Asset Pricing with Multiple Subjective Probability Measures

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December 5, 2003

Outline

- Pricing in complete and incomplete markets
- One period case
- Multiperiod case
- Conclusions

Pricing in a complete market

- All assets are replicable

- Arbitrage pricing theory gives a unique price with either:
  - The replicating portfolio
  - The unique state price density

Pricing in an incomplete market

- Not all assets are replicable

- Arbitrage pricing theory can only give bounds on the prices of non-replicable assets using either:
  - Super and sub replication
  - The infinite number of state price densities

- Soner, Shreve, and Cvitanic(1995) show that this bound can be impractically wide

3 methods for pricing in an incomplete market

- Pick one state price density out of the infinite number of possible state price densities
  - Minimize the distance to a prior density (Rubenstein(1994))
  - Minimize the squared hedge error (Follmer and Schweitzer(1991))

- Utility maximization approach. Find the price of the asset which makes an investor indifferent to holding it in their portfolio. (Davis(1997) and Monoyios(2001))

- Tighten the arbitrage pricing bounds by ruling out good-deals as well as arbitrage (restrict the set of possible state price densities)
  - Good-deals as investments with a sufficiently high Sharpe Ratio (Cochrane and Saa-Requejo(1999))

– Good-deals as investments with a sufficiently high utility (Cerny and Hodges(2001))

Real World and Subjective Probability Measures

- The real world probability measure describes the actual distribution of future payoffs and prices

- Each of the three methods mentioned above assume that this measure is known to some extent

- In practice this measure is subjective as it depends on a number of unobservable and/or stochastic factors such as consumer confidence, fiscal policy and nature

- Represent uncertainty in the real world measure with multiple subjective measures

One period case - set up

- Two times - $t = 0$ and $t = 1$

- Uncertainty is represented by K states of the world at $t = 1$ - $\omega_1, \ldots, \omega_K$

- There are M subjective probability measures, $P^1, \ldots, P^M$, each of which assigns positive probability to a subset of the K states

- The state probabilites across the probability measures are given by P

$$P = \begin{pmatrix} P^1(\omega_1) & \ldots & P^M(\omega_1) \\ \vdots & & \vdots \\ P^1(\omega_K) & \ldots & P^M(\omega_K) \end{pmatrix}$$

- There are N assets whose payoffs across the states are given by δ

$$\delta = \begin{pmatrix} \delta^1(\omega_1) & \dots & \delta^1(\omega_K) \\ \vdots & & \vdots \\ \delta^N(\omega_1) & \dots & \delta^N(\omega_K) \end{pmatrix}$$

- The expected payoffs of the assets across the probability measures are given by C

$$C = \delta P = \begin{pmatrix} E^1[\delta^1] & \dots & E^M[\delta^1] \\ \vdots & & \vdots \\ E^1[\delta^N] & \dots & E^M[\delta^N] \end{pmatrix}$$

- Prices of the assets are given by $S \in \Re^N$

- Portfolios are given by $\theta \in \Re^N$

Agents

- Agents are defined by a strictly increasing and continuous utility function $U : \Re^M_+ \rightarrow \Re$ and a finite expected endowment $e \in \Re^M_+$

- The agent's budget feasible set is given by the set of portfolios with zero initial price which lead to non-negative wealth under each subjective measure

$$X(C, S, e) = \{e + C'\theta \in \Re^M_+ : \theta \in \Re^N, \theta' S = 0\}$$

- The agent's optimization problem is to mazimize utility subject to the budget constraint

$$\max U(c)$$
$$\text{s.t. } c \in X(C, S, e)$$

Equilibrium and the economy

- There are Z agents defined by utility functions, $U_1, \ldots, U_Z$, and endowments, $e_1, \ldots, e_Z$

- The economy is given by the set of agents and the expected payoff matrix, $[(U_i, e_i)_{i=1}^{Z}, C]$

- An equilibrium for the economy is given by a portfolio for each investor and a price vector, $(\theta_1, \ldots, \theta_Z, S)$, such that θ_i solves agent $i's$ optimization problem for $i = 1, \ldots, Z$ and $\sum_{i=1}^{Z} \theta_i = 0$

Strictly acceptable opportunities and optimality

- An equilibrium can not exist if a solution to an agent's optimization problem does not exist

- **Definition:** Given the economy and a price vector, a strictly acceptable opportunity (SAO) is a $\theta \in \Re^N$ such that either:

  - $\theta' S \leq 0$ and $C'\theta > 0$
  - $\theta' S < 0$ and $C'\theta \geq 0$

- **Theorem**: A solution to an agent's optimization problem exists if and only if there are no SAOs

- Result: The absence of SAOs is a necessary condition for the existence of an equilibrium

Measure price vectors and strictly acceptable opportunities

- **Definition:** Given the economy and a price vector, a measure price vector (MPV) is a $w \gg 0$ in $\Re^M$ such that $S = Cw$

- If a MPV exists the prices of assets and portfolios are equal to the inner product of their expected payoffs with the MPV

- The i^{th} component of a measure price vector is the price of an asset that gives an expected payoff of 1 under the m^{th} probability measure and 0 under all others. Thus, w can be interpreted as Arrow-Debreu securities defined over the probabilty measures rather than the states

- **Theorem**: There is a MPV if and only if there are no SAOs

The no-arbitrage framework as a special case

- **Proposition**: If P is the $(K \times K)$ identity matrix I_K, i.e. there is one probability measure for each state placing unit mass on that state and 0 on all others, then this framework reduces to the no-arbitrage framework

- **Proposition**: If there are no SAOs, then there are no arbitrage opportunities

- Thus if there is a measure price vector then there is a state price vector

- **Proposition**: There is a MPV if and only if there is a representative state price vector (RSPV) - a state price vector q such that $q = Pw$ with $w \gg 0$

Acceptable completeness and uniqueness of the MPV

- **Definition**: The market is acceptably complete if for every $x \in \Re^M$ there is a $\theta \in \Re^N$ such that $C'\theta = x$, i.e. all assets are replicable in terms of expected payoffs in measures

- **Theorem**: The MPV is unique if and only if the market is acceptably complete
  - If the market is not acceptably complete will be an infinite number of MPVs

- **Proposition**: If the market is complete than the market is acceptably complete

Derivative pricing

- Since the absence of SAOs is a necessary condition for the existence of an equilibrium, only economies and price vectors which preclude SAOs are allowed

- Let $Q = \{C'\theta : \theta \in \Re^N\}$ denote the expected payoffs market subspace or the expected payoffs obtainable by trading in the N assets

- Let $r \in \Re^M$ denote the expected payoffs of a derivative under the M probability measures

- If $r \in Q$ then we say that the derivative is replicable (in terms of expected payoffs)

- **Proposition**: If $r \in Q$, $C'\theta = r$ and w is a MPV, then the unique price of the derivative is given by $\theta' S$ or $r' w$

- If there is a unique price in the no-arbitrage framework then there is also one in this framework. However, there may be a unique price in this framework even if there is not one in the no-arbitrage framework

- If $r \notin Q$ then upper and lower bounds for the derivative price may be obtained by super- and sub-replication

  – An upper bound for the price of the derivative is given by the following super-replication LP:

$$\min_{\theta} \theta' S$$
$$\text{s.t. } C'\theta \geq r$$

  – A lower bound for the price of the derivative is given by the following sub-replication LP:

$$\max_{\theta} \theta' S$$

$$\text{s.t. } C'\theta \leq r$$

– **Corollary:** The pricing bound obtained in this framework is at least as narrow as the pricing bound obtained in the no-arbitrage framework

Good-deal pricing

- Hansen and Jagannathan(1991) showed that placing an upper bound on allowable Sharpe ratios implied an upper bound on the volatility of discount factors consistent with observed prices

- Cochrane and Saa-Requejo(1999) termed investments with Sharpe ratios above this upper bound good-deals, and argued that such opportunities should not exist because investors would want to trade good-deals as well as arbitrages

- The assumed absence of such good-deals placed a good-deal upper bound on the volatility of discount factors. The valuation bounds resulting from the good-deal restricted set of discount factors is then shown to be narrower than those given by standard arbitrage pricing theory which considers the entire set of discount factors

- Cerny and Hodges(2001) extended the idea of good-deals to arbitrarily defined sets of investments and to sets of investments determined by utility functions

- Carr, Geman, and Madan(2001) derive this asset pricing framework by assuming that strictly acceptable opportunities should not exist because investors would want to trade such opportunities

- Thus, the method that Carr, Geman and Madan(2001) use to derive this asset pricing framework can also be can be considered a good-deal approach where a good-deal is a SAO

Multiperiod case - set up

- $T+1$ times - $t=0,\dots,T$
- Uncertainty is represented by K states of the world an a filtration F
- M subjective probability measures, $P^1,\dots,P^M$
- The price process of the N assets is given by $S=\{S_t : t=0,\dots,T\}$
- The dividend process of the N assets is given by $\delta=\{\delta_t : t=0,\dots,T\}$
- A trading strategy is given by $\theta=\{\theta_t : t=0,\dots,T\}$
- The payoff process generated by a trading strategy θ is given by $Y^\theta = \{Y_t^\theta : t=0,\dots,T\}$ where:

$$Y_t^\theta(\omega_k) = \theta_{t-1}(\omega_k)'(S_t(\omega_k) + \delta_t(\omega_k)) - \theta_t(\omega_k)'S_t(\omega_k)$$

- The expected payoff process generated by a trading strategy θ is given by $R^\theta = \{R_t^\theta : t = 0, \ldots, T\}$ where:

$$R_t^\theta(\omega_k) = \begin{pmatrix} E_{t-1}^1[\boldsymbol{Y_t^\theta}](\omega_k) \\ \vdots \\ E_{t-1}^M[\boldsymbol{Y_t^\theta}](\omega_k) \end{pmatrix} \tag{1}$$

- Let L denote the space of M dimensional processes from $t = 0, \ldots, T$

Agents

- An agent is defined by a strictly increasing and continuous utility function $U : L_+ \rightarrow \Re$ and a bounded expected endowment $e \in L_+$

- The agent's budget feasible set is given the trading strategies that lead to non-negative expected payoff processes

$$X(\delta, S, e, F) = \{e + R^{\theta} \in L_+ : \theta \in \Theta\}$$

- The agents optimzation problem is to maximize utility subject to the budget constraint

$$\max U(c)$$
$$\text{s.t. } c \in X(\delta, S, e, F)$$

Equilibrium and the economy

- There are Z agents defined by utility functions, $U_1, \ldots, U_Z$, and endowments, $e_1, \ldots, e_Z$

- The economy is given by the set of agents, the dividend process and the filtration, $[(U_i, e_i)_{i=1}^{Z}, \delta, F]$

- An equilibrium for the economy is given by a trading strategy for each agent and a price process, $(\theta_1, \ldots, \theta_Z, S)$, such that θ_i solves the i^{th} agent's optimization problem for $i = 1, \ldots, Z$ and $\sum_{i=1}^{Z} \theta_i = 0$

Strictly acceptable opportunities and optimality

- An equilibrium can not exist if a solution to an agent's optimization problem does not exist

- **Definition:** Given the economy and a price process, a SAO is a trading strategy $\theta \in \Theta$ such that $R^\theta > 0$

- **Theorem:** There is a solution to an agent's optimization problem if and only if there are no SAOs

- Result: The absence of SAOs is a necessary condition for the existence of an equilibrium

Measure price deflators and strictly acceptable opportunities

- **Definition:** Given the economy and a price process, a measure price deflator (MPD) is a process $w \in L_{++}$ such that for each $\theta \in \Theta$, $\theta_0' S_0 = \sum_{k=1}^{K} \sum_{t=1}^{T} w_t(\omega_k)' R_t^{\theta}(\omega_k)$

- **Theorem:** There are no SAOs if and only if for there is a MPD

Acceptable completeness and uniqueness of the MPD

- **Definition:** The market is acceptably complete if for every $x \in L$ there is a $\theta \in \Theta$ such that $R^\theta = x$

- **Theorem:** The MPD is unique if and only if the market is acceptably complete

Derivative pricing

- The absence of SAOs is a necessary condition for the existence of an equilibrium, only economies and price processes which preclude SAOs are allowed

- Let $Q = \{R^\theta : \theta \in \Theta\}$ denote the expected payoff process market subspace or the expected payoff processes obtainable by trading in the N assets

- Let $r \in L$ denote the expected payoff process of a derivative

- If $r \in Q$ then we say that the derivative is replicable (in terms of expected payoffs)

- **Proposition**: If $r \in Q$, $R^\theta = r$ and w is a MPD, then the unique price of the derivative is given by $\theta_0' S_0$ or $\sum_{k=1}^{K} \sum_{t=1}^{T} w_t'(\omega_k) r_t(\omega_k)$

- If $r \notin Q$ then upper and lower bounds for the derivative price may be obtained by super- and sub-replication
  - An upper bound for the price of the derivative is given by the follwing super-replication LP:

$$\min_{\theta} \theta_0' S_0$$
$$\text{s.t. } R^{\theta} \geq r$$

  - A lower bound for the price of the derivative is given by the following sub-replication LP:

$$\max_{\theta} \theta_0' S_0$$
$$\text{s.t. } R^{\theta} \leq r$$

Example Multiperiod Implementation

- European call option on a non-dividend paying stock
- Only assets are the stock and a risk-free security
- Price of the stock is assumed to follow the discrtized geometric Brownian motion:

$$S_t = S_{t-1} + \mu S_{t-1}\Delta t + \sigma S_t \Delta t \epsilon_t \tag{2}$$

- Each choice of (μ, σ) results in a different subjective measure

Generating the states

- The states are best represented in the form of a state tree

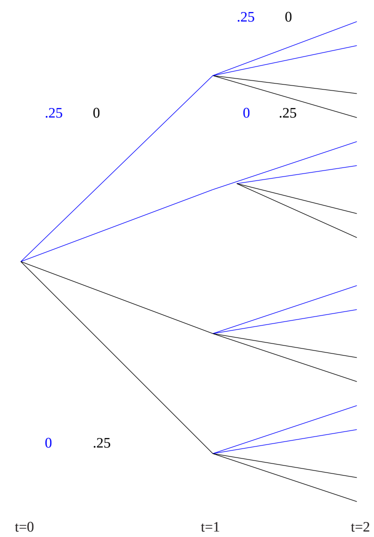

Figure 1: Multi-period State Tree

- Branches generated by Monte Carlo

- For a node with B branches $\frac{B}{M}$ branches are generated by each subjective measure

No SAOs

- The state tree must be generated without SAOs
- **Definition**: A one period SAO is a SAO over one period in the state tree
- **Theorem**: If there are no one period SAOs then there are no SAOs
- **Theorem**: There are no one period SAOs if μ is positive in at least one measure and negative in at least one other
- Result: A state tree with no SAOs can be generated by having μ positive in at least one measure and negative in at least another measure

2 times, 2 measures, 100 branches

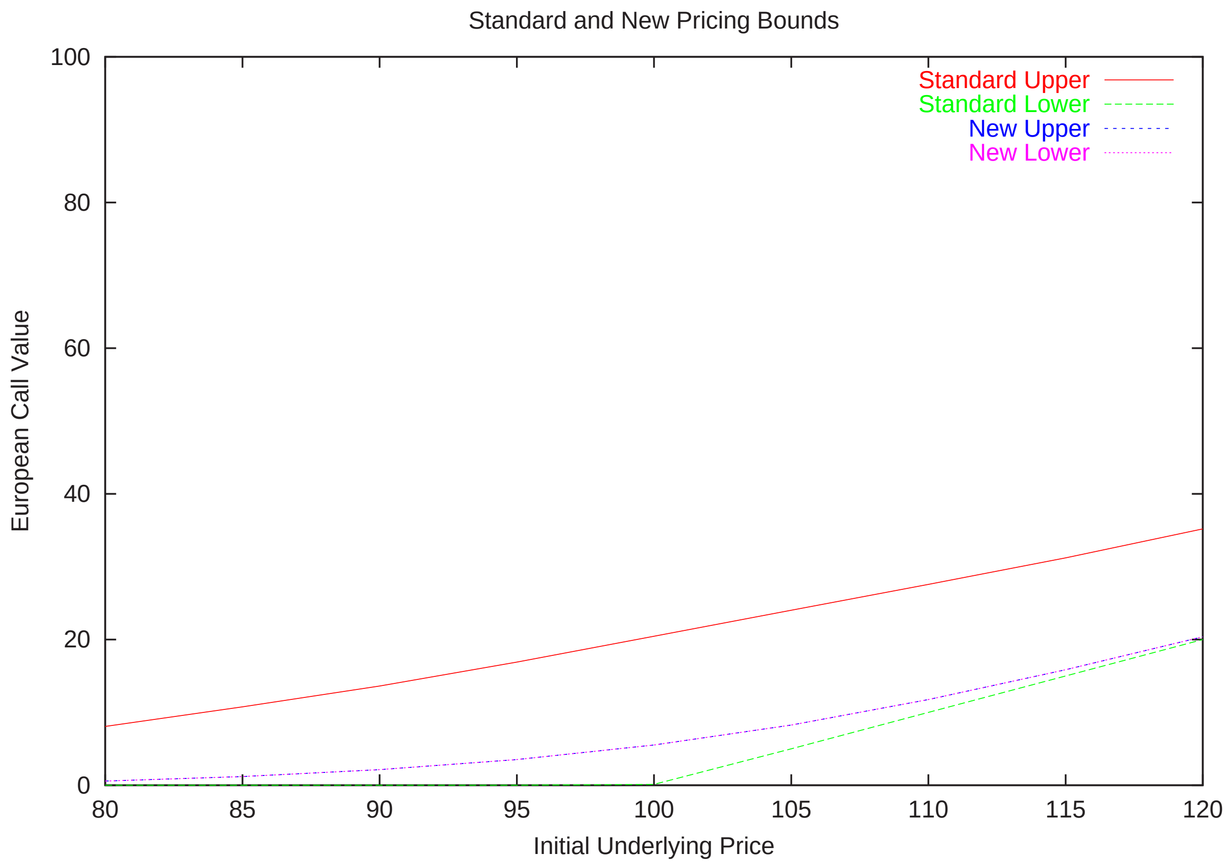

Figure 2: M=2

2 times, 3 measures, 99 branches

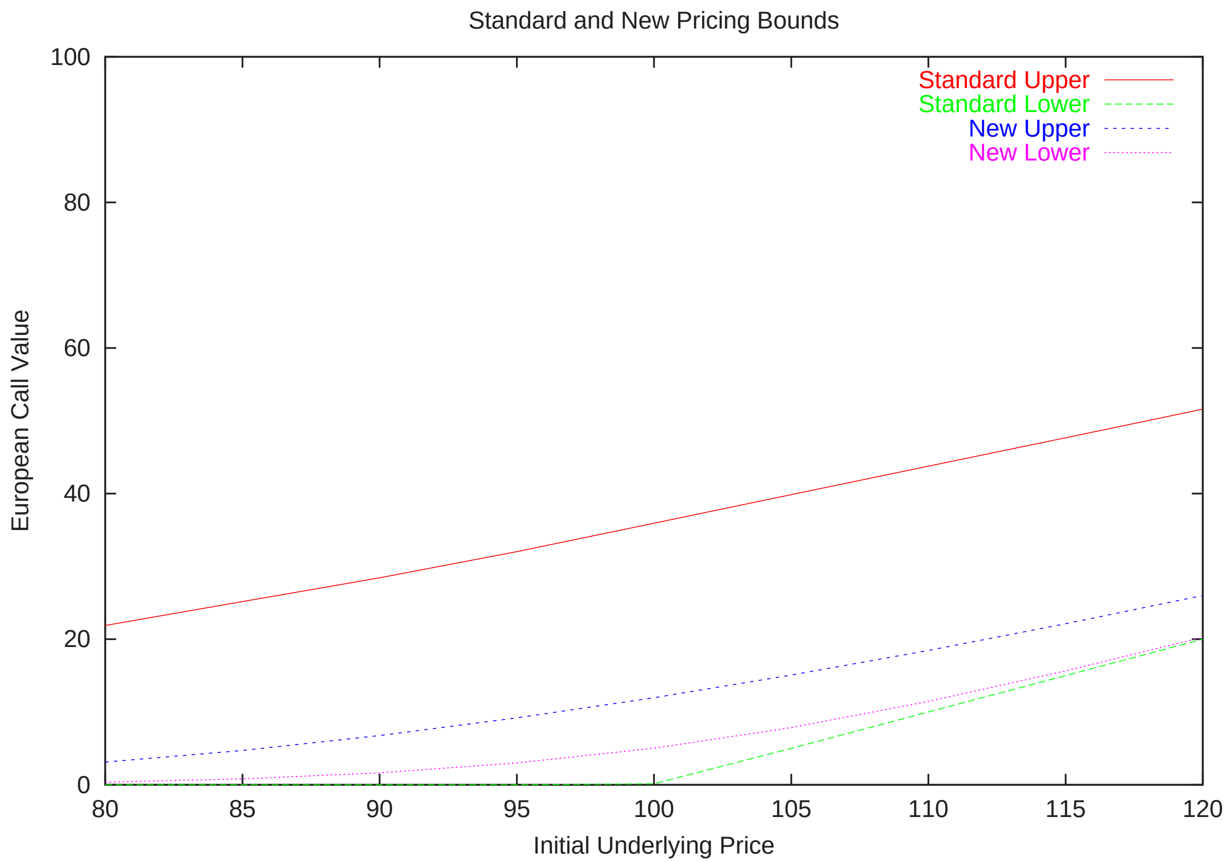

Figure 3: M=3

Conclusions

- Presented an asset pricing framework for incomplete markets which incorporates uncertainty in the real world measure using multiple subjective measures which:
  - Can be thought of as a no good-deals approach
  - Can deliver a unique price when no-arbitrage pricing can't
  - Delivers tighter bounds than arbitrage pricing theory