 2. Accuracy of alternative methods for the two-factor geometric Brownian motion model in which the analytic price is available using direct integration: Error in basis points

Impact of Volatility and Correlation

Volatility of asset 2 \ Correlation	0.5	0	-0.5
0.1	6.675496 6.675800 (0.454684)	8.494941 8.495493 (0.649928)	9.979849 9.981407 (1.561482)
0.2	7.510577 7.511055 (0.636531)	10.549590 10.550798 (1.145356)	12.870614 12.873037 (1.882598)
0.3	9.712478 9.714326 (1.901766)	13.261339 13.264996 (2.757088)	16.01200 16.18352 (3.965540)

The first value is computed using the Fast Fourier Transform method.
The second value is the analytic price computed using the conditioning technique (the one-dimensional integral is evaluated using the qromb.c routine in Numerical Recipes in C).
The third value is the error of the FFT method in basis points.

Maturity = 1.0
Interest Rate = 0.1
Initial price of Asset 1 = 100
Initial price of Asset 2 = 95
Dividend of Asset 1 = 0.05
Dividend of Asset 2 = 0.05
Volatility of Asset 1= 0.2

Strike of the spread option = 5.0

Table 3. 2-factor spread option prices across volatilities and correlations

Pricing Spread Options under Three-factor Stochastic Volatility Models

$$dS_1 = S_1\left(r\ dt + \sigma_1\sqrt{v}dW_1\right)$$

$$dS_2 = S_2\left(r\ dt + \sigma_2\sqrt{v}dW_2\right)$$

$$dv = \kappa(\mu - v)dt + \sigma_v\sqrt{v}dW_v$$

$E_Q[d\boldsymbol{W_1}\ d\boldsymbol{W_2}] = \rho dt \quad E_Q[d\boldsymbol{W_1}\ d\boldsymbol{W_v}] = \rho_1 dt \quad E_Q[d\boldsymbol{W_v}\ d\boldsymbol{W_2}] = \rho_2 dt$

- Characteristic function is known in closed-form so that the FFT method is applicable
- Benchmark with Monte Carlo and finite difference methods

Characteristic Function of the 3-Factor Model

$$\phi_T(u_1,u_2) := E_{\mathbb{Q}}\left[\exp(iu_1 s_1(T) + iu_2 s_2(T))\right]$$

$$= \exp\Big[iu_1\big(rT + s_1(0)\big) + iu_2\big(rT + s_2(0)\big)$$

$$-\frac{\kappa\mu}{\sigma_v^2}\left[2\ln\left(1-\frac{(\theta-\Gamma)(1-e^{-\theta T})}{2\theta}\right)+(\theta-\Gamma)T\right]$$

$$+\frac{2\zeta(1-e^{-\theta T})}{2\theta-(\theta-\Gamma)(1-e^{-\theta T})}v(0)\Big]$$

Computing Time of Alternative Methods

(Athlon 650 MHz with 512 MB RAM)

Fast Fourier Transform

Number of	10 Strikes		100 Strikes	
Discretisation	GBM	SV	GBM	SV
512	1.04	1.11	1.1	1.2
1024	4.28	4.64	4.48	4.83
2048	18.46	19.54	18.42	19.74
4096	74.45	81.82	76.47	81.27

Monte Carlo: 1000 Time Steps

Number of	10 Strikes		100 Strikes	
Simulations	GBM	SV	GBM	SV
10000	38.2	144.87	41.95	151.75
20000	76.22	288.09	83.81	303.31
40000	152.5	576.25	168.48	606.53
80000	304.95	1152.9	335.2	1212.76

Monte Carlo: 2000 Time Steps

Number of	10 Strikes		100 Strikes	
Simulations	GBM	SV	GBM	SV
10000	75.57	287.41	79.83	295.21
20000	157.28	574.18	159.08	590.23
40000	303.37	1149.25	317.49	1184.32
80000	606.4	2298.37	636.33	2359.05

Table 4. Computational time (seconds) of alternative methods for the two-factor Geometric Brownian motion model and the three-factor Stochastic Volatility model

Spread Option Prices by Alternative Methods

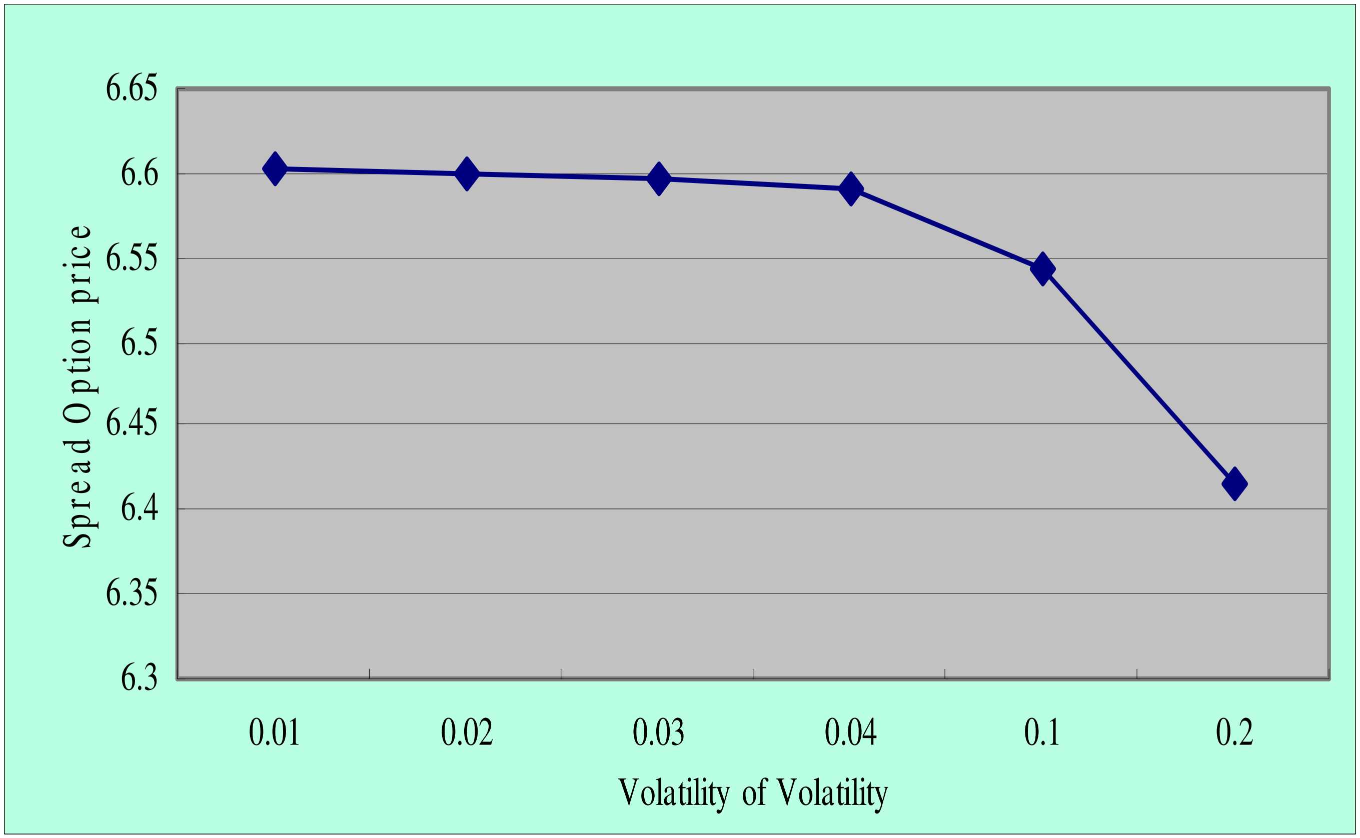
Fast Fourier Transform		
N	Lower	Upper
512	5.059379	5.068639
1024	5.062695	5.067405
2048	5.063545	5.065897
4096	5.063755	5.064492

Explicit Finite Difference		
No. of Discretisation		
Space	Time	Price
100 * 100 * 100	400	5.0845
100 * 100 * 100	1600	5.0769
100 * 100 * 100	2500	5.076
100 * 100 * 100	10000	5.0748
200 * 200 * 100	1600	5.0703
200 * 200 * 200	1600	5.0703
200 * 200 * 100	2500	5.0694
200 * 200 * 100	10000	5.0682
300 * 300 * 100	4000	5.0668

	Monte Carlo Simulation		
Number of			
Simulation	Steps	Price	error)
1280000	1000	5.052372	0.004301
1280000	2000	5.053281	0.004297
1280000	4000	5.037061	0.004286
2560000	1000	5.04989	0.003039
2560000	2000	5.051035	0.003039
2560000	4000	5.042114	0.003037
5120000	1000	5.047495	0.002148
5120000	2000	5.046263	0.002148

Table 2. Accuracy of alternative methods for the three-factor Stochastic Volatility model

Price Variation With the Volatility of the Stochastic Volatility



Stochastic Volatility parameters

$T = 1.0$

$r = 0.1$

$K = 2.0$

$S_1(0) = 100$

$S_1(0) = 98$

$\delta_1 = \delta_2 = 0.05$

$\sigma_1 = 1.0$

$\sigma_2 = 0.5$

$\rho = 0.5$

$v(0) = 0.04$

$\kappa = 1.0$

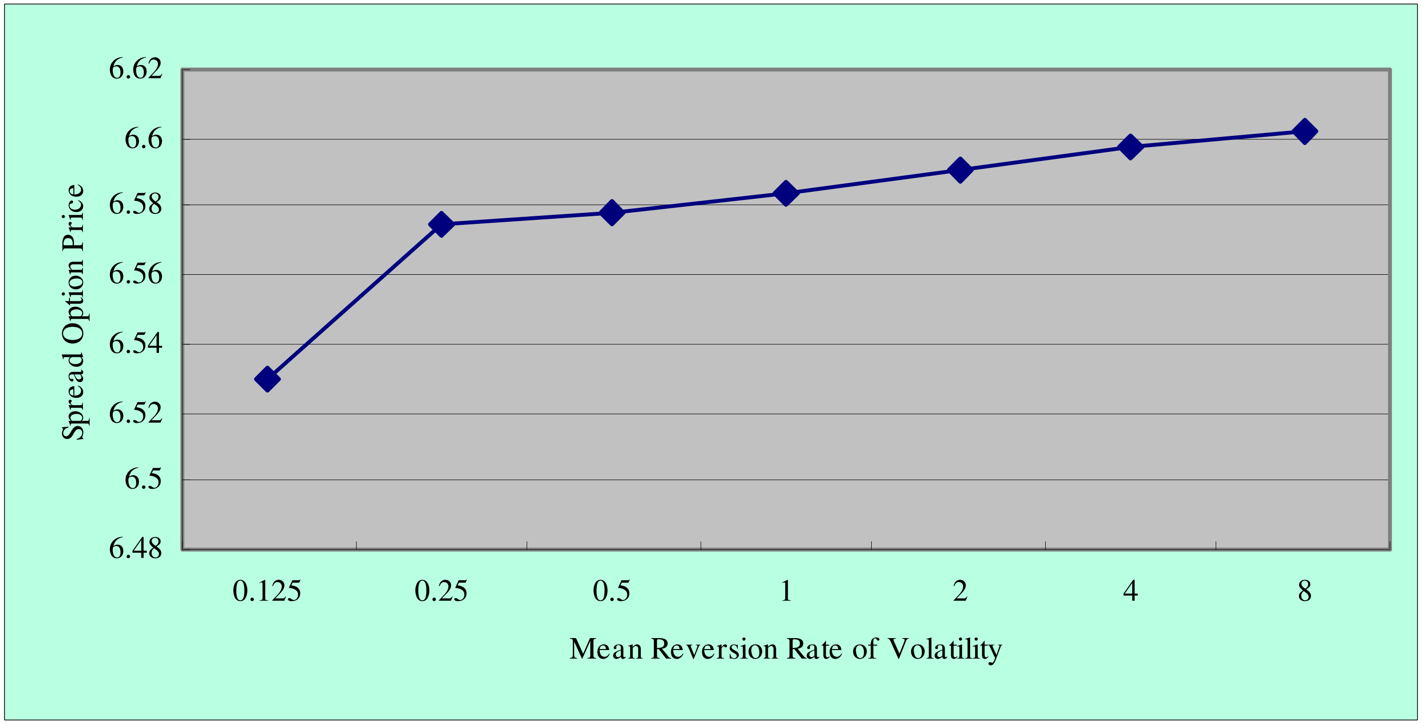
$\mu = 0.04$

$\rho_1 = -0.25$

$\rho_2 = -0.5$

Figure 2. Spread option prices under Three-factor Stochastic Volatility Model with varying volatility σ_v of the stochastic volatility V

Price Variation With the Mean Reversion Rate of Volatility



Stochastic Volatility parameters

$T = 1.0$

$r = 0.1$

$K = 2.0$

$S_1(0) = 100$

$S_1(0) = 98$

$\delta_1 = \delta_2 = 0.05$

$\sigma_1 = 1.0$

$\sigma_2 = 0.5$

$\rho = 0.5$

$v(0) = 0.04$

$\sigma_v = 0.05$

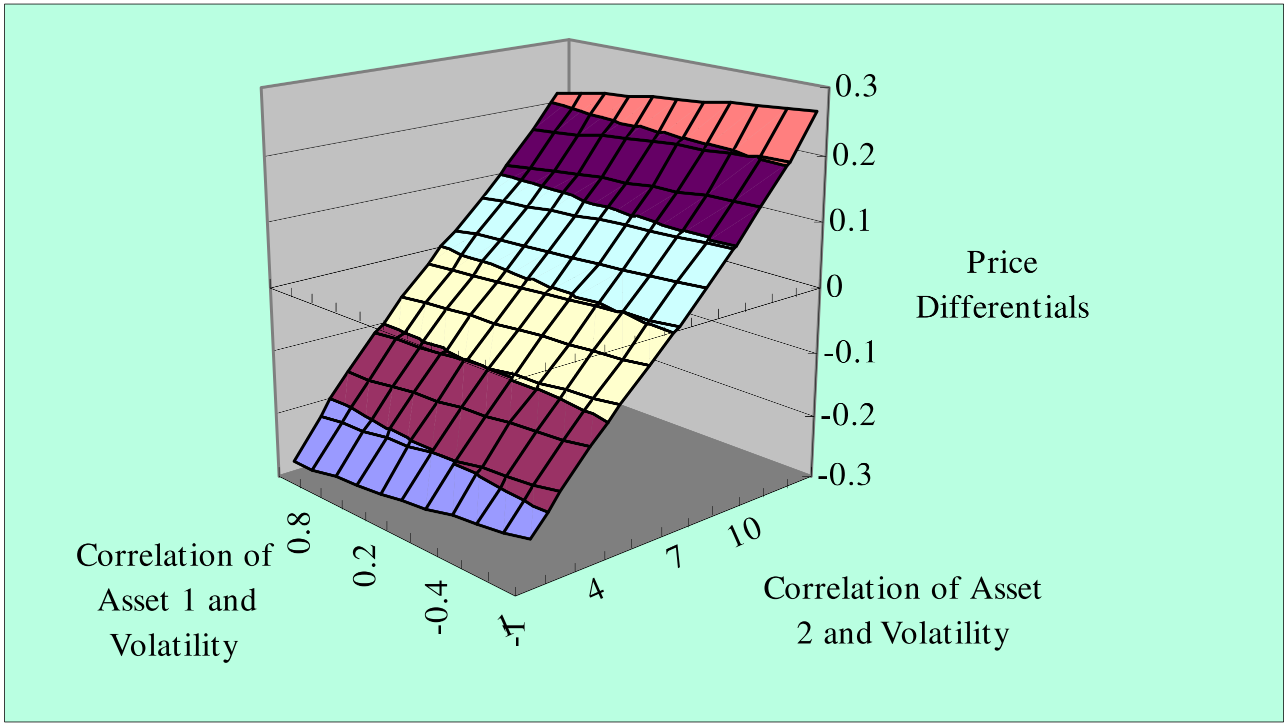
$\mu = 0.04$

$\rho_1 = -0.25$

$\rho_2 = -0.5$

Figure 3. Spread option prices under Three-factor Stochastic Volatility Model with varying mean reversion rate κ, of the stochastic volatility V

Comparison of Two-factor and Three-factor Prices



Stochastic Volatility parameters

$T = 1.0$

$r = 0.1$

$K = 2.0$

$S_1(0) = 100$

$S_1(0) = 98$

$\delta_1 = \delta_2 = 0.05$

$\sigma_1 = 1.0$

$\sigma_2 = 0.5$

$\rho = 0.5$

$v(0) = 0.04$

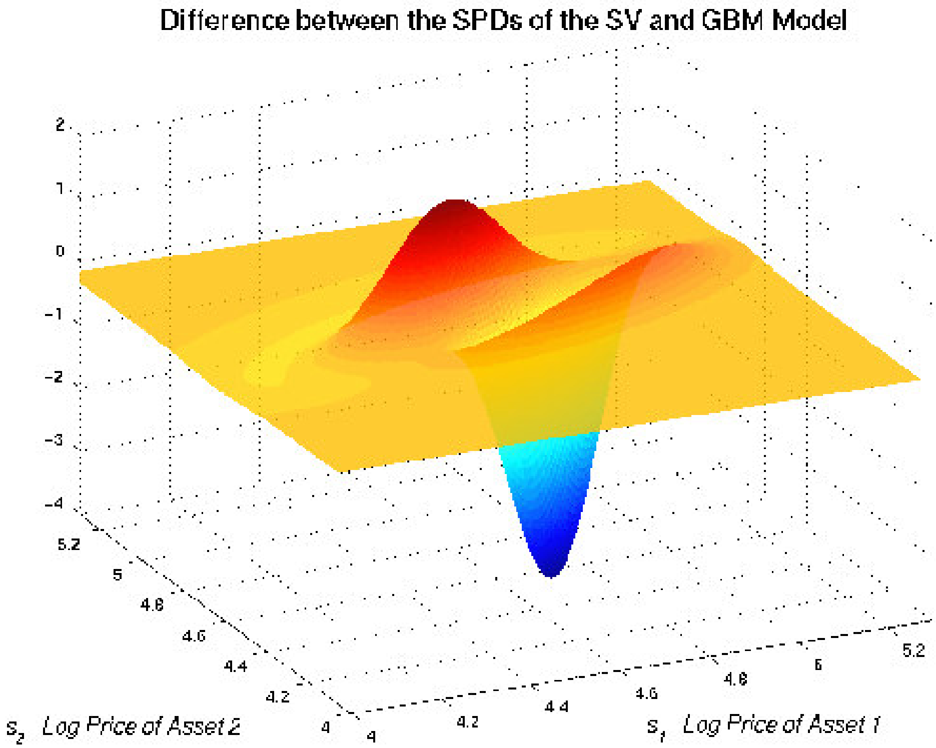
$\sigma_v = 0.05$

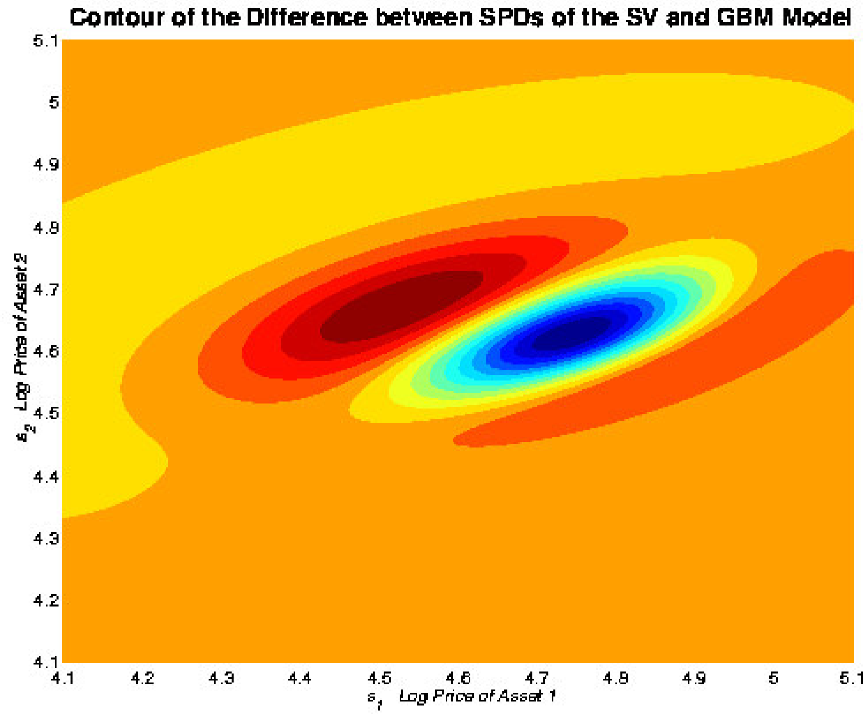
$\kappa = 1.0$

$\mu = 0.04$

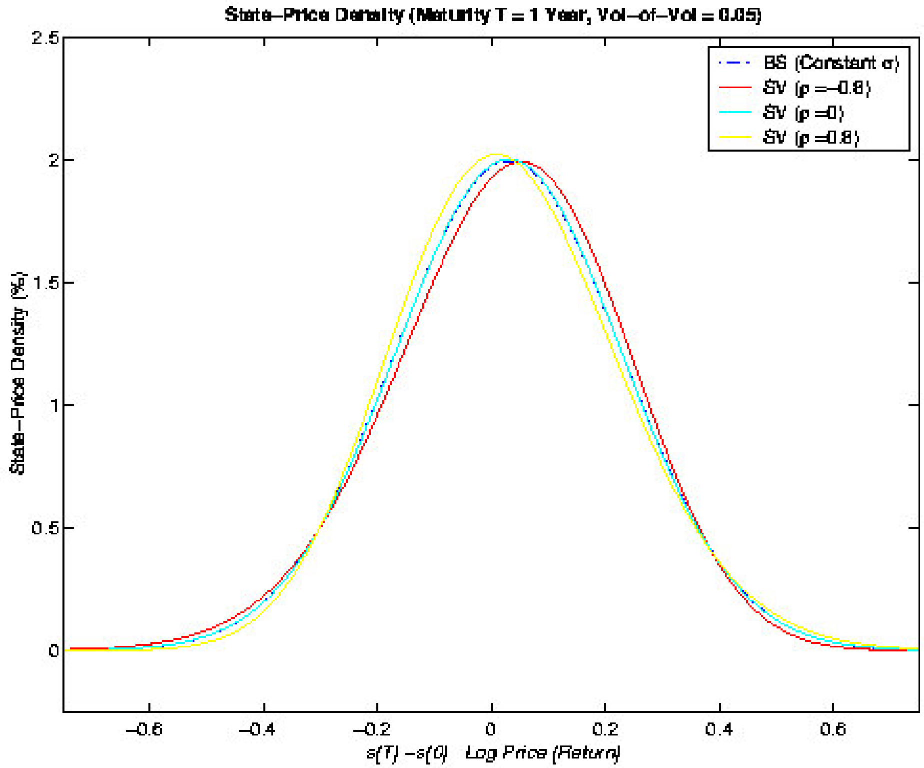
Figure 5. Price difference between Three-factor Stochastic Volatility Model and the Two-factor Geometric Brownian motion model (with implied constant volatilities and correlation)

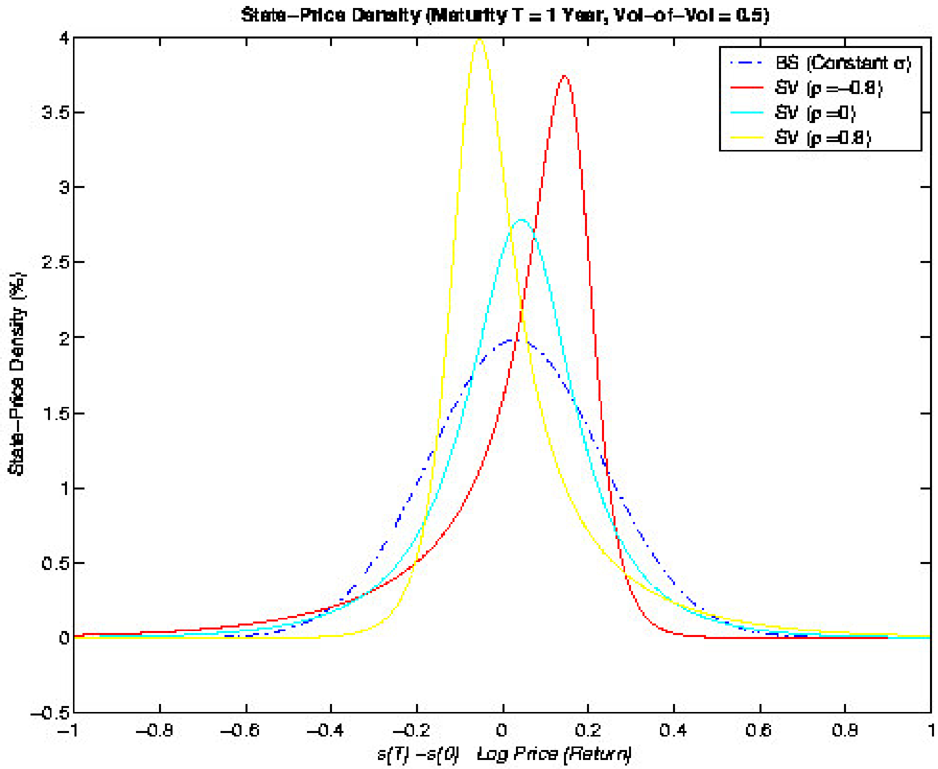
Comparison of Two-factor and Three-factor State Price Densities





State-Price Densities





State-Price Densities of Constant and Stochastic Volatility models, with volatility of (a) 5% (b) 50%

A Stochastic Volatility Model with Jumps

$$ds_1 = (r - \delta_1 - \frac{1}{2}\sigma_1^2 v - \lambda_1 \mu_1)dt + \sigma_1 \sqrt{v}\, dW_1 + \log(1 + J_1)dN_1$$

$$ds_2 = (r - \delta_2 - \frac{1}{2}\sigma_2^2 v - \lambda_1 \mu_2)dt + \sigma_2 \sqrt{v}\, dW_2 + \log(1 + J_2)dN_2$$

$$dv = \kappa(\mu - v)dt + \sqrt{v}\, dW_v$$

where N_1, N_2 are orthogonal Poisson processes independent of W_1, W_2, W_v with constant arrival rates λ_1, λ_2

- Specifically $\mathbb{Q}(dN_i(t) = 1) = \lambda_i dt$ with probability generating functions $\mathbb{E}_{\mathbb{Q}}[s^{N_i(t)}] = \exp(\lambda_i (s-1)t]$

Market Calibration

Difficult to obtain OTC spread option data

A challenging econometric task because…

- Absence of closed-form expressions for risk-neutral and objective probability density functions
- Presence of a latent variable the stochastic volatility process
- Difficult to combine the spot price and panel option price data (a highly non-linear function in the log-spot price) optimally
- Neither the Maximum Likelihood (ML) method nor Kalman filters can be applied (the common obstacle faced by SV model estimation)

Furthermore, there are two underlying assets now requiring

- Simultaneous estimation of two combined stochastic volatility models
- Calibration of a correlation surface
- Possibility of "correlation smiles" and "correlation skews"

Recent Advances in the Estimation of the Stochastic Volatility Models

Extensive interest in estimating single-asset SV models and some of the most promising econometric techniques include:

- Generalized Method of Moments (GMM):
  Hansen (1985), Pan (2000), Bollerslev & Zho (2000)
- Simulated Method of Moments (SMM):
  Duffie & Singleton (1993), Bakshi,Cao, Chen (1997, 2000)
- Efficient Method of Moments (EMM):
  Gallant & Tauchen (1996), Chernov & Ghysels (2000)

However, due to the complexity and computational burden, very few have considered the problem of estimating a *multi-asset* SV model!

Extension to Multiple Assets

Data assumed available:

- spot prices $\hat{S}_1, \hat{S}_2$
- a panel of vanilla option prices on individual assets across strikes and maturities e.g.
  - calls on asset 1: $\hat{C}_1^{n_1} \quad n_1 = 1,.., N_1$
  - calls on asset 2: $\hat{C}_2^{n_2} \quad n_2 = 1,.., N_2$

Model:

- Three-factor Stochastic Volatility: (S_1, S_2, v)

Parameters to be estimated:

$$\Theta_1 := \{\delta_1, \sigma_1, \rho_1, \kappa, \mu, \sigma_v\} \qquad \Theta_2 := \{\delta_2, \sigma_2, \rho_2, \kappa, \mu, \sigma_v\} \qquad \Theta_3 := \{\rho\}$$

- the first two sets Θ_1, Θ_2 contain structure parameters when the model is each viewed as single asset SV models
- Θ_0 contains the parameters which are crucial in determining the state-dependent correlation structure between the underlying assets

Outline of the Calibration Procedure

1. For $i = 1,2$, conduct parallel estimation of structural parameters in Θ_i based on panel data of call prices $\hat{C}_i^{n_i} \quad n_i = 1,\dots,N$ using either SMM, Ordinary Least Squares or any previously mentioned single-asset SV model estimation procedure

2. Based on the structural parameters obtained filter the unobserved volatility process V_t with an OLS procedure on the combined option data $\left\{\hat{C}_1^{n_1}(t), \hat{C}_2^{n_2}(t)\right\}$

3. Estimate the correlation-dependent parameters in Θ_0 by…

Estimation of correlation sensitive parameters I

If reliable market data on **spread option prices** are **unavailable**,

- Use the spot price time-series $\left\{\hat{S}_1(t), \hat{S}_2(t)\right\}$
- Compute the correlation function between the terminal spot prices $S_1(T), S_2(T)$ conditional on $S_1(t), S_2(t), \nu(t)$ by differentiating the characteristic functions and evaluating the second moments
- Fix the parameters in Θ_1, Θ_2 which appear in this correlation function to the estimates obtained so far so that the spot price correlation is a univariate function in $\rho \in \Theta_0$
- Manipulate ρ to take a forward view on the spot price correlation using this function or evaluate its inverse at the historical value of the spot price correlation for an optimal estimate $\hat{\rho}$ of ρ

Correlation structure between asset prices under SV

- This refers to the terminal correlation between $\boldsymbol{S}_1(T)$, $\boldsymbol{S}_2(T)$ conditional on the time-t state $x_t=(s_1(t), s_2(t), v(t))$ over the horizon [t,T]

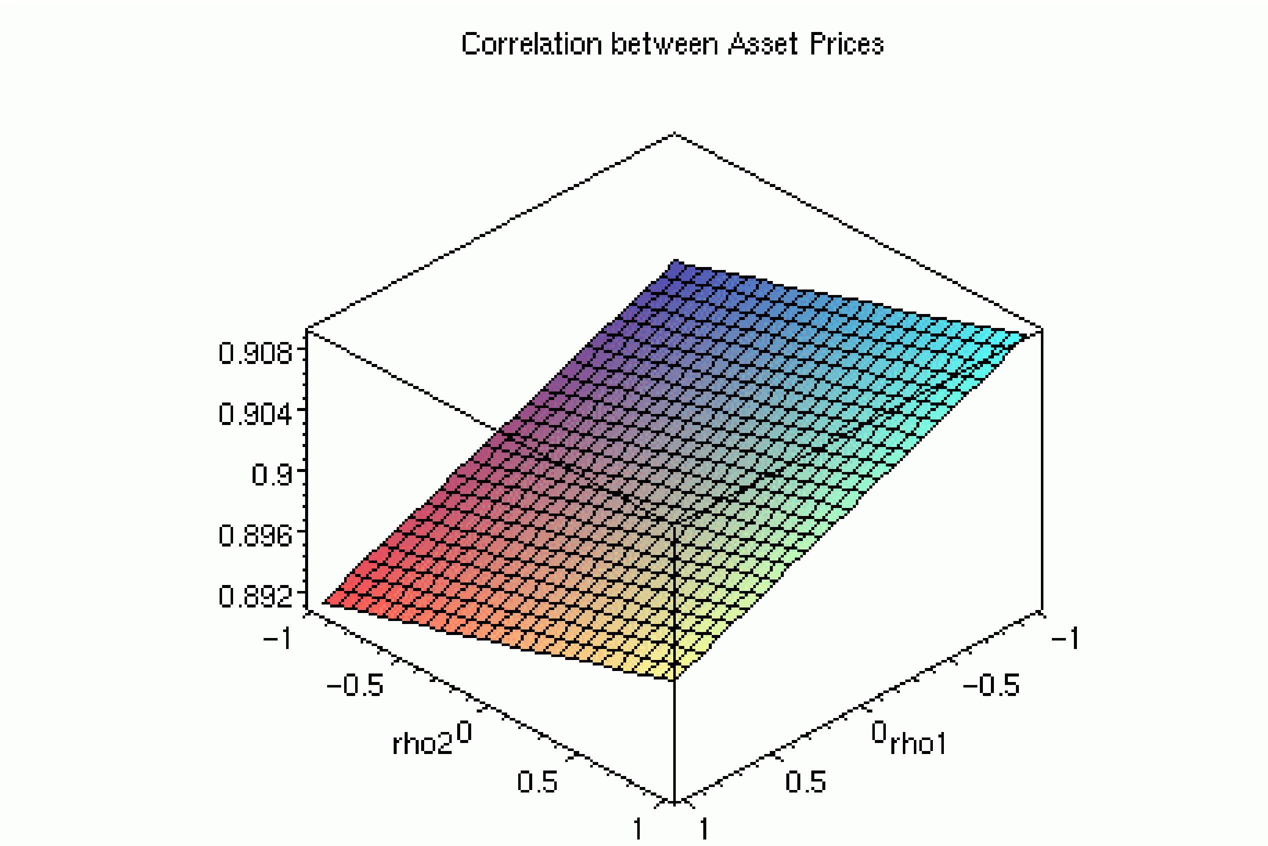
$$\rho(T) := \frac{v_{12}(T)}{\sqrt{v_{11}(T)v_{22}(T)}}$$

$$v(T) := \mathbb{E}_{\mathbb{Q}}\left[\left(s(T) - \mathbb{E}_{\mathbb{Q}}[\boldsymbol{s}(T)]\right)\left(\boldsymbol{s}(T) - \mathbb{E}_{\mathbb{Q}}[\boldsymbol{s}(T)]^{\mathrm{T}}\right)\right]$$

$$= -\left.\begin{pmatrix} \dfrac{\partial^2 \phi_T}{\partial u_1^2} - \left(\dfrac{\partial \phi_T}{\partial u_1}\right)^2 & \dfrac{\partial^2 \phi_T}{\partial u_1 \partial u_2} - \left(\dfrac{\partial \phi_T}{\partial u_1}\right)\left(\dfrac{\partial \phi_T}{\partial u_2}\right) \\ \dfrac{\partial^2 \phi_T}{\partial u_1 \partial u_2} - \left(\dfrac{\partial \phi_T}{\partial u_2}\right)\left(\dfrac{\partial \phi_T}{\partial u_1}\right) & \dfrac{\partial^2 \phi_T}{\partial u_2^2} - \left(\dfrac{\partial \phi_T}{\partial u_2}\right)^2 \end{pmatrix}\right|_{u=0}$$

- Can be calculated by differentiating the characteristic function using Maple

Correlation between Asset Prices as a Function of Correlation Parameters between Asset Prices and Volatility



$T = 1.0$

$r = 0.1$

$S_1(0) = 100$

$S_1(0) = 96$

$\delta_1 = \delta_2 = 0.05$

$\sigma_1 = 1.0$

$\sigma_2 = 0.5$

$\rho = 0.9$

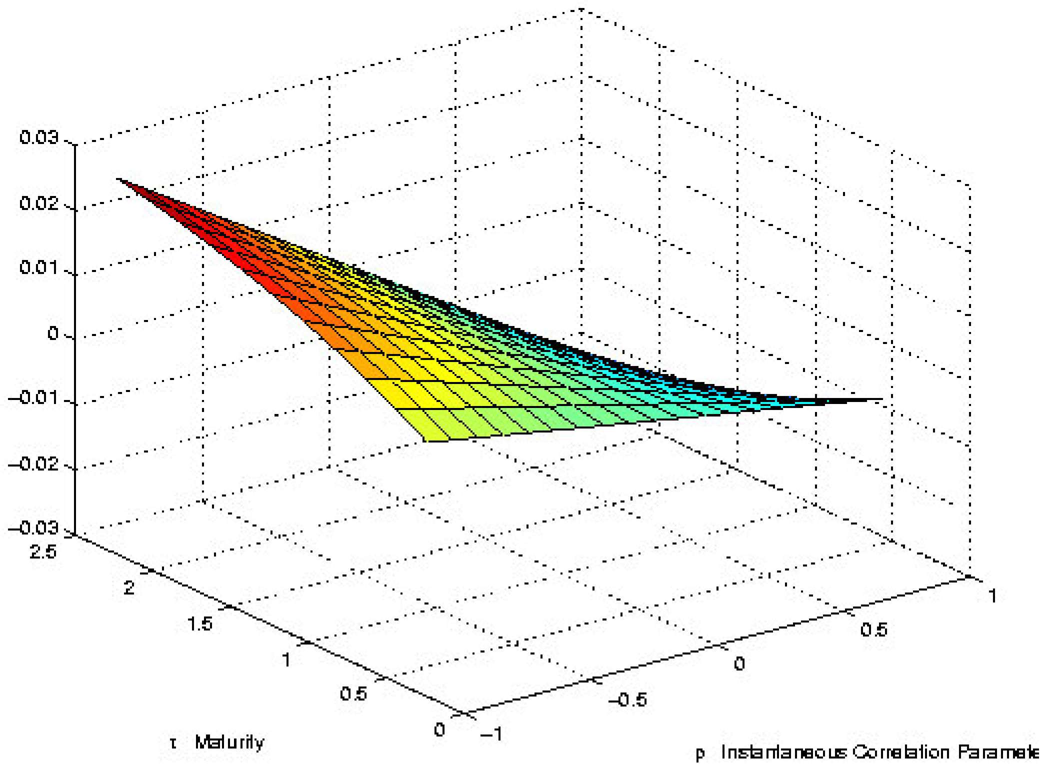
$v(0) = 0.04$

$\sigma_v = 0.05$

$\kappa = 1.0$

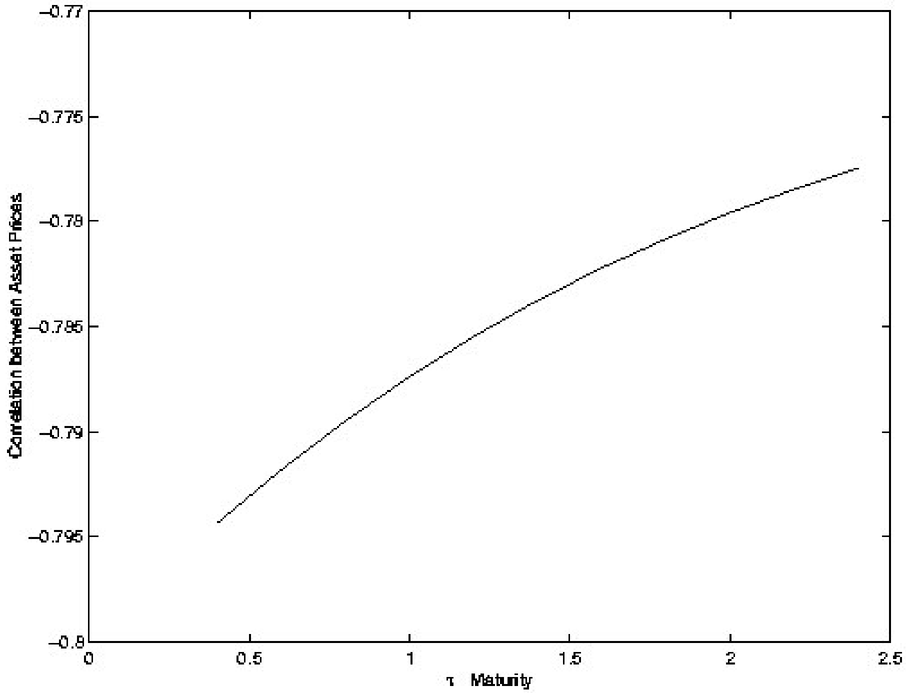
$\mu = 0.04$

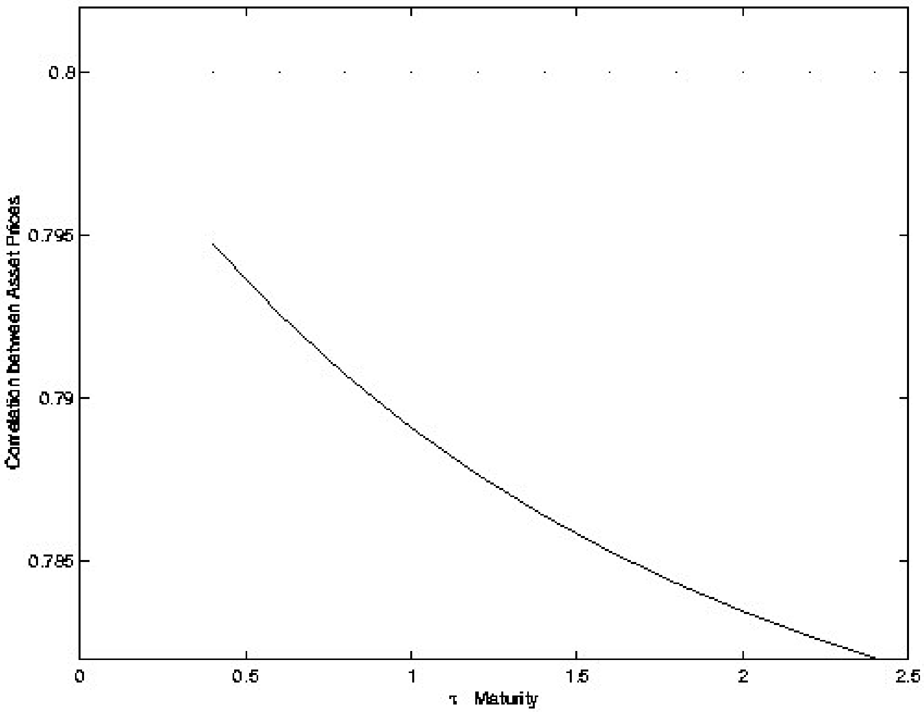
Excess Correlation at Maturity



Terminal Correlation less the Instantaneous Correlation Parameter ρ

Correlation at Maturity





Terminal Correlation between Asset Prices (a) ρ=-0.8, (b) ρ=0.8

Estimation of correlation sensitive parameters II

If spread option price data $\hat{V}^n(t)$ are observed for a range of strikes and maturities

- Take the parameter estimates and filtered volatility obtained above as fixed and compute the dependence of the theoretical spread option price $V^n(t;\rho)$ on $\rho \in \Theta_0$ using the FFT pricing method

- Minimize the sum-of-square pricing errors to obtain an estimate

$$\hat{\rho} = \arg\min_{\rho \in \Theta_0} \sum_{t=0}^{T} \sum_{n=1}^{N} \left| V^n(t;\rho) - \hat{V}^n(t) \right|^2$$

Conclusions and Future Work

- Existing approaches are unable to price spread options beyond two-factor GBM models
- The Fast Fourier Transform provides a robust method for pricing spread options with more factors under stochastic volatility and correlation, general affine models, etc
- Computation times do not increase with the number of random factors in the diffusion model
- Method is applicable to other exotic options
- Also have a 4-factor model which allows full freedom to the future correlation surface…
- Fast wavelet transform *O(N)* versus *O(N log N)* for FFT