An Asset Based Model of Defaultable Convertible Bonds with Endogenised Recovery

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Valuing a Convertible Bond

Security issued by a firm.
Firm initially all equity.

Asset value: V_t.
Equity Value: S_t.
CB value: D_t.

Principal F, maturity time T,
Coupon P_i at times t_i,
$t_1 < t \ldots t_N = T$.

Callable:
Call price C_t at call times $t \leq T$.
Redeemable:
Redemption price R_t at redemption times $t \leq T$.
Convertible:
CB holders can convert to equity at times $t \leq T$.
Get value $\kappa_t V_t$ for a conversion proportion κ_t.
κ_t subsumes dilution effects.

What we do.

Existing practitioner literature on CB behaviour.
Academic literature: sparse.

1) Construct an asset based CB model, stochastic interest rates + defaultable, with

2) Endogenised recovery.

3) Find financially consistent boundary conditions.

4) Apply sophisticated numerics.

5) Quantify the behaviour of CB prices in the model.

(Do not do empirics).

Modelling issues

1) State variable: asset value V_t or equity value S_t?
2) Stochastic interest rates?
3) Defaultable?

4) How many state variables?

Early models: asset based.

Relatively straightforward to specify.

Harder to calibrate.

Recent models: majority equity based.

Harder to avoid modelling inconsistencies,

- but does it matter?

Easier to calibrate: easier numerics (splitting)

We choose:

Asset based state variable V_t.

One -factor stochastic interest rate r_t.

Both exogenous and endogenous default.

Previous literature

		State variable	
		V_t	S_t
Stochastic interest rate	**Defaultable**	**Structural default** Brennan & Schwartz; Nyborg; Carayannopoulos; **Reduced + Structural** Bermudez & Webber;	**Credit spread** Ho & Pteffer; **Reduced form** Yigitbasioglu;
	No default	Carayannopoulos;	Epstein, Haber & Wilmott; Baroni-Adesi, Bermudez & Hatgioannides; Bermudez & Nogueiras; Zvan, Forsyth & Vetzal;
Non-stochastic interest rate	**Defaultable**	**Structural default** Ingersoll; Brennan & Schwartz;	**Reduced form** Andersen & Buffum; Takahashi, Kobayashi & Nakagawa; Ayache, Forsyth & Vetzal; Tseveriotis & Fernandes; Davis & Lischka;
	No default		McConnel & Schwartz;

Our model is V_t based, stochastic interest rate, and reduced form + structural default.

Dimensional problems

In practice:

Three or more state variables not reasonable.
Will (almost always) need numerical methods.

Effective dimension three or more?

Monte Carlo:	Hard to get acceptable accuracy. American features are trickier.
Lattices:	May get non-recombining lattices, hard to get good accuracy.
PDE methods:	For the experts only. For the rest of us: stop working.

Basic problem:

Can't solve the system of SDEs,
or approximate solutions sufficiently closely.

Our solution method (two state variables):

Tailored Lagrange-Galerkin discretisation,
Lagrange multipliers for free boundaries.

Default specifications

Two main approaches in academic credit literature:

Structural (endogenous)

Default when a barrier is hit,

eg if $V_T < F$ at maturity.

Reduced form (exogenous)

Default when hit by a default event.

Events arrival times:

- compound Poisson process jump times.

Need to specify:

How default occurs.

What happens when default occurs.

Equity based models: specification problems.

eg, require $D_t \rightarrow 0$ when $S_t \rightarrow 0$.

Our specification:

i) Firm hit exogenously by default events.

ii) Firm can choose to default at a cashflow time.

Endogenous.

The Asset process

We suppose the objective asset process is

$$dV_t = \mu_t V_{t-} dt + \sigma_V V_{t-} dz^V{}_t - w_t V_{t-} dN_t,$$

z^V, a standard Brownian motion,

N, a counting process intensity λ_t,

Firm defaults (exogenously) at first jump time of N.
w_t is proportional loss to V_{t-} on default (deterministic).

Use accumulator account numeraire $B_t = \exp(\int_o^t r_s ds)$.
Then

$$\mu_t = r_t + \underline{\lambda}_t w_t,$$

where $\underline{\lambda}_t = \lambda_t(1-\gamma)$ is jump intensity under the EMM.

We assume $w_t = w$ is a constant.
We model $\underline{\lambda}_t$ directly.

The hazard rate $\underline{\lambda}_t$.

We set

$$\underline{\lambda}_t = \underline{\lambda}(V_t, r_t),$$

a function of V_t and r_t.

For concreteness we choose:

$$\underline{\lambda}(V_t, r_t) = \lambda.\exp(-aV_t - br_t), \quad \text{for } \lambda, a, b \geq 0.$$

$\underline{\lambda}_t$ does not go to ∞ when $V_t \to 0$,
however, $D_t \to 0$ in any case.

$\underline{\lambda}_t$ decreases when V_t and r_t increase.
Other specifications possible...

(We are not too concerned over exact specification).

Stochastic interest rates

Apparently desirable.
Problem with total numbers of state variables.
In practice: need a one-factor interest rate model.

Other literature:

Uses CIR, BDT, etc.

We use (extended) Vasicek,

$$dr_t = \alpha(\mu(t) - r_t)dt + \sigma dz^r_t, \quad dz^r_t dz^V_t = \rho dt.$$

1) Simple.
2) Extended Vasicek can fit to term structure.
3) Makes boundary conditions easier.
4) Get helpful explicit solutions for options.

We do not mind that rates could become negative.
In fact, this is very helpful numerically.

Endogenous (Structural) Default

Firm can choose to default at a cashflow date:

A redemption date,

A coupon or principal payment date.

At a default date (endogenous or exogenous):

CB holders can choose to convert.

Default can occur when V_τ much greater than F.

In this case CB holders would choose to convert:

Default is ‘technical’ - wouldn’t actually occur.

The Bondholders Claim

Defaults time τ (exogenously or endogenously).
The bondholders have a claim against the firm, F_τ.

We set:

$$F_t = \begin{cases} F + P_N, & \text{if } t = T, \\ F + P_i, & \text{if } t = t_i, \\ F, & \text{if } t \neq t_i \text{ for some } i, \\ & \text{(eg if default is exogenous or at a redemption time)} \end{cases}$$

Note:

1) Could choose to put in accrued interest if defaults at a non-coupon date.
2) On a redemption date could set

$$F_t = F \vee R_t.$$

Recovery on Default

Default at time τ, CB holders have claim of F_τ.

Actually recover F^*_τ.

How to determine.

In practice: Firms often go into reorganisation.

Get protection against bondholders.

We suppose:

Firm given respite period length s.

Claims deferred until end of respite period.

At time $\tau + s$ claim is worth

$$F^*_{\tau+s} = V_{\tau+s} \wedge F_\tau = F_\tau - (F_\tau - V_{\tau+s})_+.$$

No intermediate cashflows, then

$$F^*_\tau = Pv(F_\tau) - p(V_\tau, F_\tau)$$

for a put on V_τ with strike F_τ. (Pv is over period s).

<u>Recovery endogenised</u>.

CB holders may choose to convert upon default.

CB value upon default is $D^*_\tau = \kappa_\tau V_\tau \vee F^*_\tau$.

Elaboration

Here, claim accrues no interest,
could assume that claim <u>does</u> accrue interest.

Firm asset volatility:
May increase upon default.
Set to σ^* to compute put value.

Specifying default:
Freedom to choose (s,σ^*).
Fit to observed recovery rates etc.

N stops after first default time:
firm defaults only once.
V_t has GBM after default.
Interest rates Vasicek, $\rho = 0$,
then $p(V_\tau,F_\tau)$ has explicit solution.

We use this even when $\rho \neq 0$:
Effect is slight.

More on Exercise times

Not necessarily exercisable at all times.

If callable then convertible.

In numerical examples:

Suppose redeemable only at coupon dates,

Continuously callable and convertible.

For concreteness:

Call and redemption prices accrue interest.

$$C_t = C + \frac{t - t_i}{t_{i+1} - t_i} P_{i+1}, \quad t_i \le t < t_{i+1},$$

$$R_t = R + \frac{t - t_i}{t_{i+1} - t_i} R_{i+1}, \quad t_i \le t < t_{i+1},$$

for constant C and R.

Obtaining a PDI

By Ito:

$$dD_t = \Big(\frac{\partial D}{\partial t} + (r_t + \underline{\lambda}_t w_t)V_t\frac{\partial D}{\partial V} + \tfrac{1}{2}\sigma_V{}^2 V_t{}^2\frac{\partial^2 D}{\partial V^2}$$

$$+ \mu_r\frac{\partial D}{\partial r} + \tfrac{1}{2}\sigma_r{}^2\frac{\partial^2 D}{\partial r^2} + \rho\sigma_V V_t\mu_r\frac{\partial^2 D}{\partial V\partial r}\Big)dt$$

$$+ \sigma_V V_t\frac{\partial D}{\partial V}dz^V{}_t + \sigma_r\frac{\partial D}{\partial r}dz^r{}_t$$

$$+ \Delta D_t(V_{t-})$$

where $\Delta D_t(V_{t-})$ is change of CB value on a jump, ie, when default occurs, at time t,

$$\Delta D_t(V_{t-}) = D^*{}_t(V_t) - D_{t-}(V_{t-}).$$

When the EMM D_t/B_t is a martingale, hence

$$r_t D_t = \frac{\partial D}{\partial t} + (r_t + \underline{\lambda}_t w_t)V_t\frac{\partial D}{\partial V} + \tfrac{1}{2}\sigma_V{}^2 V_t{}^2\frac{\partial^2 D}{\partial V^2}$$

$$+ \mu_r\frac{\partial D}{\partial r} + \tfrac{1}{2}\sigma_r{}^2\frac{\partial^2 D}{\partial r^2} + \rho\sigma_V V_t\mu_r\frac{\partial^2 D}{\partial V\partial r}$$

$$+ \underline{\lambda}_t E_{t-}[D^*{}_t(V_t) - D_{t-}(V_{t-})]$$

Deterministic loss and recovery?

PDI is:

$$(r_t + \underline{\lambda}_t)D_t = \underline{\lambda}_t D^*_t + \frac{\partial D}{\partial t} + (r_t + \underline{\lambda}_t w_t)V_t\frac{\partial D}{\partial V} + \tfrac{1}{2}\sigma_V{}^2 V_t{}^2\frac{\partial^2 D}{\partial V^2} + \mu_r\frac{\partial D}{\partial r} + \tfrac{1}{2}\sigma_r{}^2\frac{\partial^2 D}{\partial r^2} + \rho\sigma_V V_t \mu_r\frac{\partial^2 D}{\partial V \partial r}$$

where

$D^*_t = D^*_t(V_t) = \kappa_t V_t \vee F^*_t$,

$F^*_t = Pv(F_\tau) - p(V_\tau, F_\tau)$,

$V_t = (1-w_t)V_{t-}$.

Interest rate constant? Reduces to:

$$(r_t + \underline{\lambda}_t)D_t = \underline{\lambda}_t D^*_t + \frac{\partial D}{\partial t} + (r_t + \underline{\lambda}_t w_t)V_t\frac{\partial D}{\partial V} + \tfrac{1}{2}\sigma_V{}^2 V_t{}^2\frac{\partial^2 D}{\partial V^2}$$

(as in Ayache, Forsythe and Vetzal).

Inequality Constraints and Auxiliary Conditions

At time T:

$D_T = D^*_T = \kappa_T V_T \vee F^*_T$.

At redemption and call times:

$(F^*_t \wedge R_t) \vee \kappa_t V_t \le D_t \le (V_t \wedge C_t) \vee \kappa_t V_t$

At coupon dates t:
(Exercise conditions constant across coupon dates).

Firm might call just before t,
CB holder might redeem just after t.
Firm acts to maximise equity value.
Critical value V^*_{t-}, $D^*_{t-}(V^*_{t-}) = P_t + D^*_t(V^*_{t-})$.

If $V_{t-} < V^*_{t-}$, firm defaults, pays no coupon,

$D_t(V^*_{t-}) = D^*_{t-}(V_{t-}) = F^*_{t-} \vee \kappa_{t-} V_{t-}$,

If $V_{t-} \ge V^*_{t-}$, no default, firm pays the coupon,

$V_t = V_{t-} - P_t$,

$D_t(V_t) = P_t + D_{t-}(V_{t-})$, and

$(F^*_t \wedge R_t) \vee \kappa_t V_t \le D_t \le (V_t \wedge C_t) \vee \kappa_t V_t$.

This Model:

Avoids problems with equity based models.
 - Financial consistent boundary conditions.

Includes a stochastic interest rate

Has ‘realistic’ default.

Has endogenised recovery; two intuitive parameters.

Can investigate its behaviour.
Need to solve numerically.