An Asset Based Model of Defaultable Convertible Bonds with Endogenised Recovery

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November 26, 2003

First version: February 2003

Abstract

We describe a two factor valuation model for convertible bonds when the firm may default. The underlying state variables are the asset value of the firm and the short riskless interest rate. Default can occur exogenously, or endogenously at a time a cash payment is made by the bond. We endogenize the recovery value of a defaulted bond through assumptions concerning the character of the reorganization period following default.

We use a tailored Lagrange-Galerkin discretization, coupled with a Lagrange multiplier method for free boundaries, to value convertibles in the model. Our framework enables us to specify numerically and financially consistent boundary conditions and inequality constraints.

We investigate the affect of changing the default, recovery and loss specification. The affect of introducing a stochastic interest rate is quantified, and asset and interest rate delta and gammas are found.

In our example we find that the value of the convertible bond is relatively insensitive to the initial asset value. Its sensitivity to interest rate changes is about one fifth less than that of a corresponding defaultable straight bond, chiefly due to the presence of the conversion feature.

*Ana Bermudez gratefully acknowledges the support of Fundacion Caixa Galicia, Spain.

1 Introduction

Convertible bonds (CBs) are widely issued securities that enable issuers to obtain relatively cheap finance in exchange for up-side gain. Only recently, however, have models begun to take unpredictable default of the issuer into account. This paper proposes a two-factor model for CBs in which the issuer may default. Our state variables are the firm's asset value and the short interest rate. The CB defaults either at the unpredictable jump time of a counting process, or when, potentially, the firm is required to make a cashflow to the CB. We endogenise recovery upon default into our model by assuming that the firm can invoke temporary protection against its creditors, leading to a quantification of the recovered value of the claim against the firm.

Let V_t denote the value of a firm's assets, S_t the value of its equity and D_t the value of the firm's debt, so that $V_t = S_t + D_t$. We assume the firm has a single debt class composed of convertible bonds.

Valuation models for convertible bonds fall into several categories, depending on the state variables of the model and how default is specified. Existing models take as their primary state variable either the asset value V_t or the stock value S_t, they may include or not a second state variable r_t representing interest rate risk, and default risk may or may not be modelled.

The early models of convertible bonds (Ingersoll (1977) [18] and Brennan and Schwartz (1977) [7]) follow Merton (1973) [22] in using V_t with geometric Brownian motion as the sole state variable. McConnel and Schwartz (1996) [21] have S_t as the sole state variable. Brennan and Schwartz (1980) [8] and more recently Nyborg (1996) [23] and Carayannopoulos (1996) [9] include in addition a stochastic interest rate. Brennan and Schwartz and Nyborg assume the short rate follows a mean reverting log-normal process; Carayannopoulos assumes the short rate follows the Cox, Ingersoll and Ross (1985) [10] model. Default risk is usually incorporated structurally by capping payouts to the bond by the value of the firm.

The majority of the recent literature uses S_t with geometric Brownian motion as the main state variable, incorporating either an interest rate variable, or default, or both. A variety of one-factor interest rate models have been used. The Vasicek (1977) [27] or else the extended Vasicek (Hull and White (1990) [17]) models are used by Epstein, Haber and Wilmott (2000) [13], Barone-Adesi, Bermudez and Hatgioannides (2003) [4], Bermudez and Nogueiras (2003) [5], and Davis and Lischka (1999) [11]. Ho and Pteffer (1996) [16] use the Black, Derman and Toy (1990) [6] model; and Zvan, Forsyth and Vetzal (1998) [31] and Yigitbasioglu (2002) [29] use the Cox, Ingersoll and Ross model.

Of these papers, only Davis and Lischka (1999) [11] and Yigitbasioglu (2002) [29] also allow for default (although Ho and Pteffer incorporate a credit spread). In general, credit risk models fall into two main categories, structural and reduced form. In structural models such as Zhou (1997) [30] and Longstaff and Schwartz (1995) [20], default occurs when a state variable, usually V_t, breaches a barrier level. It is necessary to specify the process for V_t, the location of the barrier, and the form and amount of recovery upon default. In reduced form models, such as Finger (2000) [14] and Jarrow, Lando and Turnbull (1997) [19], default is exogenous, occurring at the jump time of a counting process, N_t, with jump intensity λ_t. Reduced form models were analyzed by Duffie and Singleton (1999) [12]. The main issues in reduced form models are the specification of processes for the riskless short rate r_t, the hazard rate λ_t, and the loss rate w_t. Our model has both reduced form and strucural features.

Dimensional problems restrict the use of more than two factors in a model. Single factor models that incorporate default risk and do so in the reduced form framework include Andersen and Buffum (2003) [1], Takahashi, Kobayahashi and Nakagawa (2001) [25] and Ayache, Forsyth and Vetzal (2002) [2], (2003) [3].[1] Davis and Lischka (1999) [11] present a reduced form modelling framework with several graphical comparisons. Tseveriotis and Fernandes (1998) [26] and Yigitbasioglu (2002) [29], extended by Ayache, Forsyth and Vetzal (2002) [2], (2003) [3], define default risk via a credit spread. They split the CB value into a bond part and an equity part each with its own discount rate. Tseveriotis and Fernandes and Yigitbasioglu impose a fixed credit spread between the discount rates. Ayache, Forsyth and Vetzal and Takahashi, Kobayahashi and Nakagawa determine the credit spread via a hazard rate in a reduced form framework. All of these models are equity based, with S_t as the main state variable.

Care is required to specify correctly what happens to the convertible bond when default occurs. When the underlying state variable is the asset value V_t it is relatively easy to so do in a logically consistent way. When the state variable is the equity value S_t considerable difficulties may arise. For instance boundary conditions are hard to specify in a financially consistent manner; some models, for instance, may not require that when $S_t \to 0$ the bond value goes to zero.

To avoid specification problems inherent in models based upon S_t we choose V_t as our primary state variable, taking care to impose financially consistent boundary conditions. We are not aware of other reduced form specifications that model default when V_t is the state variable.

We obtain values by solving numerically a partial differential inequality (PDI) using a finite element and duality method. These methods have previously been used by Vazquez (1998) [28], Forsyth *et al.* (2002) [15], Zvan *et al.* (1998) [31] and Zvan *et al.* (2001) [32]. Variational inequalities, which are fundamental to our numerical method, provide an excellent framework to deal with existence and uniqueness issues, as well as for numerical analysis. Finite element methods offer some computation advantages compared to finite differences and lattice methods.

We use the method of Barone-Adesi, Bermudez and Hatgioannides (2003) [4] and Bermudez and Nogueiras (2003) [5]. Finite elements are used to discretise in space and the method of characteristics to discretise in time, yielding a tailored Lagrange-Galerkin discretization. The method of characteristics copes better with the degenerate nature of financial PDIs, avoiding instabilities that typically arise with other discretisation schemes.

In order to deal with the free boundaries arising from early exercise, we use

[1] Takahashi, Kobayahashi and Nakagawa also discuss a structural model of default.

a duality method over the variational formulation of the discretised problem, essentially a Lagrange multiplier method. This scheme implicitly incorporates the early exercise conditions, rather than explicitly applying them after evolving backwards in time at each step. This improves convergence.

We find that our recovery specification allows a wide range of behaviour upon default. For an illustrative example we find that the value of the convertible bond in a particular case is relatively insensitive to both the initial asset value and to the initial interest rate. Its value is most sensitive to the specification of a conversion feature and least sensitive to the specification of a call feature. Adding a convertibility feature increases the value of a non-convertible bond more than subsequent the introduction of a redemption feature.[2]

The second section of the paper describes the CB valuation model. The third section describes the numerical method. The fourth section presents numerical results, and the final section concludes.

2 An Asset Based Model for Convertible Bond Valuation

Suppose that the value V_t of the firm's assets follows a jump-augmented geometric Brownian motion under the objective measure,

$$\mathrm{d}V_t = \mu_t V_{t\_}\mathrm{d}t + \sigma_V V_{t\_}\mathrm{d}z_t^V - w_t V_{t\_}\mathrm{d}N_t, \tag{1}$$

where z_t^V is a standard Brownian motion and N_t is a counting process with intensity λ_t. w_t is a proportional loss. N_t models exogenous default events. At a jump time τ for N_t the asset value falls by a proportion w_τ,

$$V_\tau = V_{\tau\_}\left(1 - w_\tau\right). \tag{2}$$

Since we focus on asset risk and interest rate risk we assume that w_t is non-stochastic. Under the equivalent martingale measure (EMM) associated with the accumulator numeraire $B_t = \exp\left(\int_0^t r_s \mathrm{d}s\right)$ the relative price V_t/B_t is a martingale so

$$\mathrm{d}V_t = \left(r_t + \bar{\lambda}_t w_t\right) V_{t\_}\mathrm{d}t + \sigma_V V_{t\_}\mathrm{d}z_t^V - w_t V_{t\_}\mathrm{d}N_t, \tag{3}$$

where r_t is the instantaneous short rate, $\bar{\lambda}_t = \lambda_t\left(1 - \gamma\right)$ is the jump intensity under the EMM and $\bar{\lambda}_t w_t$ is the compensator for the jump component of V_t.

We suppose that the firm has issued a convertible bond with market value D_t at time t. The bond matures at time T with face value F. At certain times t_i, $i = 1, \ldots, N$, $t_N = T$, it pays coupons of size P_{t_i}, and $V_{t_i} = V_{t_i -} - P_{t_i}$. At certain times up to and including time T the bond may be converted to equity. Its value upon conversion at time t is $\kappa_t V_t$ where κ_t is the proportion of the

[2] Our illustrative example is out of the money. Other specifications have a variety of possible behaviours.

firms asset value acquired by the debt holders.[3] Dilution effects are absorbed into κ_t.

We assume that the CB may be both callable and redeemable with call price C_t and redemption price R_t at certain times t. On any particular date the CB need be neither callable nor redeemable but we assume that if it is callable on some date then it is also convertible on that date. In the sequel we suppose that the call price and redemption price are imputed to accrue interest on coupons and that if $t_i \le t < t_{i+1}$ for coupon payment dates t_i the call price and redemption price are set to be

$$C_t = C + \frac{t - t_i}{t_{i+1} - t_i} P_{t_{i+1}}, \tag{4}$$

$$R_t = R + \frac{t - t_i}{t_{i+1} - t_i} P_{t_{i+1}}, \tag{5}$$

for constants C and R.

If the bond is redeemed, or if a coupon or principal is to be paid, we suppose that the firm may choose to default. We assume that if the firm defaults, whether exogenously or endogenously, the CB holders may choose to convert.

Our default specification has both reduced form and structural elements. Sumarising, a default event may occur in one of two ways. Firstly, when the counting process N_t jumps the firm is supposed to have been hit by an unexpected exogenous default. Secondly, when a claim is made against the firm, specifically when the CB is redeemed or when a coupon or principal payment is due to be made, the firm may choose to default.

We first give a detailed specification of the components of the model. Then we display the PDI obeyed by the convertible bond value in this framework, and its boundary conditions.

2.1 Detailed Specification of the Model

To specify a model we need to define what happens to the CB value when default occurs, define the hazard rate process $\bar{\lambda}_t$, and provide an interest rate model. We consider each of these in turn. Finally we bring together the separate components into a fully specified model with a consistent set of boundary conditions and inequality constraints.

2.1.1 The default event and recovery values

So far no assumptions have been made about what happens upon default. We now assume that at the time τ of a default event the firm loses the right to call the debt, and that CB holders may no longer redeem the debt, but upon default the CB holders have the option to convert.[4] Write D^*_τ for the value of the CB

[3] Unlike Ayache, Forsyth and Vetzal (2002) [2] and Tsiveriotis and Fernandes (1998) [26] we account for the affect of conversion upon the value of the firm's equity. At conversion, the firm's total value is unchanged but it becomes all equity.

[4] We see below it may indeed be optimal for the CB holders to do so.

at a default time τ and F_τ^* for the recovery value of the CB at time τ. Since bondholders have the option to convert in the event of default we have

$$D_\tau^* = \max\left\{F_\tau^*,\ \kappa_\tau V_\tau\right\}. \tag{6}$$

Now we consider the recovery value F_τ^* of the convertible bond upon default.

Two main assumptions are made in the credit literature about the recovery value F_τ^* of a defaulted bond. The first is to suppose that the ratio $l_\tau = \left(D_{\tau\_} - F_\tau^*\right)/D_{\tau\_}$, the loss in the event of default, may be modelled and so determine F_τ^* from $D_{\tau\_}$. The second supposes that F_τ^* is a function of the riskless present value to time τ of the face value F.

Each of these assumptions has some attractions, but neither attempts to model the recovery process, regarding recovery values as exogenously determined and separately estimated. Most models of convertibles assume that the recovery value is a fraction of either the bond principal F (for example Andersen and Buffum (2003) [1], Davis and Lischka (1999) [11]), or the market price of the CB just prior to default $D_{\tau\_}$ (for example Takahashi, Kobayahashi and Nakagawa (2001) [25], or, in splitting schemes, some proportion of the bond part of the CB (for example Ayache, Forsyth and Vetzal (2003) [3]).[5]

We endogenise recovery into our model.

In practice default may occur when the firm value is significantly greater than the value of its obligations, a feature allowed in our model. The outcome of default is to put the firm into reorganization during which time it receives protection against the claims of its creditors. The effect is that even though theoretically the firm may have the capacity to fulfill the claims against it, in practice the values of the claims may be considerably less than their face values.

We operationalise this as follows. We interpret a default event simply as a trigger that puts the firm into reorganization, giving the firm protection against its creditors. Upon default at time τ the bondholders have a claim of value F_τ against the firm where

$$F_\tau = \begin{cases} F + P_T, & \text{if } \tau = T \text{ is at the maturity time } T, \\ F + P_{t_i}, & \left\{\begin{array}{l}\text{if default is at a coupon payment time, } \tau = t_i, \\ \text{or a redemption date coinciding with a coupon date,}\end{array}\right. \\ F, & \left\{\begin{array}{l}\text{if default is exogenous,} \\ \text{or at a redemption date not coinciding with a coupon date.}\end{array}\right. \end{cases} \tag{7}$$

Alternatively we could assume, for instance, that F_τ contains accrued interest, or that on a redemption date $F_\tau = \max\{F,\ R_\tau\}$.

We suppose that the protection offered by reorganization grants the firm a grace period of length s after default such that during this period the bondholders no longer have the right to enforce default but the firm has the option at any time during this period to choose to default. At a put time $t \geq \tau$, the recovery value of the CB is $F_t^* = \min\{F_\tau, V_t\} = F_\tau - (F_\tau - V_t)_+$, where we

[5] Ayache, Forsyth and Vetzal (2002) [2] discuss in detail default issues in equity based models.

suppose that the bond holders' claim does not earn interest. Hence, given no disbursements or refinancing, at a default time τ the value of debt is

$$\begin{aligned} F_\tau^* &= \mathrm{Pv}\,(F_\tau) - p\,(V_\tau, F_\tau)\,, \text{ where} &(8)\\ V_\tau &= V_{\tau_-}\,(1 - w_\tau)\,, &(9)\end{aligned}$$

for a put option p.[6] Effectively the bondholders are forced to give a put option to the firm allowing the firm to annul the bondholders' claim of F_τ by transferring the firm to the bondholders.

For simplicity we suppose that default is a unique event. Once default has occurred we suppose the firm value follows a geometric Brownian motion but that the event of default influences the growth of firm's future asset value. If we suppose that both the CB and the firm's equity continue to trade during reorganization, then under the EMM the instantaneous return to the firm's assets is still the short rate r_t. However, it seems reasonable to suppose that the volatility of V_t may change, perhaps increasing, as a consequence of default. Denote the post-default volatility by σ^*. The recovery value $F_\tau^*\,(V_\tau)$ at default time τ is thus determined by two parameter values, s and σ^*, each with a natural interpretation.

In the exposition that follows we employ a simplifying assumption. We suppose the firm may exercise the put only at the time $\tau + s$, so p becomes a European put. Effectively, after a default event, the bondholders are obliged to wait a period s before receiving a payment of $\min\{F_\tau, V_{\tau+s}\}$. This assumption enormously decreases the complexity and cost of finding numerical solutions to (13).

A feature of our formulation is that at a default time τ the CB holders never receive the amount of their claim, F_τ. In practice at the maturity time T there will be a range of asset values where the firm will not default but where bondholders will not convert, receiving instead an amount equal to F_T. In our model if the bondholders do not convert they recover F_T^*, which can be significantly less that F_T. This behaviour could be inappropriate at moderate levels of the firm's assets. We can overcome this problem by allowing s to depend on $\frac{V_T}{F_T}$ so that $s \sim 0$ when $V_T \gg F_T$. However, this would complicate the model and in any case for in the money CBs the effect is likely to be slight.

When at a default time τ the asset value $V_\tau \gg F_\tau$ is high we refer to 'technical default', since the CB will be converted.

2.1.2 The Interest Rate Model

We assume the interest rate model is extended Vasicek. This model combines tractability with the flexibility to calibrate to a pre-specified initial term structure. The short rate process under the EMM is

$$\mathrm{d}r_t = \alpha\,(\theta(t) - r_t)\,\mathrm{d}t + \sigma_r \mathrm{d}z_t^r, \tag{10}$$

[6] We could also assume that the claim does earn interest, in which case $F_t^* = \min\{\mathrm{Fv}\,(F_\tau)\,, V_t\} = \mathrm{Fv}\,(F_\tau) - (\mathrm{Fv}\,(F_\tau) - V_t)_+$, where $\mathrm{Fv}\,(F)$ stands for the future value of F, so that $F_\tau^* = F_\tau - p\,(V_\tau, \mathrm{Fv}\,(F_\tau))$.

where $\theta(t)$ can be chosen so that model spot rates coincide with market spot rates. We set $\mu_r \equiv \mu(t,r) = \alpha(\theta(t) - r_t)$ for the drift of r and write ρ for the correlation between z_t^r and z_t^V, $\mathrm{d}z_t^r \mathrm{d}z_t^V = \rho \mathrm{d}t$.

The Vasicek model allows rates to become negative (with small probability). An alternative would be to use the CIR model in which rates are certain to remain non-negative. However we will see below that choosing the Vasicek model allows us to simplify the specification of boundary conditions.

In the Vasicek model when $\rho = 0$ there is a simple explicit solution for $p(V,r)$.[7]

2.1.3 The Hazard Rate Process

We do not model the risk-adjusted hazard rate $\bar{\lambda}_t$ with its own specific risk. Instead we suppose that $\bar{\lambda}_t \equiv \bar{\lambda}(V_t, r_t)$ is a deterministic function of V_t and r_t. We assume that $\bar{\lambda}_t$ decreases as both V_t and r_t increase. In principle a credit spread model implicitly determines an intensity function. In their single factor reduced form model Takahashi, Kobayahashi and Nakagawa explicitly assume that $\bar{\lambda}_t \equiv \bar{\lambda}(S_t,) = a + bS_t^{-c}$. Andersen and Buffum (2003) [1] discuss several functional forms for $\bar{\lambda}_t$.

We allow $\bar{\lambda}_t$ to depend on V_t and r_t. For concreteness we choose the functional form

$$\bar{\lambda}(V_t, r_t) = \lambda \exp\left(-\left(aV_t + br_t\right)\right), \qquad a, b \geq 0. \tag{11}$$

The coefficients λ, a and b control the background default rate and the sensitivity of $\bar{\lambda}_t$ to V_t and r_t. Default risk decreases as V_t increases. As $r_t \to -\infty$, $\bar{\lambda}_t \to \infty$ so that default becomes inevitable.

Note that (11) does not require $\bar{\lambda}_t$ to go to infinity when V_t goes to zero. However a consequence of our formulation is that $D_t < V_t$ for all t, so that D_t goes to zero as V_t goes to zero without any constraint on $\bar{\lambda}_t$.

2.2 A PDI for a Convertible Bond

We need to specify both the PDI, its boundary conditions and inequality constraints.

2.2.1 The PDI

By Itō's lemma (Protter (1995) [24]) the process followed by D_t is

$$\begin{aligned} \mathrm{d}D_t = \Bigg(& \frac{\partial D}{\partial t} + \left(r_t + \lambda_t^* w_t\right) V_t \frac{\partial D}{\partial V} + \frac{1}{2}\sigma_V^2 V_t^2 \frac{\partial^2 D}{\partial V^2} \\ & + \mu_r \frac{\partial D}{\partial r} + \frac{1}{2}\sigma_r^2 \frac{\partial^2 D}{\partial r^2} + \rho \sigma_r \sigma_V V_t \frac{\partial^2 D}{\partial V \partial r} \Bigg) \mathrm{d}t \\ & + \sigma_V V_t \frac{\partial D}{\partial V} \mathrm{d}z_t^V + \sigma_r \frac{\partial D}{\partial r} \mathrm{d}z_t^r + \Delta D_t\left(V_{t_-}\right), \end{aligned} \tag{12}$$

[7] In fact in our numerical work we use this formula even when $\rho \neq 0$. The error introduced is slight (over our range of values of ρ) and the numerical burden is considerably reduced.

where $\Delta D_t\left(V_{t\_}\right) = D_t^*\left(V_t\right) - D_{t\_}\left(V_{t\_}\right)$ is the change in the value of the convertible bond if a jump, hence a default, occurs at time t.

Under the EMM the relative price D_t/B_t is a martingale. Imposing this condition we find,

$$r_t D_t = \frac{\partial D}{\partial t} + \left(r_t + \bar{\lambda}_t w_t\right) V_t \frac{\partial D}{\partial V} + \frac{1}{2}\sigma_V^2 V_t^2 \frac{\partial^2 D}{\partial V^2} + \mu_r \frac{\partial D}{\partial r} + \frac{1}{2}\sigma_r^2 \frac{\partial^2 D}{\partial r^2} + \rho\sigma_r\sigma_V V_t \frac{\partial^2 D}{\partial V \partial r} + \bar{\lambda}_t \mathbb{E}_{t\_}\left[D_t^*\left(V_t\right) - D_{t\_}\left(V_{t\_}\right)\right] \tag{13}$$

Since we assume deterministic loss and recovery conditions this becomes

$$\begin{aligned}\left(r_t + \bar{\lambda}_t\right) D_t = \bar{\lambda}_t D_t^* + \frac{\partial D}{\partial t} + \left(r_t + \bar{\lambda}_t w_t\right) V_t \frac{\partial D}{\partial V} + \frac{1}{2}\sigma_V^2 V_t^2 \frac{\partial^2 D}{\partial V^2} \\ + \mu_r \frac{\partial D}{\partial r} + \frac{1}{2}\sigma_r^2 \frac{\partial^2 D}{\partial r^2} + \rho\sigma_r\sigma_V V_t \frac{\partial^2 D}{\partial V \partial r}\end{aligned} \tag{14}$$

where in our formulation $D_t^* = \max\left\{F_t^*,\ \kappa_t V_t\right\}$ and $F_t^* = \mathrm{Pv}\left(F_t\right) - p$ for a put $p \equiv p\left(V_t, F_t\right)$ where $V_t = V_{t-}\left(1 - w_t\right)$.

If V_t is the sole state variable this becomes

$$\left(r_t + \bar{\lambda}_t\right) D_t = \bar{\lambda}_t D_t^* + \frac{\partial D}{\partial t} + \left(r_t + \bar{\lambda}_t w_t\right) V_t \frac{\partial D}{\partial V} + \frac{1}{2}\sigma_V^2 V_t^2 \frac{\partial^2 D}{\partial V^2} \tag{15}$$

which is the form of equation (43) in Ayache, Forsyth and Vetzal (2003) [3].

2.2.2 Inequality Constraints and Auxiliary Conditions for the PDI

We need to specify the final payoff to the convertible bond at time T and payoffs at intermediate times $0 \leq t < T$. We also specify inequality constraints and other conditions on the bond's value.

At the Final Exercise Time T At times when a cash payment has to be made to the bond the firm has the option to default. At the final time T the firm will default if $V_{T\_} < F_T$. If $V_{T\_} \geq F_T$ we suppose the firm acts to maximise the value of equity by minimising the value of the CB. Since the bond may convert upon default, we have

$$D_T = D_T^* = \max\left\{F_T^*,\ \kappa_T V_T\right\}. \tag{16}$$

There is a critical asset value $V_T^* < F_T$ such that $F_T^* = \mathrm{Pv}\left(F_T\right) - p\left(V_T, F_T\right) = \kappa_T V_T^*$. The CB holders will convert if $V_T > V_T^*$, whether or not the firm elects to default.

At Redemption and Call dates Consider a redemption date t at which the CB is not callable and which no coupon is paid. Redemption is at the option of the bondholders. If the bond is redeemed the firm has the option to default. If the firm defaults the CB holders have the option to convert. Hence

$$\max\left\{\min\left\{F_t^*,\ R_t\right\},\ \kappa_t V_t\right\} \leq D_t. \tag{17}$$

At a call date t there is an upper bound on the value of the bond. If it is called at time t the CB holders have the option to convert so

$$\begin{array}{ll} \kappa_t V_t \leq D_t \leq \min\left\{V_t, C_t\right\}, & V_t < C_t/\kappa_t, \\ D_t = \kappa_t V_t, & V_t \geq C_t/\kappa_t. \end{array} \tag{18}$$

We can combine (17) and (18) into a single expression

$$\max\left\{\min\left\{F_t^*,\ R_t\right\},\ \kappa_t V_t\right\} \leq D_t \leq \max\left\{\min\left\{V_t, C_t\right\},\ \kappa_t V_t\right\}, \tag{19}$$

where R_t is set to zero on a non-redemption date, C_t is set to $+\infty$ on a non-call date, and κ_t is set to zero on a non-conversion date.

At a Coupon Date Suppose a coupon of size P_t is due to be paid at time t and that no exercise conditions are invoked so that $D_t^* = F_t^*$. We suppose that the firm acts to maximise the value of equity. The firm has the choice of paying the coupon or defaulting so the equity value is

$$S_t = \left\{ \begin{array}{ll} \left(V_{t_-} - P_t\right)_+ - D_t, & \text{if the coupon is paid,} \\ V_{t_-} - F_t^* & \text{if the firm defaults,} \end{array} \right. \tag{20}$$

so $S_t = \max\left\{\left(V_{t_-} - P_t\right)_+ - D_t,\ V_{t_-} - F_t^*\right\} > 0$.

There is a critical value $V_{t_-}^* > P_t$ with $D_{t_-}^*\left(V_{t_-}^*\right) = P_t + D_t\left(V_{t_-}^*\right)$, such that the firm defaults if $V_{t_-} < V_{t_-}^*$ and pays the coupon otherwise. When $V_{t_-} \geq V_{t_-}^*$ we have $D_{t_-}\left(V_{t_-}^*\right) \geq P_t$. Then, if there are no exercise conditions,

$$0 < D_t\left(V_t\right) = \left\{ \begin{array}{ll} D_{t_-}\left(V_{t_-}\right) - P_t, & V_{t_-} \geq V_{t_-}^*, \\ F_t - p\left(V_{t_-}\right), & V_{t_-} < V_{t_-}^*. \end{array} \right. \tag{21}$$

Now suppose that exercise features are present. If we assume for simplicity that $R_t = R_{t-}$, $C_t = C_{t-}$ and $\kappa_t = \kappa_{t-}$, and that the CB specifies that $F_t = F_{t_-}$, then the firm may choose to call just before the coupon is paid, but the CB holders will not elect to redeem or convert until after the coupon is paid. Then if $V_{t_-} \geq V_{t_-}^*$ the firm does not default and

$$\max\left\{\min\left\{F_t^*,\ R_t\right\},\ \kappa_t V_t\right\} \leq D_t\left(V_t\right) \leq \max\left\{\min\left\{V_t, C_t\right\},\ \kappa_t V_t\right\}. \tag{22}$$

Otherwise, if $V_{t_-} < V_{t_-}^*$ the firm defaults and $D_t\left(V_t\right) = D_{t_-}^* = \max\left\{F_{t_-}^*,\ \kappa_{t_-} V_{t_-}\right\}$.

3 The Solution Method

Several numerical methods have been used in the literature to obtain CB values. Ho and Pteffer (1996) [16] use a two factor lattice. Epstein, Haber and Wilmott (2000) [13], and Yigitbasioglu (2002) [29] use finite difference methods. Zvan, Forsyth and Vetzal (1998) [31] and Barone-Adesi, Bermudez and Hatgioannides (2003) [4] use finite element schemes.

We use a finite element and duality method, with time discretised by the method of characteristics, to solve the PDI (14). Bermudez and Nogueiras (2003) [5] give a full description of the numerical method. They present an algorithm for a general two factor PDI and apply it to valuation problems in finance. We summarise the general setting.

On a spatial domain Ω and for given measurable functions f, A_0, B_i, $A_{i,j}$, H, R_1 and R_2 of x_1, x_2 and t, the algorithm finds functions $D(x_1, x_2, t)$ and $P(x_1, x_2, t)$ such that in $\Omega \times (0, T)$

$$P = f + A_0 D + \frac{\partial D}{\partial t} + \sum_{i=1}^{2} B_i \frac{\partial D}{\partial x_i} + \sum_{i,j=1}^{2} A_{i,j} \frac{\partial^2 D}{\partial x_i \partial x_j}, \tag{23}$$

$$R_1 \leq D \leq R_2, \tag{24}$$

$$D(x_1, x_2, T) = H(x_1, x_2). \tag{25}$$

The Lagrange multiplier P has the property that

$$R_1 < D < R_2 \Longrightarrow P = 0, \tag{26}$$

$$D = R_1 \Longrightarrow P \leq 0, \tag{27}$$

$$D = R_2 \Longrightarrow P \geq 0. \tag{28}$$

In the region where $P = 0$ the equality (26) holds. The surfaces separating the regions where $P < 0$, $P = 0$ and $P > 0$ are the free boundaries.

In our case we have

$$x_1 = r_t, \tag{29}$$

$$x_2 = V_t, \tag{30}$$

$$A_{11} = \frac{1}{2}\sigma_r^2,\ A_{12} = A_{21} = \frac{1}{2}\rho\sigma_r\sigma_V V_t,\ A_{22} = \frac{1}{2}\sigma_V^2 V_t^2, \tag{31}$$

$$B_1 = \mu_r,\ B_2 = \left(r_t + \bar{\lambda}_t w_t\right) V_t, \tag{32}$$

$$A_0 = -\left(r_t + \bar{\lambda}_t\right),\ f = \bar{\lambda}_t D_t^*, \tag{33}$$

Early exercise features are modelled by the functions R_1 and R_2 and at a call or redemption date, for instance,

$$R_1(r_t, S_t, t) = \max\left\{\min\left\{F_t^*,\ R_t\right\},\ \kappa_t V_t\right\}, \tag{34}$$

$$R_2(r_t, S_t, t) = \max\left\{\min\left\{V_t,\ C_t\right\},\ \kappa_t V_t\right\}, \tag{35}$$

and H is determined by the payoff function of the CB.

In the next section we specify the boundary conditions require by the numerical method.

3.1 Asymptotic Boundary Conditions

For numerical purposes we need to solve the PDI on a finite domain $\Omega = \Omega^r \times \Omega^V$ where $\Omega^r = [r_{\min}, r_{\max}]$, $\Omega^V = [0, V_{\max}]$ and $V_{\max} \geq R_t, C_t$ at all times when these are defined. At the boundaries of the solution domain Ω we need to supply boundary conditions to our solution method. We suppose that asymptotic approximations can be applied at $r_{\min}$, $r_{\max}$ and $V_{\max}$.

Four boundary conditions are required. The convertible bond literature is often not explicit about the boundary conditions used. In our framework the asset boundary conditions, at $V_t = 0$, $V_{\max}$ are straightforward, as are the conditions at $r_{\max}$. There are problems if one tries to supply a boundary condition at $r_{\min} = 0$. Instead we choose $r_{\min} < 0$, a natural assumption in the Vasicek model where interest rates are not constrained to be positive.[8]

We suppose the final condition and inequality constraints are given by (16), (19) and (22) and explore asymptotic conditions. Since the PDI is solved backwards in time we re-formulate (21) and (22). On a coupon date t we first compute a value D_{t_+}, notionally the CB value immediately after the coupon has been paid, checking exercise conditions at time t_+, post-coupon. We then find D_{t_-}, the CB value immediately before the coupon is paid, and check exercise conditions at time t_-, pre-coupon. Then we continue iterating backwards.

Over the coupon payment time we have

$$\begin{aligned} V_{t_-} &= V_{t_+} + P_t, && (36) \\ \widetilde{D}_{t_-}\left(V_{t_-}\right) &= \begin{cases} D_{t_+}\left(V_{t_-} - P_t\right) + P_t, & V_{t_-} \geq V^*_{t_-}, \\ D^*_{t_-}, & V_{t_-} < V^*_{t_-}. \end{cases} && (37) \end{aligned}$$

then the exercise condition is

$$\max\left\{\min\left\{\widetilde{D}_{t_-}\left(V_{t_-}\right),\ R_t\right\},\ \kappa_t V_{t_-}\right\} \leq D_{t_-} \leq \max\left\{\min\left\{V_{t_-}, C_t\right\},\ \kappa_t V_{t_-}\right\}. \tag{38}$$

First, consider a convertible bond with no coupons, convertible only at time T, nowhere callable or redeemable, where default occurs only at time T with no proportional loss and is modelled by setting the bond payoff to be $D_T = \max\left\{\min\left\{V_T, F\right\}, \kappa_T V_T\right\}$. We call this a simple CB. In this case the convertible value decomposes into the value of a defaultable straight bond and a call on the firm's assets, with explicit solution

$$D_t = V_t - c_t\left(V_t, F\right) + \kappa_T c_t\left(V_t, F/\kappa_T\right) \tag{39}$$

where

$$\begin{aligned} c_t\left(V_t, F\right) &= V_t N\left(d_1\right) - \mathrm{Pv}\left(F\right) N\left(d_2\right), && (40) \\ d_1 &= \frac{1}{\sigma\sqrt{T-t}} \ln\left(\frac{V_t}{\mathrm{Pv}\left(F\right)}\right) + \frac{1}{2}\sigma\sqrt{T-t}, && (41) \\ d_2 &= d_1 - \sigma\sqrt{T-t}, && (42) \end{aligned}$$

[8] We find that it is possible to choose $r_{\min} < 0$ so that asymptotic conditions apply and the computation of $\bar{\lambda}_t$ does not cause overflow.

Boundary:		$V \to 0$	$V \to \infty$	$r \to -\infty$	$r \to \infty$
Dirichlet:	D_t	0	$\kappa_T V_t$	V_T	$\kappa_T V_t$
Neumann:	$\frac{\partial D_t}{\partial V}$	1	κ_T	1	κ_T
	$\frac{\partial D_t}{\partial r}$	0	0	0	0

Table 1: Boundary conditions, simple CB

$N(\cdot)$ is the cumulative normal distribution function and $\mathrm{Pv}(F)$ is the present value computed in the Vasicek term structure model.

Asymptotic Dirichlet and Neumann boundary conditions can be computed and are given in table 1. Note that limits as $(V, r) \to (+\infty, -\infty)$ depend upon the direction of the limit. We found it best to use a Neumann boundary conditions at the $r \to -\infty$ boundary, and Dirichlet boundary conditions at the other three boundaries.

Now consider the general CB.

High asset value As $V \to \infty$ the CB effectively becomes an equity instrument and it will either be converted, or converted when it is called, at an optimal time t^* that does not depend upon V or the coupon stream. Then, ignoring the possibility of default, the bond value is

$$D_t = \kappa_{t^*} V_t + \widehat{P}_t^{t^*} \tag{43}$$

where $\widehat{P}_t^{t^*} = \sum_{t \le t_i \le t^*} \mathrm{Pv}(P_{t_i})$ is the value at time t of the future coupons received up to time t^*.

In general (43) may be hard to compute, but when conversion terms are constant t^* is the first available conversion date. If continuous conversion is possible $t^* = t$ and $D_t = \kappa_t V_t$.

Low asset value When $V = 0$ we have $D_t = 0$.

High interest rate As $r \to \infty$ for $t < T$ the present value of the principle F goes to zero. Cash becomes irrelevant and the time value of money is expressed in returns to the asset process. The payoff to the CB at time T is effectively $\kappa_T V_T$, and default is irrelevant. Payoffs, if received in cash, will be used to immediately buy the asset.

As before, suppose the bond is exercised by one party or the other at an optimal time t^*. It may be optimal to redeem if R_t/V_t is large enough, or to call when $V_t < C_t/\kappa_t$ if future values of κ_t are large enough. At time t^* we have $D_{t^*} = \kappa_{t^*}^* V_{t^*}$ where $\kappa_{t^*}^* = \max\{C_{t^*}/V_{t^*},\ \kappa_t\}$ if the bond was called and $\kappa_{t^*}^* = \max\{\kappa_{t^*},\ \min\{D_{t^*}^*,\ R_{t^*}\}/V_{t^*}\}$ if the bond is redeemed. Since cashflows are immediately used to buy equity, $\kappa_{t^*}^*$ is the effective conversion ratio at time t^*. For high r,

$$D_t = \kappa_{t^*}^* V_t + \widehat{V}_t^{t^*} \tag{44}$$

Exercise parameter values			
	Convertibility	Callability	Redeemability
Base:	0.1	22	18
High:	0.15	24	19
Low:	0.05	21	16

Table 2: Exercise parameter value

where $\widehat{V}_{t_1}^{t_2} = V_t \sum_{t_1 \leq s \leq t_2} P_s/V_s$ is the value at time t_1 of asset rebased future coupons received up to time t_2.

As before, this simplifies if conversion terms, *et cetera*, are constant and continuous, and we may set $D_t = \kappa_t^* V_t$.

Low interest rate When $r \to -\infty$ the asset value becomes irrelevant and cash values dominate. Default occurs at the first cashflow date, if not sooner. CB holders will wait for a default event at some time τ and then take over the firm, so $D_t = \mathbb{E}_t \left[\mathrm{Pv}\left(V_\tau \right) \right] = V_t$.

We see that, with possible slight modification, table 1 gives the correct boundary conditions for a general convertible bond.

4 Numerical results

In this section we first benchmark the model, investigating the convergence properties of the finite element method. We then explore the affect upon CB values of altering parameter values within the model, looking particularly at the exercise conditions, asset and interest rate values and parameters, and the default parameters s, σ^* and λ.

Each parameter has a base case value, and a high and a low value. These are given in tables 2, 3 and 4. For the base case we suppose that the CB has $T = 5$ years to maturity with face value $F = 20$. The CB may be converted at any time with indirect conversion ratio $\kappa_t \equiv \kappa = 0.1$. The CB pays a coupon of 0.6 every half year. It is callable and redeemable at any time with the call and redemption prices determined from (4) and (5) with $C = 22$ and $R = 18$. The initial asset value is $V_0 = 100$ and initial interest rate is $r_0 = 0.06$. For the default intensity function we set $\lambda = 0.1$, $a = 0.03$, $b = 3$, so that for middling values of V_t and r_t, $\bar{\lambda}_t$ has about the same sensitivity to changes in each. Other parameter values are given in the tables.

We note that with this specification the convertible bond is out of the money and that in the base case the likelihood of exogenous default is relatively low.

For the numerical method we use four mesh specifications of increasing resolution. Mesh 1 is the coarsest with just 20 space steps in the interest rate dimension, 40 in the asset dimension, and 50 time steps up to time $T = 5$. Each successive mesh doubles both the number of space steps in each dimension and the number of time steps so that the finest mesh, mesh 4, has 160 interest rate

Process parameters							
	For r				For V		Corr.
Parameter:	r_0	α	θ	σ_r	V_0	σ_V	ρ
Base:	0.06	0.2	0.06	0.02	100	0.25	0.1
High:	0.07	0.21	0.07	0.025	105	0.30	0.15
Low:	0.05	0.19	0.05	0.015	95	0.20	0.05

Table 3: Process parameter values

Default, recovery and loss parameters						
	Default			Recovery		Loss
Parameter:	λ	a	b	s	σ^*	w
Base:	0.1	0.03	3	1	0.35	0.4
High:	0.2	0.06	6	5	0.45	0.6
Low:	0.01	0.003	0.3	0.25	0.25	0.2

Table 4: Default and recovery parameter values

steps and 320 assset steps, and 400 times steps up to five years.

4.1 Benchmarking

We benchmark to a simple CB whose value is given by (39), investigating convergence. Domain bounds are set to be $\Omega^r = [-1, 1]$ and $\Omega^V = [0, 800]$. Ω^V corresponds to roughly a 99.9% confidence interval on V_T. We give L^2 errors over both the entire domain Ω and also over a narrower region of interest $\widehat{\Omega} = \widehat{\Omega}^r \times \widehat{\Omega}^V$, where $\widehat{\Omega}^r = [0, 0.15]$ and $\widehat{\Omega}^V = [25, 400]$. $\widehat{\Omega}^V$ is roughly a 99% confidence interval on V_T. $\widehat{\Omega}$ reflects a range of values of r and V likely to be observed in practice and so the error on $\widehat{\Omega}$ is likely to be more representative.

The results are presented in table 5. Two sets of results are shown. The top panel uses analytical values on the boundary, the bottom panel uses asymptotic approximations, as given in table 1. In each case three of the boundaries are Dirichlet and fourth, at the lower boundary for r, is Neumann. 'Error TD' is the error on the entire domain Ω; 'Error RI' is the error on the region of interest, $\widehat{\Omega}$. 'Factor' is progressive error reduction factor in moving to a finer mesh level from the preceding mesh level. Times are in seconds.[9]

We see that using both analytical and asymptotic boundaries the convergence rate is not as fast as the theoretical rate of 2, although on the region of interest the convergence rate is much faster than on the whole domain. Errors are significantly less, by a factor of 100, on the region of interest compared to the total domain. Errors are greater on the total domain with asymptotic boundary conditions, but they are of the same order of magnitude. On the region of interest the errors for analytical and asymptotic boundary conditions are the same

[9] The implementation was in Fortran 77 run on a 2.4 Mhz Pentium IV PC, with no special speed-ups.

Errors and Convergence, Analytical Boundary Conditions					
Mesh	Error TD	Factor	Error RI	Factor	Time
1	$6.1E{-}02$	-	$3.1E{-}03$	-	3
2	$3.8E{-}02$	$1.6E{+}00$	$1.4E{-}03$	$2.2E{+}00$	17
3	$2.2E{-}02$	$1.7E{+}00$	$7.4E{-}04$	$1.8E{+}00$	154
4	$1.2E{-}02$	$1.8E{+}00$	$4.5E{-}04$	$1.6E{+}00$	1648

Errors and Convergence, Asymptotic Boundary Conditions					
Mesh	Error TD	Factor	Error RI	Factor	Time
1	$6.7E{-}02$	-	$3.1E{-}03$	-	2
2	$4.7E{-}02$	$1.4E{+}00$	$1.4E{-}03$	$2.2E{+}00$	15
3	$3.4E{-}02$	$1.4E{+}00$	$7.4E{-}04$	$1.8E{+}00$	146
4	$2.8E{-}02$	$1.2E{+}00$	$4.5E{-}04$	$1.6E{+}00$	1648

Table 5: Error and convergence

to two significant figures, supporting our use of asymptotic boundary conditions in the sequel.

Subsequent tables are computed using mesh 4 and asymptotic boundary conditions. All specifications lie within the region of interest so, in line with the errors reported in table 5, CB values are reported to 3 decimal places. With early exercise possible, a typical execution time is around 6700 seconds, relatively independent of the CB specification.

4.2 The Recovery Specification

We investigate the consequences of our recovery specification, interpreting it by computing the implied recovery ratio $\delta(V_t, r_t)$ defined as

$$\delta(V_t, r_t) = \mathbb{E}_t \left[\frac{F_\tau^*}{F} \mid V_t, r_t \right] \tag{45}$$

for a default time τ. δ is the proportion of face value the bondholders can expect to recover in the event of default if they do not convert.[10] We compute δ for a simple CB (so that default is only at maturity and the loss rate is zero). Table 6 shows δ for a variety of initial conditions and recovery specifications. The entry in bold is the base case.

Our example is out of the money and the value F_T^* is approximately equal to the present value at time T of F_T paid at time $T + s$.

We see that the most important factor for expected recovery is the length of the reorganisation period, followed by the interest rate and then the initial asset value. Changing the volatility parameter has little effect when the reorganisation period is short, but has an effect comparable in size to the asset value change when s is longer.

[10] When the CB is in the money default is technical and the CB will be converted.

Recovery parameters		Initial values of (V_t, r_t)				
		$(100, 0.06)$	$(80, 0.05)$	$(120, 0.05)$	$(80, 0.07)$	$(120, 0.07)$
(s, σ^*)	$(1, 0.35)$	**0.941**	0.942	0.945	0.937	0.939
	$(0.25, 0.25)$	0.985	0.985	0.986	0.984	0.984
	$(0.25, 0.45)$	0.985	0.985	0.986	0.983	0.984
	$(5, 0.25)$	0.746	0.752	0.755	0.736	0.738
	$(5, 0.45)$	0.721	0.715	0.736	0.704	0.722

Table 6: The Implied Recovery Rate: Simple CB

Recovery parameters		Initial values of (V_t, r_t)		
		$(25, 0.06)$	$(30, 0.06)$	$(35, 0.06)$
(s, σ^*)	$(1, 0.35)$	0.854	0.885	0.904
	$(0.25, 0.25)$	0.913	0.942	0.959
	$(0.25, 0.45)$	0.904	0.935	0.953
	$(5, 0.25)$	0.675	0.697	0.711
	$(5, 0.45)$	0.577	0.606	0.628

Table 7: The Implied Recovery Rate, Riskier CB

Table 7 gives recovery rates for riskier CBs issued at a much lower asset value. The implied recovery rates are significantly smaller. With low asset values the put value is not negligible; default is no longer technical and CB holders will not find it optimal to convert, instead obtaining only the expected recovery rates given in the table. The affect of σ^* is now significant.

4.3 Exercise Conditions

We investigate the affect of the presence or absence of the various exercise conditions. We consider a riskless coupon bond with default and various combinations of exercise conditions added in, ending with the full specification of the base case CB. We also give an approximation to the value of $\partial D_t/\partial r$ found by central difference from CB values computed at different initial values of the interest rate.

Table 8 shows the results. 'Def' is defaultable (with recovery), 'Con' is convertible, 'Red' is redeemable and 'Call' is callable. Δ and Γ are the CB delta and gamma respectively.[11] The riskless bond values are Vasicek values computed analytically and shown for comparison. The base case value of the CB is 19.326, shown in bold.

With our specification and model parameters, the presense of default reduces the value of the corresponding riskless bond by about 5%. The bond has a high credit risk stemming from a relatively high endogenous default rate.

Adding the conversion feature increases the value of the CB by a percent or two. The effect is greater at higher levels of the initial interest rate. The

[11] These are reported 'plain', without division by the conversion ratio.

Exercise Conditions				V_0	r			$\partial D_t/\partial r$
Def	Con	Red	Call		0.05	0.06	0.07	
Riskless Bond				100	20.569	19.992	19.432	−56.9
				95	19.306	18.853	18.373	−46.6
√	-	-	-	100	19.311	18.858	18.378	−46.7
(Defaultable bond)				105	19.315	18.862	18.381	−46.7
				Δ	8.6E-04	8.7E-04	8.2E-04	
				Γ	−4.3E-05	−4.4E-05	−4.1E-05	
				95	19.527	19.153	18.752	−38.8
√	√	-	-	100	19.565	19.204	18.816	−37.5
				105	19.601	19.256	18.881	−36.0
				Δ	7.4E-03	1.0E-02	1.3E-02	
				Γ	−2.5E-05	1.6E-05	6.0E-05	
				95	19.624	19.295	18.948	−33.8
√	√	√	-	100	19.649	19.329	18.989	−33.0
				105	19.675	19.365	19.034	−32.0
				Δ	5.1E-03	7.0E-03	8.6E-03	
				Γ	3.7E-05	8.8E-05	1.5E-04	
				95	19.525	19.150	18.748	−38.9
√	√	-	√	100	19.563	19.201	18.812	−37.6
				105	19.599	19.252	18.877	−36.1
				Δ	7.4E-03	1.0E-02	1.3E-02	
				Γ	−2.5E-05	1.7E-05	6.0E-05	
				95	19.623	19.293	18.945	−33.9
√	√	√	√	100	19.647	**19.326**	18.986	−33.1
(Base case)				105	19.673	19.362	19.030	−32.1
				Δ	5.0E-03	7.0E-03	8.5E-03	
				Γ	3.6E-05	8.8E-05	1.5E-04	

Table 8: Effect of Exercise Features

introduction of the call feature slighlty reduces the value of the CB, and the redemption feature increases the value of the CB by about half a percent.

Asset deltas are small and vary only a little as the initial interest rate changes. Introducing conversion to a straight defaultable bond increases the delta by a factor of 10. The call feature does not greatly affect the asset delta, and adding a redemption feature slightly reduces the asset delta.

$\partial D_t/\partial r$, the CB's rho, indicates the sensitivity of the CB to changes in the initial value of the interest rate. The conversion feature reduces the CB's rho by about a fifth. The redemption feature reduces rho still further, but in this case the call feature has relatively little affect. Allowing the riskless bond to become defaultable reduces rho by roughly 20% and adding additional optionality reduces it further. For this CB, additional optionality effectively decreases the interest rate exposure of the CB and increases (little) its asset value exposure.

Convertibility		Red.	Call	
			Low	High
Low	19.155	Low	19.195	19.208
High	19.764	High	19.323	19.332

Table 9: Sensitivities to changes in exercise conditions

	r-parameters				V-parameters		Corr.
Parameter:	r	α	θ	σ_r	V	σ_V	ρ
High:	18.986	19.326	19.190	19.293	19.362	19.375	19.335
Low:	19.647	19.327	19.458	19.352	19.293	19.271	19.318
Delta:	−33.1	−0.09	−13.4	−5.9	0.01	1.03	0.18
Gamma:	−197	0.00	−47.7	−303	0.00	−2.75	0.10

Table 10: Sensitivities to changes in parameter values

Table 9 shows the affect of changing exercise conditions. Since the CB is out of the money, changing the convertibility condition has a small effect. Increasing either the redemption level or the call level has an even smaller affect.

4.4 Parameter Deltas and Gammas

We investigate the sensitivity of the base case CB to changes in parameter values. We value the CB at the higher value and lower value of each parameter. The delta and gamma are then computed by central difference. Results are given in table 10. r is the initial value of the stochastic Vasicek interest rate. Later, table 13 considers the effect of changes to r where r is a constant interest rate.

Deltas are very small. σ_r has a greater delta than σ_V, but when σ_V is scaled by V (to make it comparable to an absolute volatility) the affect upon D is much smaller. Increasing the correlation ρ slightly increases the bond value. θ, the level to which r_t reverts, has a relatively large delta since it reflects the longer term value of r_t.

4.5 The Default Specification

We explore the consequences of changing the default specifications. Table 11 summarises the results. For base case values and each of four sets of default likelihood parameters we consider high and low values of the recovery parameters s and σ^*. The base case value is shown in bold.

For short s the value of the volatility parameter σ^* makes very little difference. For long s the effect is much more pronounced.

Increasing λ decreases the value of the CB. The effect is significant but not large, since default is infrequent and the loss in the event of default is not large.

Increasing b has no effect on D when a is large. When a is small increasing b increases the value of the CB when s is long, and decreases it otherwise.

Recovery Parameter		Default parameters						
		(λ, a, b)	(a, b)				λ	
		$(0.1, 0.03, 3)$	$(0.003, 0.3)$	$(0.003, 6)$	$(0.06, 0.3)$	$(0.06, 6)$	0.01	0.2
(s, σ^*)	$(1, 0.35)$	**19.326**	19.285	19.291	19.331	19.331	19.331	19.321
	$(0.25, 0.25)$	19.567	19.585	19.576	19.567	19.567	19.567	19.567
	$(0.25, 0.45)$	19.566	19.583	19.575	19.566	19.566	19.566	19.566
	$(5, 0.25)$	16.999	16.836	16.882	17.013	17.014	17.014	16.983
	$(5, 0.45)$	16.550	16.331	16.392	16.570	16.571	16.572	16.528

Table 11: Sensitivities to changes in default parameters

Recovery Parameters		Loss rate, w		
		0.2	0.4	0.6
V_0	80	19.204	19.203	19.189
	100	19.326	19.326	19.322
	120	19.482	19.482	19.481

Table 12: Effect of Different Loss Rates

Increasing a increases the value of the bond except when s is short.

We look at the affect upon the CB value of varying the loss rate, w, for different initial asset values. Table 12 presents the results.

We see that the loss rate has very little affect for this CB, since the rate of exogenous default in the base case is quite low.

4.6 The Effect of a Stochastic Interest Rate

We have seen the effect upon the bond value of changes is the parameters of the interest rate process. We can also test to find the extend of the affect upon the bond price of a stochastic interest rate. By setting $\sigma_r = 0$ and $r = \theta$ we effectively make r non-stochastic. We investigate the presence of a stochastic interest rate in more detail. Table 13 gives the results, looking at several sets of initial conditions.[12]

Since the coupon rate is close to current are future interest rate levels the CB price remains relatively stable as T increases. As we have seen elsewhere, the value of the CB is relatively insensitive to the initial asset value V_0 but more so to the conversion parameter κ.

Comparing to table 10, we see that making r constant at its initial value has the effect of increasing the value of the CB when r starts high and decreasing the value when r starts low. Consistent with this, increasing the interest rate volatility when the rate is stochastic decreases the value of the CB, except for short times to maturity.

[12] When T increases or decreases, the time step Δt is held constant and the number of time steps is varied.

Initial Conditions		Stochastic r			Constant r		
		σ_r			r		
		0.015	0.02	0.025	0.05	0.06	0.07
T	1	18.868	18.873	18.880	19.231	18.863	18.502
	5	19.352	19.326	19.293	20.071	19.342	18.782
	10	19.431	19.376	19.327	20.088	19.485	18.867
V_0	95	19.316	19.293	19.261	20.061	19.294	18.730
	100	19.352	**19.326**	19.293	20.071	19.342	18.782
	105	19.390	19.362	19.329	20.078	19.390	18.840
κ	0.05	19.160	19.155	19.132	19.997	19.041	18.527
	0.1	19.352	19.326	19.293	20.071	19.342	18.782
	0.15	19.781	19.764	19.743	20.147	19.843	19.521

Table 13: Effect of a stochastic interest rate

5 Conclusions

In this paper we have introduced a two-factor model for defaultable convertible bond pricing. The state variables are the firm asset value and the short interest rate. Default can be exogenous, at the jump time of a counting process, or endogenous at times that the firm must make a cash payment. We endogenise recovery into the model by supposing that upon default the firm enters a re-organisation period.

We price convertible bonds by solving numerically a PDI using finite elements to discretise in space and a method of characteristics to discretise in time. Early exercise is dealt with using a duality method in the variational formulation of the discretised problem.

Care has been taken to specify correctly the boundary conditions in the model, ensuring that these are financially and numerically consistent.

We have investigated the effect of introducing a stochastic interest rate and we have explored the consequences of our default, recovery and loss specification, finding that a wide range of recovery levels are possible, linked to a natural interpretation of the recovery process.

The sensitivity of the CB value to changes in the initial values of the asset and the interest rate have been investigated. We have found that the CB has a very small asset delta and a low sensitivity to the initial interest rate. The conversion feature increases the CB asset delta by a factor of 10.

We believe that the modelling framework presented in this paper is flexible and more realistic than formulations based upon the firm equity value as a state variable. Our endogenised recovery specification potentially allows a greater ability to estimate recovery values from the market.

We conclude that the flexible specifiation of this model may give it greater potential to explain empirical CB values than existing models in the literature.

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