Ruin, Operational Risk and How Fast Stochastic Processes Mix

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Based on joint work with:

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Basel Committee for Banking Supervision ([3], [7])

- Basel Accord (= Basel I) (1988)
  - credit risk

- Amendment to Basel I (1996)
  - market risk
  - netting
  - derivatives, Value-at-Risk based

- Basel II (1998–2005/6)
  - (internal) rating models for credit risk
  - increased granularity
  - new risk category: operational risk

- Increased collaboration between insurance- and banking supervision: integrated risk management

Definition(s) of operational risk:

- (non)definition (early): the complement of market risk

- coming from DFA: the company specific risk, uncorrelated with capital markets, non-systematic part (frictional costs)

- current definition in use through Basel II:
  Operational risk is the risk of losses resulting from inadequate or failed internal processes, people and systems or from external events.

Some examples:

- Barings, £700,– Mio

- Allied Irish (Allfirst subsidiary), US$700,– Mio

- Bank of New York, US$140,– Mio

Disclosed figures:

- 2001 Annual Reports, disclosure for economic capital for operational risk:
  - Deutsche Bank: € 2.5 Bio
  - JP Morgan-Chase: US$ 6.8 Bio

- Estimated total losses 2001 in USA: US$ 50 Bio

September 2001 BIS Quantitative Impact Study:

- credit (51%), market (23%), operational (16%), other (10%)

Three Pillar concept of Basel II:

- Pillar I: Minimal Capital Requirement

- Pillar II: Supervisory Review Process

- Pillar III: Market Discipline Requirement

These apply to both credit- as well as operational risk

Pillar I (Minimal Capital Requirement) for Operational Risk

- The Basic Indicator Approach:

  - $RC(OR) = \alpha \, GI$

- The Standardized Approach:

  - $RC(OR) = \sum_{i=1}^{8} \beta_i \, GI_i$

- The Advanced Measurement Approach:

  - $RC(OR) = \sum_{i=1}^{8} \sum_{k=1}^{7} \gamma_{i,k} \, e_{i,k}$

  - $RC(OR) = \sum_{i=1}^{8} \rho_{i,k}$

Eight standardized business lines:

- Corporate Finance; Trading and Sales; Retail Banking; Payment and Settlement; Agency Services; Commercial Banking; Asset Management; Retail Brokerage

Seven loss types:

- Internal Fraud; External Fraud; Employment Practices and Workplace Safety; Clients, Products and Business Practices; Damage to Physical Assets; Business Disruption and System Failure; Execution, Delivery and Process Management

In total: 56 categories to model!

Some critical remarks:

- business risk (though very important) is explicitly excluded

- distinguish between

  - repetitive versus non-repetitive losses

  - low frequency/high impact versus high frequency/low impact

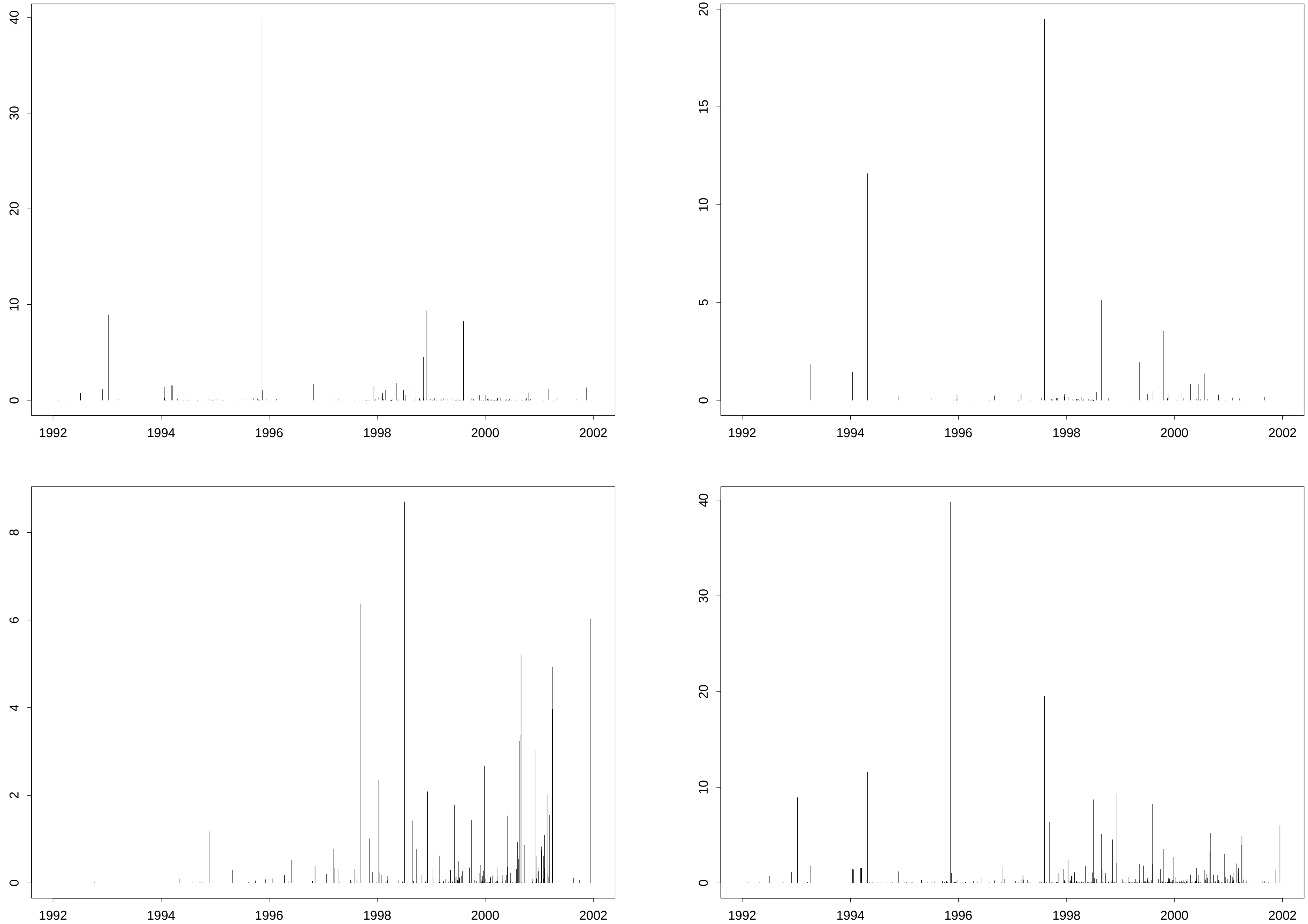
- lack of data, data pooling (?), near misses (??)

- Pillar II very important

- for the moment: qualitative $\gg$ quantitative

- overall complexity (Comptroller of the Currency)

Some data ([5])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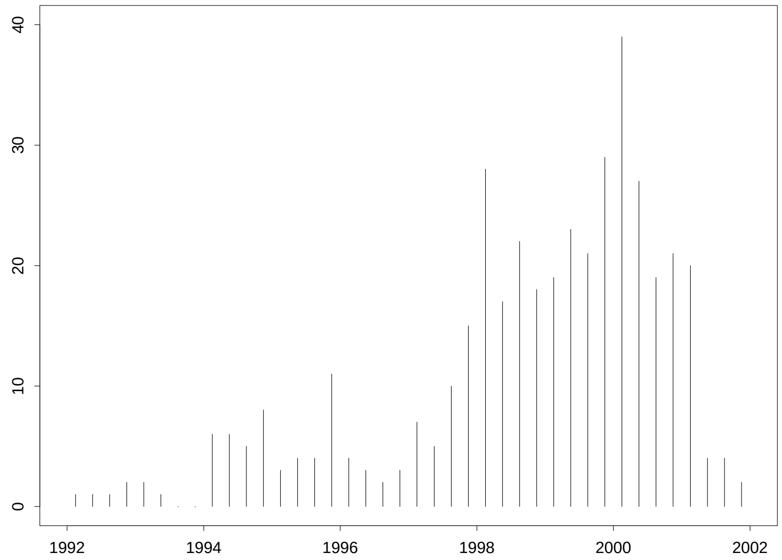
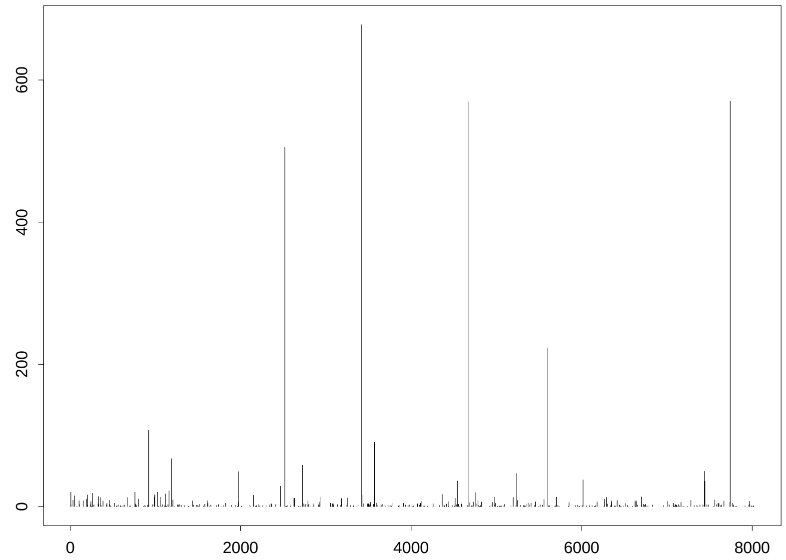
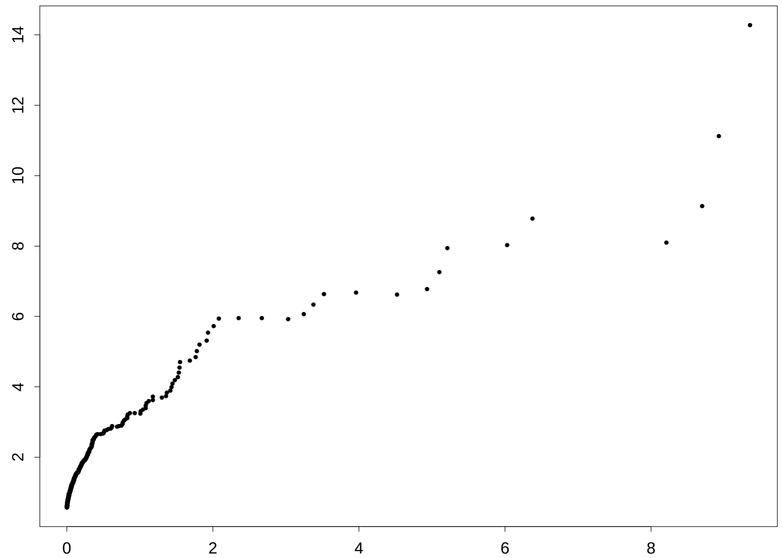
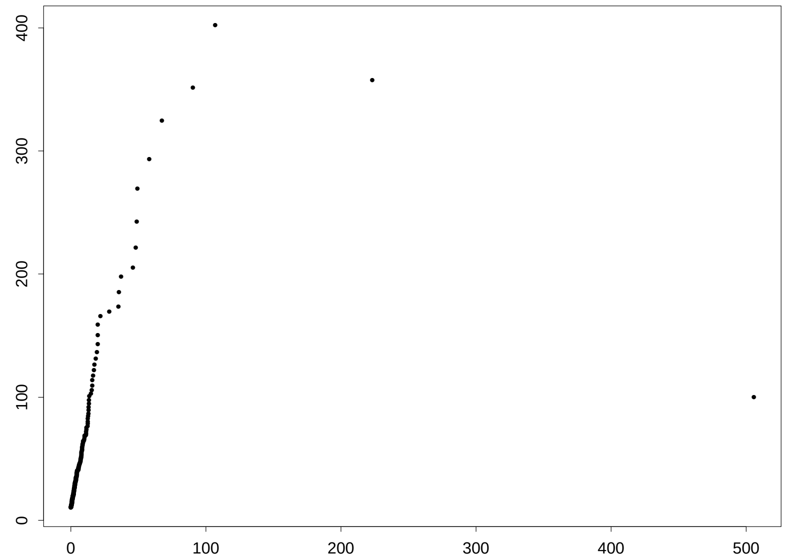


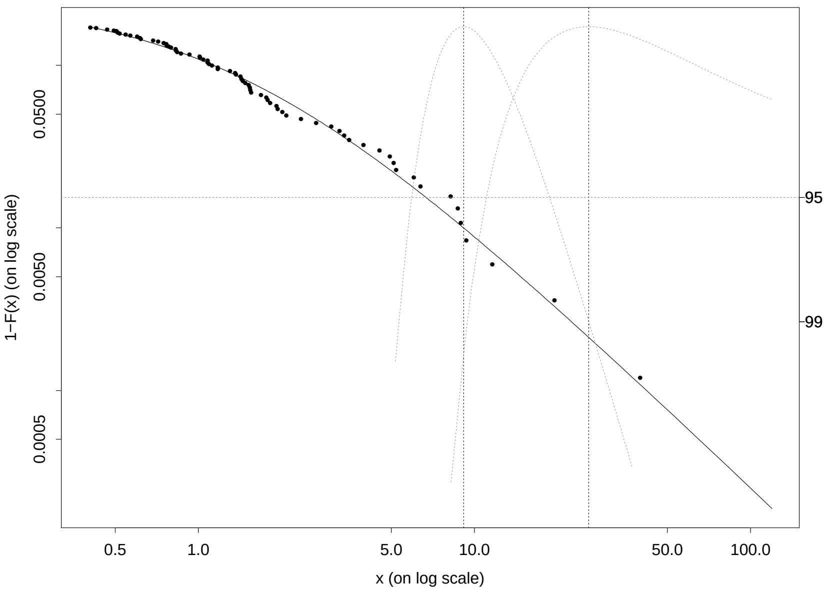


operational risk

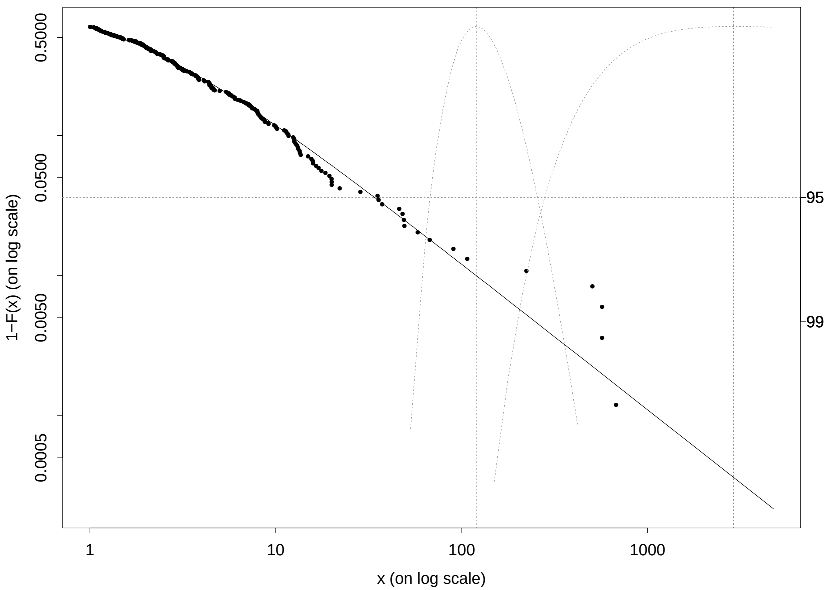
fire loss







operational risk



fire loss

A mathematical (actuarial) model:

- Operational Risk loss database (for each business line)

$$\{X_k^{t,i} : t = 1, \ldots, T; i = 1, \ldots, 7; k = 1, \ldots, N^{t,i}\}$$

  t (years), i (loss type), k (number of losses)

- Truncation

$$X_k^{t,i} = X_k^{t,i} \, I_{\{X_k^{t,i} > d^{t,i}\}}$$

  and (random) censoring

- a further index indicating business line can be introduced (deleted for this talk)

Loss amounts:

- $L_t = \sum_{i=1}^{7} \sum_{k=1}^{N^{t,i}} X_k^{t,i}, \; t = 1, \ldots, T$

- $L_t = \sum_{i=1}^{7} L_{t,i}$

Pillar I modelling:

- F_{L_t} and $F_{L_{t_i}}, \; i = 1, \ldots, 7$

- risk measurement (e.g. for L_t)

  $\text{OR-VaR}_{T+1}^{1-\alpha} = F_{L_{T+1}}^{\leftarrow}(1-\alpha)$, α (very) small

  $\text{OR-CVaR}_{T+1}^{1-\alpha} = E(L_{T+1} \mid L_{T+1} > \text{OR-VaR}_{T+1}^{1-\alpha})$

Question: Suppose we have calculated risk measures $\rho^i_{T+1,1-\alpha}$, $i = 1, \ldots, 7$, for each risk category. When can we consider

$$\sum_{i=1}^{7} \rho^i_{T+1,1-\alpha}$$

as a "good" risk measure for the total loss L_{T+1}?

Answer: Ingredients

- (non-) coherence of risk measures (Artzner, Delbaen, Eber, Heath framework)
- optimization problem: given $(\rho^i_{T+1,1-\alpha})_{i=1,\ldots,7}$, what is the worst case for the overall risk for L_{T+1}?

  Solution: using copulas in [4] and references therein
- aggregation of banking risks ([1])

(Methodological) link to risk theory:

- operational risk process

$$V_{i,t} = u_i + p_i(t) - L_{t,i}, \;\; t \geq 0$$

for some initial capital u_i and a premium function $p_i(t)$ satisfying

$$P(L_{t,i} - p_i(t) \to -\infty) = 1$$

- given $\epsilon > 0$, calculate $u_i(\epsilon)$ so that

$$P(\inf_{T \leq t \leq T+1} (u_i(\epsilon) + p_i(t) - L_{t,i}) < 0) \leq \epsilon \tag{1}$$

$u_i(\epsilon)$ is a risk capital charge (internal)

Solving for (1) is difficult:

- complicated loss process $(L_{t,i})_{t \geq 0}$
- heavy-tailed case
- finite horizon $[T, T+1]$

Hence:

- only approach possible: Monte Carlo
- rare event simulation
- non-standard situation, see [2]!

From a mathematical point of view:

- heavy-tailed ruin estimation for general risk processes

Classical Cramér-Lundberg model (new notation):

- $Y(t) = \sum_{k=1}^{N(t)} Y_k,\ t \geq 0$ where

  - (Y_k) iid $\sim F_Y$, independent of $(N(t)) \sim HPOIS(\lambda)$

  - NPC: $\lambda\, E(Y_1) < c$

- risk process $\{u + ct - Y(t) : t \geq 0\}$

- infinite-horizon ruin probability:

$$\begin{aligned}\Psi_1(u) &= P(\inf_{t\geq 0}(u + ct - Y(t)) < 0) \\ &= P(\sup_{t\geq 0}(Y(t) - ct) > u)\end{aligned}$$

  hence tail-probability of ultimate supremum

- NPC: $P(\lim_{t\to\infty}(Y(t) - ct) = -\infty) = 1$

In the heavy-tailed Cramér-Lundberg case:

$$1 - F_Y(y) \sim y^{-\beta-1}L(y) \Rightarrow \Psi_1(u) \sim cte\, u^{-\beta}L(u) \qquad (2)$$

$$(\beta \geq 0,\ L \text{ s.v.},\ y \to \infty) \qquad\qquad (u \to \infty)$$

Question: how general does (2) hold?

Solution: given a general stochastic process $\{Y(t) : t \geq 0\}$ for which we have that for some $c > 0$

- $P(\lim_{t\to\infty}(Y(t) - ct) = -\infty) = 1$, and
- $\Psi_1(u) = P(\sup_{t\geq 0}(Y(t) - ct) > u) \sim u^{-\beta}L(u),\ \beta \geq 0$
  $(u \to \infty)$

Starting from $(Y(t))$ define a more general process $(Y(\Delta(t)))$ using the notion of time change:

- $(\Delta(t))$ is a right-continuous process, non-decreasing, Δ and Y are both defined on the same probability space $(\Omega, \mathcal{F}, P)$ and $\Delta(0) = 0$

Define a new ruin function:

$$\Psi_{\Delta}(u) = P(\sup_{t \geq 0}(Y(\Delta(t)) - ct) > u)$$

How sensitive is ruin as a function of Δ?

More precisely:

Questions:

- under which (extra) conditions on Y and Δ does ruin behave similarly in both models, i.e.

$$\lim_{u \to \infty} \frac{\Psi_{\Delta}(u)}{\Psi_1(u)} = 1$$

- examples

- “link” to operational risk

References: [5] and [6]

Remark: why using time change?

- actuarial tool (Lundberg, Cramér (1930's)):

  inhomogeneous Poisson $\rightarrow$ homogeneous Poisson

- W. Doeblin (1940):

  Itô's formula via time change

- Olsen's Θ-time in finance (1990's):

  market data follows (geometric) BM in Θ-time

- Monroe's Theorem (1978):

  every semi-martingale can be written as a time changed BM

Conclusion: very powerful tool!

Solution to our problem:

- basic assumption: $\lim_{t\to\infty} \frac{\Delta(t)}{t} = 1,\ P -$ a.s.

- crucial: how fast does this convergence hold (mixing)

  $\forall \epsilon > 0 : g_\epsilon(u) = P(|\frac{\Delta(t)}{t} - 1| > \epsilon$ for some $t > u)$

- and define for $\epsilon > 0$ the perturbed ruin function

  $\Psi_{1,\epsilon}(u) = P(\sup_{t\geq 0}(Y(t) - c\,\epsilon\, t) > u)$

The solution very much depends on the behaviour of $g_\epsilon(u)$ and $\Psi_{1,\epsilon}(u)$ for $\epsilon > 0$.

The following basic assumptions hold in most cases:

(A1) (no early ruin in the original process)

$$\lim_{\delta\searrow 0} \limsup_{u\to\infty} \frac{P(\sup_{0\le t\le \delta u}(Y(t) - ct) > u)}{\Psi_1(u)} = 0$$

(A2) (a continuity assumption for ruin in the original process)

$$\lim_{\epsilon\searrow 1} \limsup_{u\to\infty} \frac{\Psi_1(u)}{\Psi_{1,\epsilon}(u)} = \lim_{\epsilon\nearrow 1} \liminf_{u\to\infty} \frac{\Psi_1(u)}{\Psi_{1,\epsilon}(u)} = 1$$

Theorem ([6])

Assume (A1) and (A2) hold, and that

$$\Psi_1(u) \sim u^{-\beta} L(u),\ \ u \to \infty, \beta \geq 0.$$

If (mixing condition)

$$\forall \epsilon > 0, \delta > 0 : \lim_{u \to \infty} \frac{g_\epsilon(\delta u)}{\Psi_1(u)} = 0$$

and either

i) Δ is continuous with probability 1,

or

ii) $\exists\, a \geq 0 : Y(t) + a(t)$ is eventually non-decreasing with probability 1,

then

$$\lim_{u \to \infty} \frac{\Psi_\Delta(u)}{\Psi_1(u)} = 1.$$

Reformulation:

"If the mixing rate of Δ is fast enough, i.e. $\frac{\Delta(t)}{t} \to 1$ fast enough measured with respect to the original ruin probability Ψ_1, then the ruin probability of the time-changed process Ψ_Δ is not affected by the time change."

Further results:

- slow mixing $\Rightarrow$ ruin is affected

- several examples (motivated by operational risk)

Example (Ingredients, details in [6])

- $\{Z_n : n \geq 0\}$ irreducible Markov chain on $\{1, \ldots, K\}$, stationary distribution function (π_i)
- $\{F_j : j = 1, \ldots, K\}$ holding time dfs with means (μ_i), finite
- take (r_i) so that $\sum_{j=1}^{K} r_j \mu_j \pi_j = \sum_{j=1}^{K} \mu_j \pi_j$
- time change $\Delta(0) = 0, \frac{d\Delta(t)}{dt} = r_j$ if (Z_n) at t is in j
- key assumption (heavy-tailed holding times):

$$\exists\, \overline{F}(x) \in RV(-\gamma), \gamma > 1 \text{ and } \lim_{x\to\infty} \frac{\overline{F}_j(x)}{\overline{F}(x)} = \Theta_j \in [0, \infty)$$

Theorem:
$$\lim_{u\to\infty} \frac{g_\epsilon(u)}{u\overline{F}(u)} = \frac{1}{\epsilon\gamma\overline{\mu}}\Big[\sum_{j\in J_+(\epsilon)} \Theta_j \pi_j (r_j - 1 - \epsilon)(r_j - 1)^{\gamma-1} + \sum_{j\in J_-(\epsilon)} \Theta_j \pi_j (1 - r_j - \epsilon)(1 - r_j)^{\gamma-1}\Big],$$

where $\epsilon > 0$ s.t. $\{j = 1, \ldots, K : |r_j - 1| = \epsilon\} = \emptyset$
and $J_+(\epsilon) = \{j = 1, \ldots, K : r_j > 1 + \epsilon\}$, $J_-(\epsilon) = \{j = 1, \ldots, K : r_j > 1 - \epsilon\}$

Conclusion

- at the moment, qualitative (Pillar II) handling of operational risk is more useful than quantitative (Pillar I) modelling

- actuarial methods are useful

- more data are needed

- interesting source of mathematical problems

- challenges: choice of risk measures, aggregation of risk measures

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