Volatility and Correlation Modelling

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Application to Portfolio Optimisation and Risk

by

Frederick Bourgoin

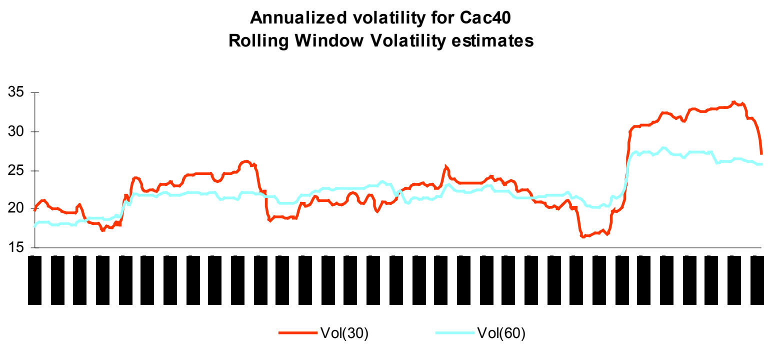
Barclays Global Investors - London

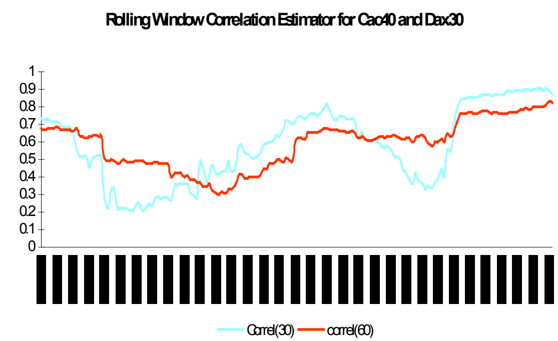
Email:frederick.bourgoin@barclaysglobal.com

Why everyone want to measure volatility and correlation?

- To measure how volatile are the markets,
- To measure risk adjusted performance for traders or fund managers,
- To perform risk management on portfolios of assets,
- To perform optimal asset allocation,
- To find optimal hedge ratios from asset to another,
- To price basket options, spread options, quantos,…
- To derive risk premiums from the markets,
- ….

Rolling Volatility and Correlation :
First problem : the choice of the rolling window





- Which length should we use to have an estimates: 30, 60 or 120 days?
- Ghost features are appearing :
  - When big moves happened in the market they impact the volatility estimates when they appear in the rolling window and then their effect vanishes when they are dropped from the rolling window, this means that we are overstating the values of volatilities when these points are at the beginning of the sample.
  - In a 30day rolling window volatility the information stays 30 days in the volatility.

GARCH Models: Basics

- Since Engle (82) and Bollerslev (86), the world went into GARCH (Generalized Autoregressive Conditionnally Heteroscedastic). What does it mean in practice?
  - It means that the model is estimated conditionally upon the information set available today (we call it a filtration in statistics and/or a market participant's mind).
  - All GARCH processes allow for Fat Tail in the unconditionnal distribution of the asset return.
  - Time varying volatility (volatility clusters)
    - highly volatile period tend to be followed by highly period,
    - tranquil periods tend to follow tranquil periods.
  - The volatility dynamics will depend on past values of itself and unexpected shocks from the markets.

GARCH Models: First Model

- Formally it means :

$$A(L)x_t = B(L)\varepsilon_t \qquad x_t \to ARMA(r,m)$$

$$\varepsilon_t / \Omega_{t-1} \to N(0, h_t)$$

$$h_t = \omega + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 + \sum_{j}^{p} \beta_j h_{t-j}$$

$$A(L) = \sum_{k=1}^{r} a_k L^k \qquad B(L) = \sum_{l=1}^{m} b_l L^l$$

$$\omega \geq 0, \alpha_i \geq 0, \beta_j \geq 0 \quad with \ \sum_{i=1}^{q} \alpha_i + \sum_{j}^{p} \beta_j \leq 1$$

- In practice although, x_t tend to follow a pure ARCH familly process with conditional mean zero.

$$x_t = \varepsilon_t \sqrt{\mathrm{h_t}} \quad \varepsilon_t / \Omega_{t-1} \to N(0,1) \quad h_t = \omega + f\left(\varepsilon_{t-1}^2, \ldots, \varepsilon_{t-q}^2 \, ; h_{t-1}, \ldots, h_{t-p}\right)$$

GARCH Models: The GARCH Family

ARCH(p) : $h_t = \omega + \sum_{i=1}^{p} \alpha_i \varepsilon_{t-i}^2$

GARCH(p,q) : $h_t = \omega + \sum_{i=1}^{p} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{q} \beta_j h_{t-j}$

EGARCH(p,q) : $Log(h_t) = \omega + \sum_{j=1}^{q} \beta_j \ Log(h_{t-j}) + \sum_{i=1}^{p} \left(\gamma_i \frac{\varepsilon_{t-i}}{\sqrt{h_{t-i}}} + \alpha_i \left[\frac{|\varepsilon_{t-i}|}{\sqrt{h_{t-i}}} - \sqrt{\frac{2}{\pi}} \right] \right)$

GJR(p,q) : $h_t = \omega + \sum_{j=1}^{q} \beta_j h_{t-j} + \sum_{i=1}^{p} \left(\alpha_i \varepsilon_{t-i}^2 + \gamma_i S_{t-i}^{-} \varepsilon_{t-i}^2 \right) \quad S_{t-i}^{-} = \begin{cases} 1 & if \ \ \varepsilon_{t-i} < 0 \\ 0 & Otherwise \end{cases}$

AGARCH(p,q) : $h_t = \omega + \sum_{i=1}^{p} \alpha_i \left(\varepsilon_{t-i} + \gamma_i \right)^2 + \sum_{j=1}^{q} \beta_j h_{t-j}$

NAGARCH(p,q) : $h_t = \omega + \sum_{i=1}^{p} \alpha_i \left(\varepsilon_{t-i} + \gamma_i \sqrt{h_{t-i}} \right)^2 + \sum_{j=1}^{q} \beta_j h_{t-j}$

VGARCH(p,q) : $h_t = \omega + \sum_{i=1}^{p} \alpha_i \left(\frac{\varepsilon_{t-i}}{\sqrt{h_{t-i}}} + \gamma_i \right)^2 + \sum_{j=1}^{q} \beta_j h_{t-j}$

GARCH Models : Answers
How can one compare
GARCH and RiskMetrics Volatilities

- JP Morgan as introduced Riskmetrics in 1994, a new way of measuring volatilities and correlations for risk management purposes.
- Let's recall RiskMetrics volatility estimator :

$$\sigma_t^2 = \frac{x_t + \lambda x_{t-1} + \lambda^2 x_{t-2} + \cdots + \lambda^n x_{t-n}}{1 + \lambda + \lambda^2 + \cdots + \lambda^n} \xrightarrow{n \to \infty} (1-\lambda)\sum_{i=0}^{\infty} \lambda^i x_{t-i} = (1-\lambda) x_t + \lambda \sigma_{t-1}^2$$

- and the GARCH(1,1) conditional variance equation : $\sigma_t^2 = \omega + \alpha \varepsilon_{t-1}^2 + \beta \sigma_{t-1}^2$
- Conclusion : RiskMetrics volatility estimator is a nested in the GARCH equation with Lambda = 0.94 for daily data
  - $\omega = 0 \quad \alpha = 1 - \lambda \quad \beta = \lambda \quad \Rightarrow \quad \alpha + \beta = 1$ It is a special case of IGARCH(1,1)
  - So we can test if the RiskMetrics volatilities are valid by just verifying if the restrictions holds

GARCH Models: Answers
How can one compare GARCH and RiskMetrics Volatilities

RiskMetrics© Test :

FX or Index	KhiSquared(3)	Significance Level
Dax30	284.951	0.00000
Ftse100	20.279	0.00015
N500	102.488	0.00000
Smi	252.645	0.00000
Cac40	73.420	0.00000
Mib30	191.101	0.00000
Ibex35	202.145	0.00000
Sp500	1,208.415	0.00000
Dem	62.097	0.00000
Gbp	79.146	0.00000
Jpy	58.839	0.00000
Chf	70.634	0.00000
Frf	78.209	0.00000
Itl	41.903	0.00000
Esp	42.538	0.00000

- None of financial time series tested here have a volatility dynamic that follow an exponential smoothing "a la RiskMetrics©"
- RiskMetrics volatility methodology doesn't provide a mean reverting term structure of forecast to a long term average volatility. This is due to the fact that Omega=0 and Alpha+Beta=1
- RiskMetrics all bad? no will explain why later.

Multivariate GARCH Models : WHY ?

- Modelling a single asset return volatility is unfortunately not as useful as modelling an entire covariance structure of multiple assets because :
  - Risk manager are very interested to measure accurately the correlations between each asset in their portfolio to perform VaR calculation,
  - Bond or Equity managers want to dynamically allocated money efficiently between assets using portfolio optimisation principles by enhancing the Markowitz framework with a GARCH correlation structure,
  - The option trader who need to price basket options which are neither liquid nor quoted in the market and need an accurate measure of correlation between each assets,
  - ….

Multivariate GARCH Models : Scalar GARCH

- The scalar GARCH generalise the exponential smoother in an important fashion :

$$h_{ij,t} = \omega_{ij} + \alpha \varepsilon_{i,t-1} \varepsilon_{j,t-1} + \beta h_{ij,t-1} \quad \forall i, j$$

- If there is N assets in the portfolio, the model has $\frac{N^2}{2} + \frac{7N}{2}$ parameters
- in matrix notation :

$$vech(H_t)' = \begin{vmatrix} h_{11,t} \\ h_{12,t} \\ h_{22,t} \end{vmatrix} = \begin{vmatrix} \omega_{11} \\ \omega_{12} \\ \omega_{22} \end{vmatrix} + \begin{vmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{vmatrix} \cdot \begin{vmatrix} \varepsilon_{1,t-1}^2 \\ \varepsilon_{1,t-1}\varepsilon_{2,t-1} \\ \varepsilon_{t-1}^2 \end{vmatrix} + \begin{vmatrix} \beta & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \beta \end{vmatrix} \cdot \begin{vmatrix} h_{11,t-1} \\ h_{12,t-1} \\ h_{22,t-1} \end{vmatrix}$$

- This is the simplest multivariate GARCH model because the whole covariance matrix depend on 2 parameters for its dynamic and n(n+1)/2 parameters for its average, and if W is semi definite positive H(t) will be semi definite positive, this is very appealing!!!

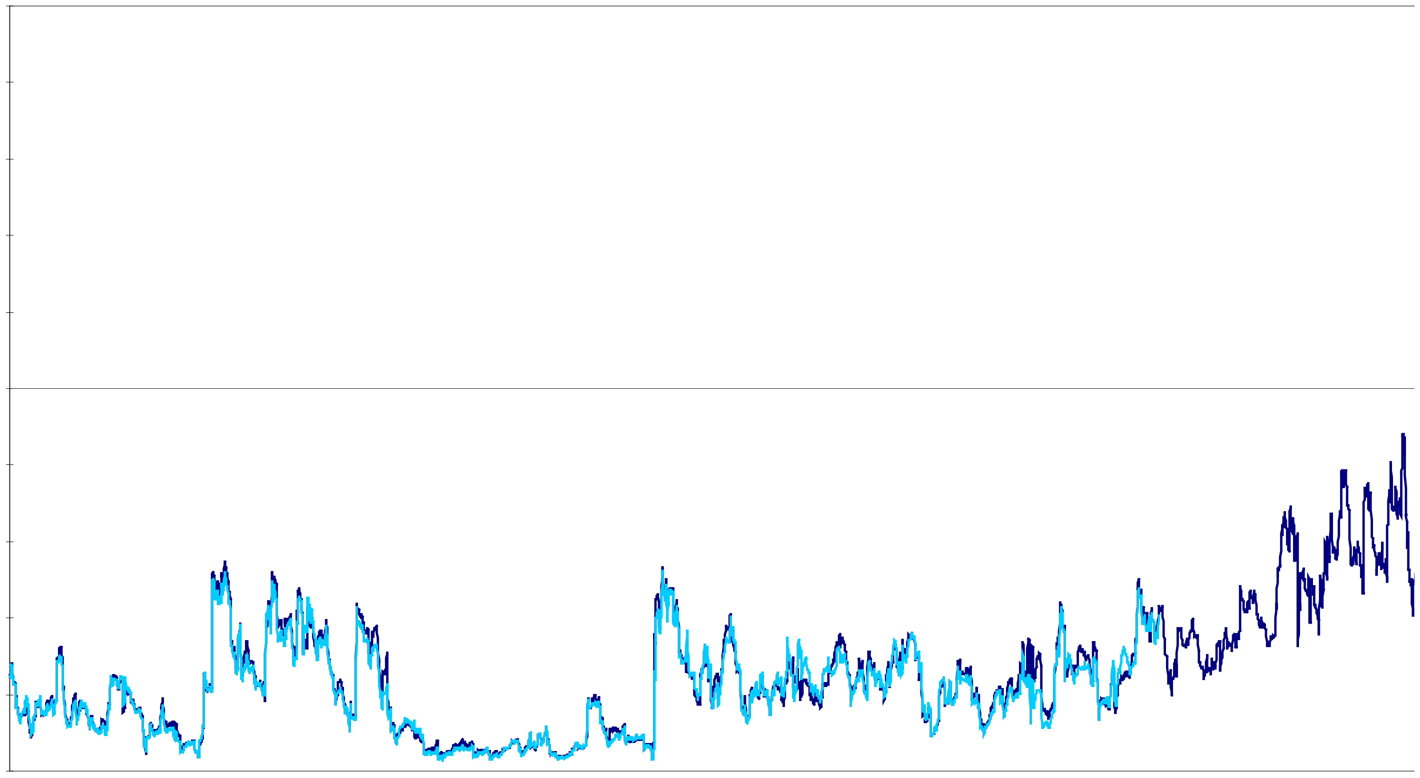
Multivariate GARCH Models : Scalar GARCH Pros & Cons

- Critics :
  - All volatilities and correlations have the same dynamics,
  - There is no distinction between permanent and transitory volatilities and correlations,
  - There is no asymmetry response to the unexpected news,
  - There is no feedback from different volatilities and correlations to others.
- Advantages :
  - easy interpretation, easy to compute for large portfolios
  - Variance targeting can be used to simplify the model even further by replacing omega(ij) by its long term covariate, recall the long term volatility :

$$\bar{\sigma}^2 = \frac{\omega}{1-\alpha-\beta} \Leftrightarrow \omega = (1-\alpha-\beta)\bar{\sigma}^2 \Rightarrow \bar{\sigma}_{ij}{}^2 = \frac{\omega_{ij}}{1-\alpha-\beta} \Leftrightarrow \omega_{ij} = (1-\alpha-\beta)\bar{\sigma}_{ij}^2$$

  - In this set-up the model as only two parameters!!!

Multivariate GARCH Models : Scalar GARCH : An example



Multivariate GARCH Models : How does RiskMetrics© fit in this framework?

- RiskMetrics© daily covariance is as follows :

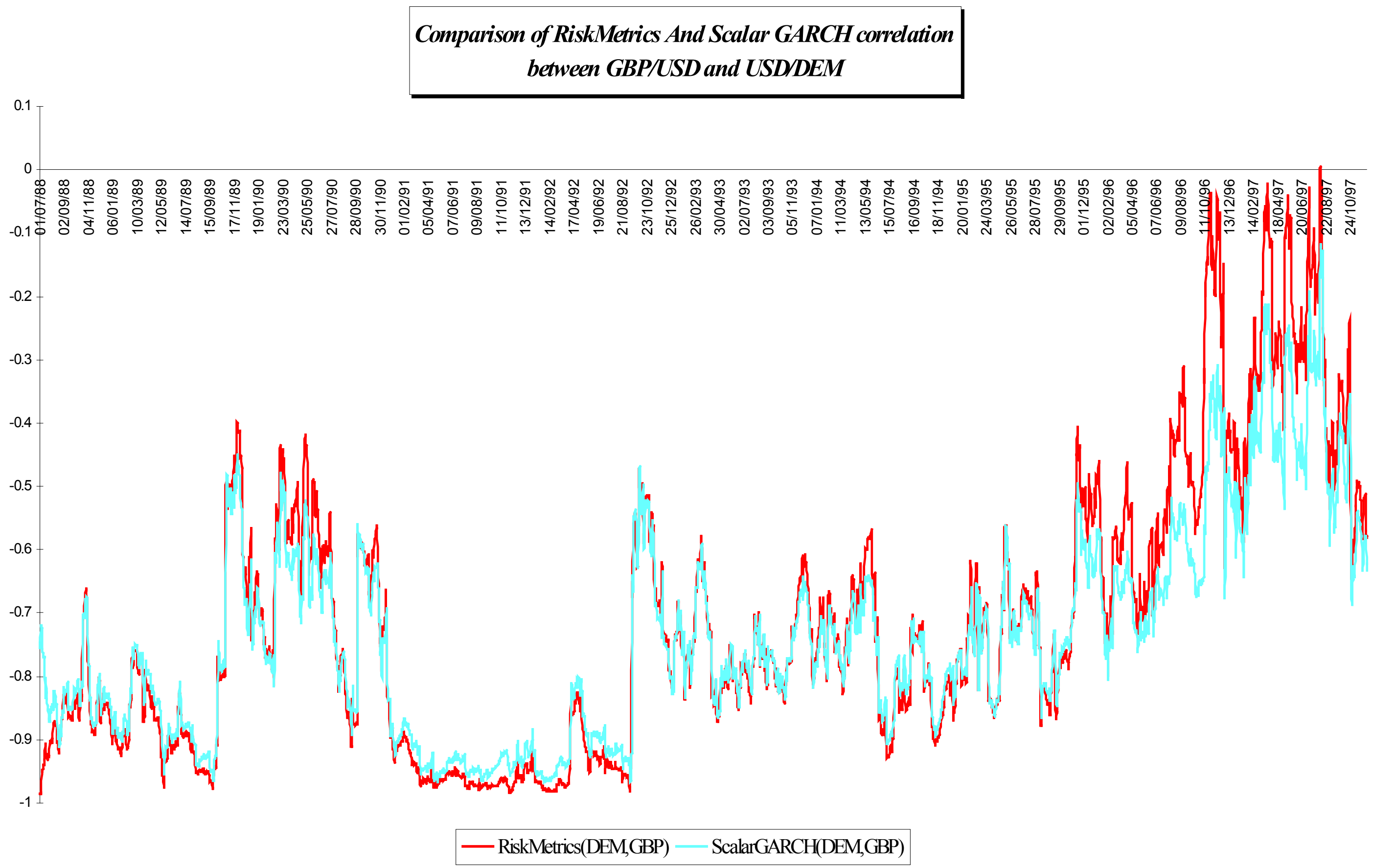
$$h_{ij,t} = 0.06\varepsilon_{i,t-1}\varepsilon_{j,t-1} + 0.94h_{ij,t-1} \quad \forall i, j$$

- It is very appealing to calculate for very large covariance matrix (few hundred assets) that's why it has been widely adopted for risk management purpose but it doesn't mean revert to a long term value and shocks are persistent in the future.
- The RiskMetrics© covariance is embedded in the scalar GARCH, so we can test it, using the previous example by :

$$H_0 = \begin{cases} \omega_{ij} = 0 \\ \alpha = 0.06 \\ \beta = 0.94 \end{cases} \qquad H_A \neq H_0$$

The Chi Squared Statistic is rejected at 1% (X(12)=10801!!!)

Multivariate GARCH Models : How does RiskMetrics© fit in this framework?



Multivariate GARCH Models : The Diagonal Vech model

- As we just mentioned earlier, we can have hij(t) in same form as in the univariate case but this time each covariance component can have its own dynamic (i.e. alpha and beta), the model look like this :

$$h_{ij,t} = \omega_{ij} + \alpha_{ij}\varepsilon_{i,t-1}\varepsilon_{j,t-1} + \beta_{ij}h_{ij,t-1} \quad \forall i,j$$

- In matrix format, for a 2 asset model this yields to :

$$vech(H_t)' = \begin{vmatrix} h_{11,t} \\ h_{12,t} \\ h_{22,t} \end{vmatrix} = \begin{vmatrix} \omega_{11} \\ \omega_{12} \\ \omega_{22} \end{vmatrix} + \begin{vmatrix} \alpha_{11} & 0 & 0 \\ 0 & \alpha_{12} & 0 \\ 0 & 0 & \alpha_{22} \end{vmatrix} \cdot \begin{vmatrix} \varepsilon^2_{1,t-1} \\ \varepsilon_{1,t-1}\varepsilon_{2,t-1} \\ \varepsilon^2_{t-1} \end{vmatrix} + \begin{vmatrix} \beta_{11} & 0 & 0 \\ 0 & \beta_{12} & 0 \\ 0 & 0 & \beta_{22} \end{vmatrix} \cdot \begin{vmatrix} h_{11,t-1} \\ h_{12,t-1} \\ h_{22,t-1} \end{vmatrix}$$

- If there is N assets in the portfolio, the model has $\frac{3N(N+1)}{2}$ parameters

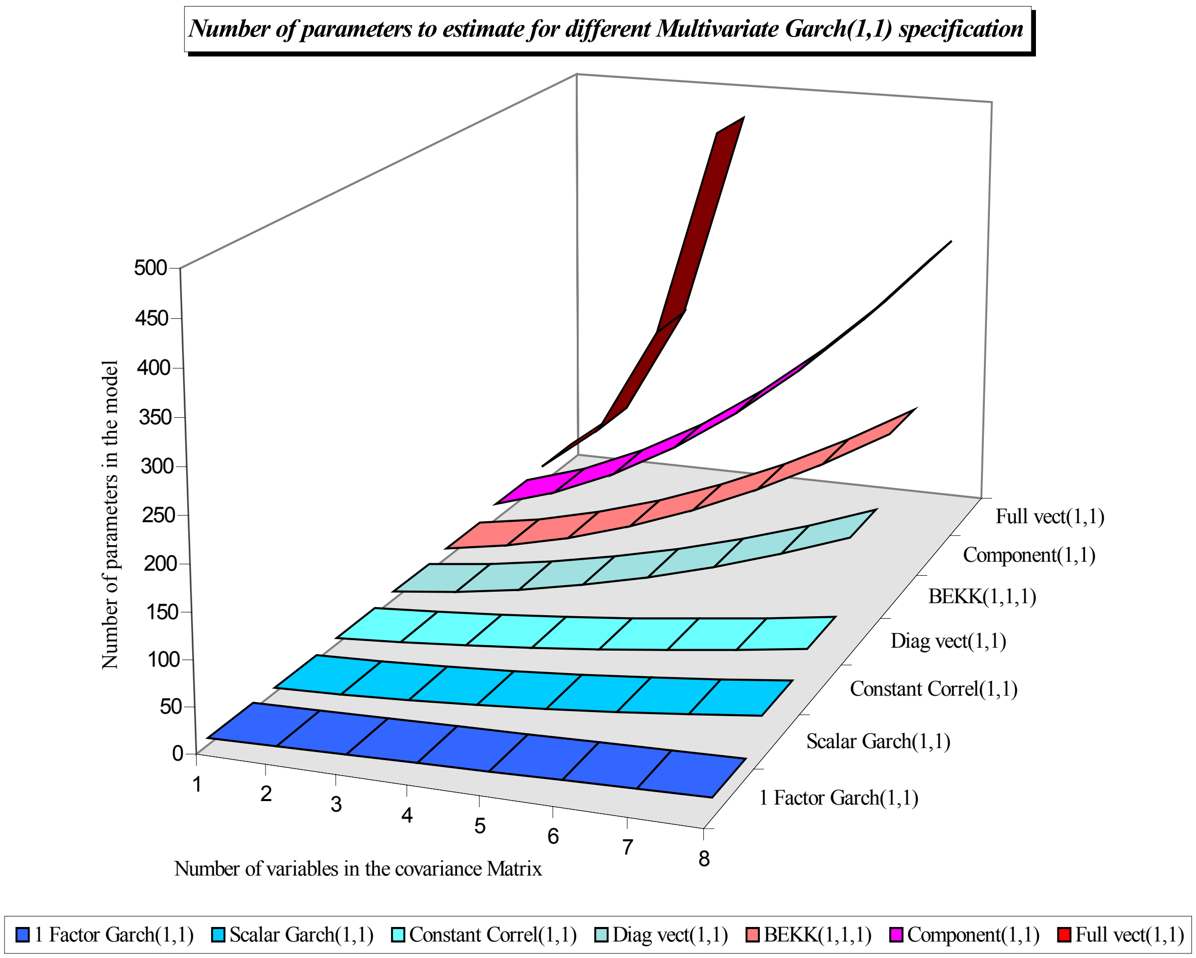
Multivariate GARCH Models : The Diagonal Vech model

- Advantages :
  - easy interpretation, easy to compute for large portfolios
- Disadvantages :
  - No improvement compare to N(N+1)/2 univariate GARCH models,
  - No constraints to ensure the positive definitiveness of the covariance matrix (we can tweak the model to avoid this).
  - No interactions between asset covariances means :
    - no causality in variance from one asset to another,
    - no co-persistence.

Multivariate GARCH Models : Comparison between Scalar & Diagonal model

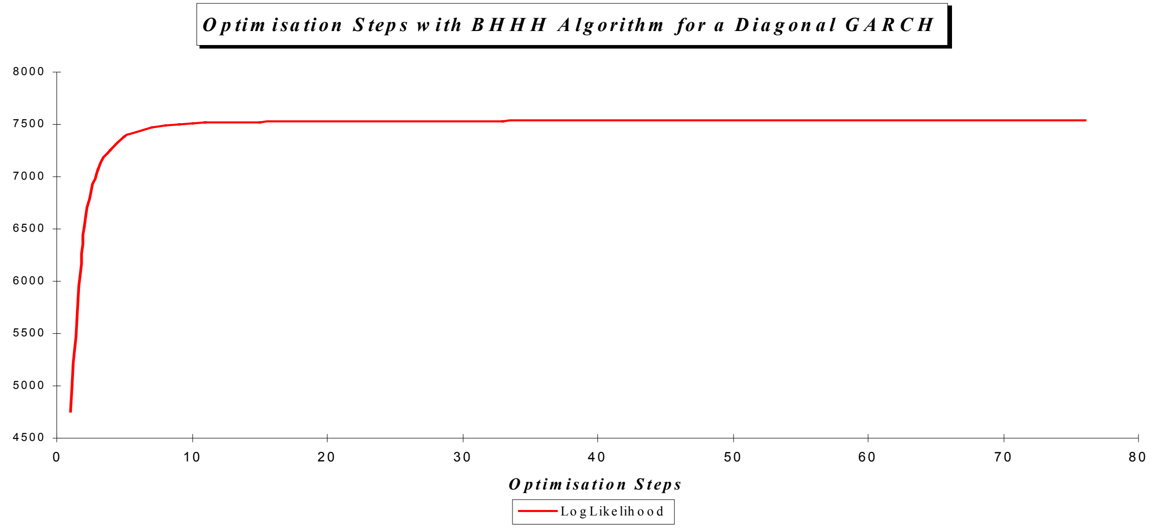
- As the Scalar GARCH model is nested into the Diagonal GARCH model there is several ways to statistically test if the Scalar GARCH is the right model knowing that the alternative model is the Diagonal GARCH :
  - Likelihood Ratio Test
  - Chi Squared and Fisher Test by constraining the parameters of the diagonal model
- What is the conclusion ?
  - Chi Squared(30) is 4892 !!! we reject the simple scalar GARCH!!!

Multivariate GARCH Models : The estimation problem : The number of parameters



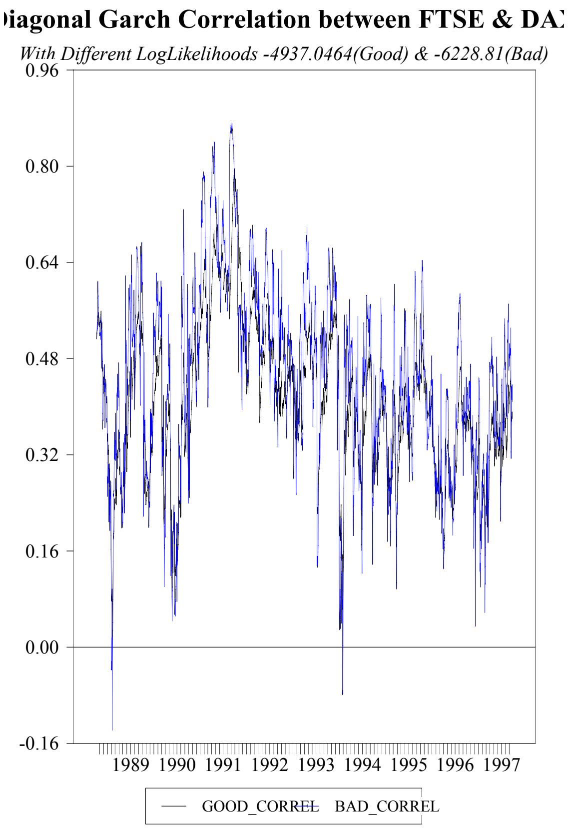
Multivariate GARCH Models : Evaluating the usability and the limitations of Multivariate GARCH models.

- Two problems rises when estimating Multivariate GARCH models :
  - The number of parameters are rising quite dramatically as we saw earlier,
  - The function we are maximising (the likelihood function of the multivariate model) is very flat near the optimum which poses some convergence problems for the algorithm (BHHH, BFGS, LM, …)



Multivariate GARCH Models : Evaluating the usability and the limitations of Multivariate GARCH models.

- Flat Maximum Likelihood functions yields to optimisation problems which translates into bad estimation of correlation and volatilities.
- Badly optimised Multivariate GARCH gives more noisy correlation dynamics with more extremes which are unrelated to market changes.
- This chart shows the correct estimation (until convergence is obtained) of a diagonal model and an optimisation stopped at an earlier stage of the optimisation process.



Multivariate GARCH Models : Evaluating the usability and the limitations of Multivariate GARCH models.

- As we saw unfortunately few models are manageable for more than a half a dozen assets in the covariance dynamics.
- The Maximum Likelihood estimation runs into convergence problems because of its structural behaviour.
- But it is the best methodology available to measure correlation dynamics between assets.
- So what's the solution for LARGE SCALE COVARIANCE MATRIX?
  - We have to reduce the number of estimated parameters.

Multivariate GARCH Models : What are the Alternatives for very large Dimensions ? The RiskMetrics© case

- JP Morgan launched RiskMetrics© as a benchmark for measuring Value at Risk. This methodology is based on a Variance / Covariance approach, hence the calculation of a correlation matrix is necessary.
- As we saw earlier, JPM use a single decay factor to estimate correlation matrix

$$H(\lambda) = \begin{vmatrix} \sigma_{11}^2(\lambda) & \sigma_{12}^2(\lambda) & \sigma_{13}^2(\lambda) \\ \sigma_{12}^2(\lambda) & \sigma_{22}^2(\lambda) & \sigma_{23}^2(\lambda) \\ \sigma_{13}^2(\lambda) & \sigma_{23}^2(\lambda) & \sigma_{33}^2(\lambda) \end{vmatrix}$$

- The decay factor is NOT estimated but chosen ad hoc.
- A single decay factor for the whole correlation structure is appealing because the matrix will remain positive definite under very weak conditions.

Multivariate GARCH Models : What are the Alternatives for very large Dimensions ? The RiskMetrics© case

- Currently there is 481 Assets (115200+ Correlations) in the RiskMetrics© universe, it is indeed suitable for very large scale problem.
- It is very EASY to calculate !!!
- It is very FAST to calculate !!!
- As we show earlier, even in the simplest multivariate GARCH framework, the RiskMetrics© methodology is strongly rejected, so what is the solution ?
  - Estimate the decay factor via Maximum Likelihood techniques will tremendously improve the model and will keep the “nice” feature of the model (SDP). There will be one parameter to find for the optimizer so it should done quickly …
  - Fat Tail models.

Multivariate GARCH Models : What are the Alternatives for very large Dimensions ? The orthogonal GARCH model

- Let suppose we have N time series (Y) from which we want to model the time varying covariances.
- The principal component analysis of the unconditional covariance matrix of the normalised data X gives us the SVD : P = X.W, where P is the orthonormal matrix of the eigenvectors ordered by decreasing values of their respective eigenvalues $\lambda_1 > \lambda_2 > \cdots > \lambda_1$. $X = PW'$
- The PCA representation of the normalised data is given by :

$$X_i = \omega_{i1} P_1 + \omega_{i2} P_2 + \cdots + \omega_{ik} P_k$$

- For the raw data we have : $Y = \mu + P\Phi \quad \Phi = \left(\varphi_{ij}\right) = \omega_{ij}\sigma_i$
- So the variance-covariance of Y is :
- We ONLY need to calculate univariate GARCH volatilities and not a covariances!

$$V(Y_t) = E(Y_t Y_t') = \Phi' V(P'P)\Phi = \Phi' V(\Lambda)\Phi$$

$$= \Phi' \begin{pmatrix} V_t(P_1) & 0 & \cdots & 0 \\ 0 & V_t(P_2) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & V_t(P_N) \end{pmatrix} \Phi$$

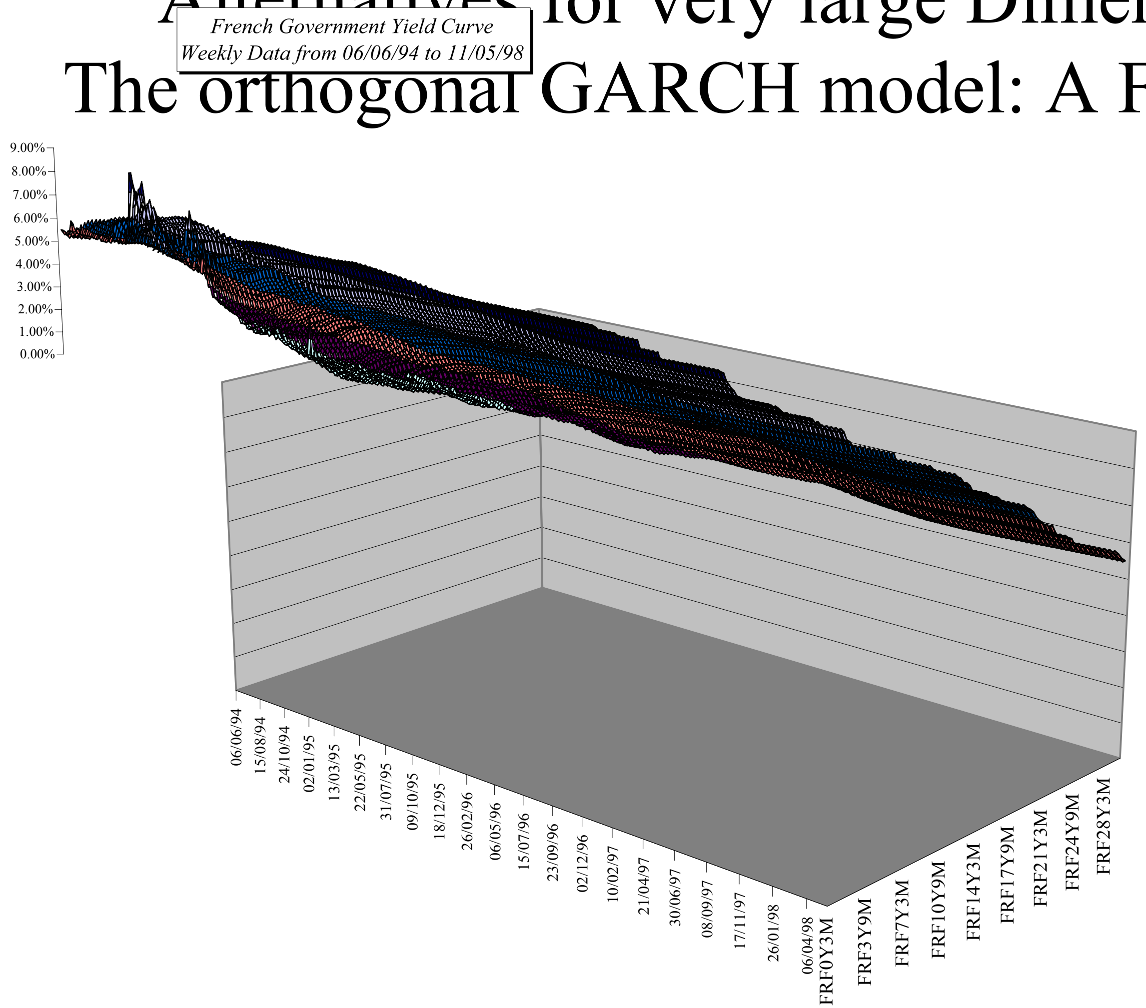
Multivariate GARCH Models : What are the Alternatives for very large Dimensions ? The orthogonal GARCH model

- The covariance matrix will always be positive definite because all the volatilities of the principal component factors are strictly positive.
- The specification of the univariate model of each PCA is left to the analyst.
  - An asymmetric model will be suitable for Stocks and bonds,
  - A symmetric model will be good enough for G15 FX rates
- The number of parameters required by the model is depending on the univariate specification, i.e., GARCH(1,1) then : 3*N
- Because we use PCA we can control the amount of noise we want to filter from the raw data by allowing only K<N PCA factors to enter the reconstruction of the covariance matrix.
  - For Bonds, a two factors model will capture at least 95% of the information.

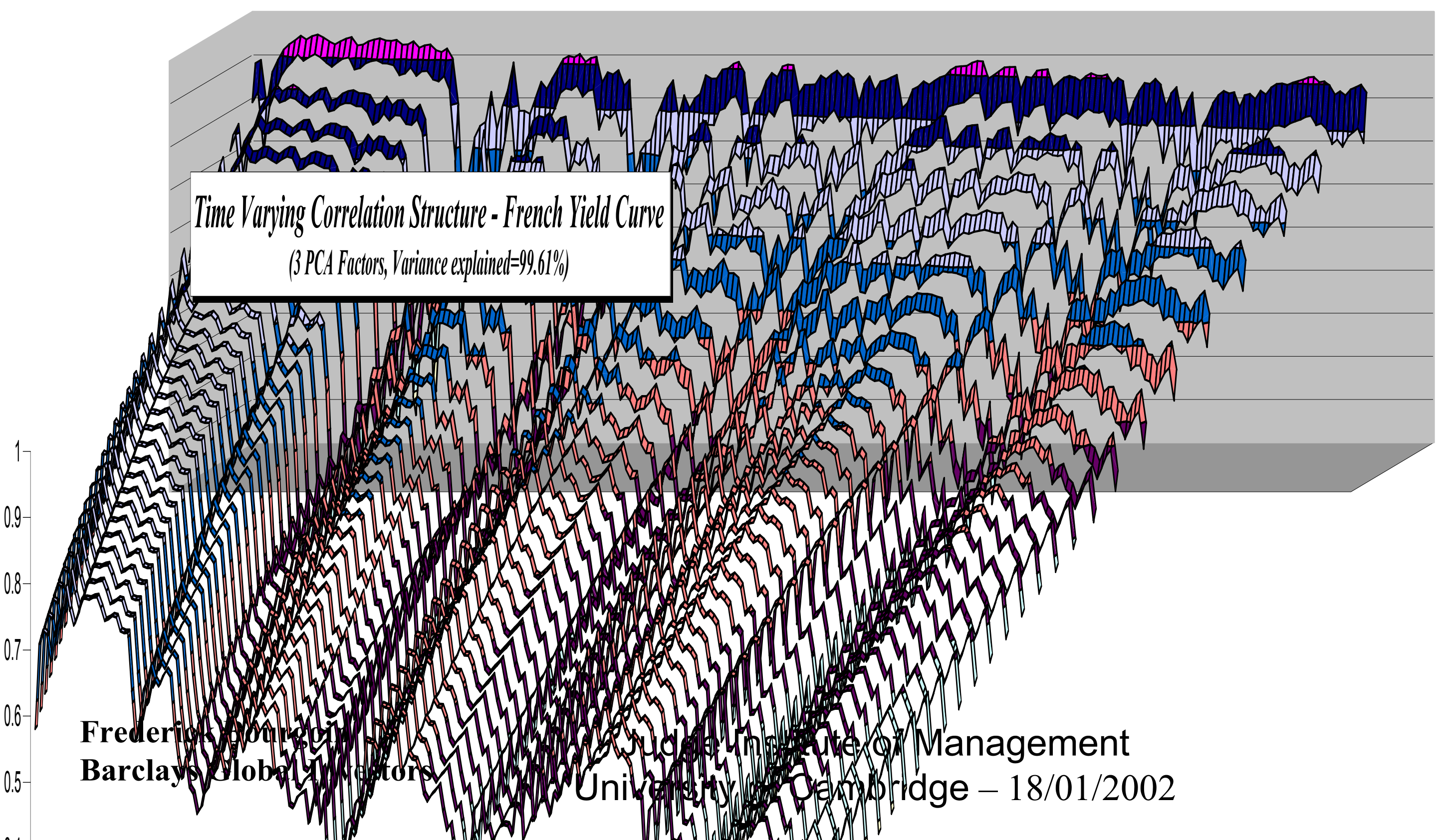
Multivariate GARCH Models : What are the Alternatives for very large Dimensions ? The orthogonal GARCH model

- Advantages :
  - The number of parameters required is very small compared to other multivariate specification,
  - Any univariate specification can be used : symmetric or asymmetric,
  - The covariance matrix is at least always Semi definite positive,
  - A filtering process is applicable to reduce the noise in the covariance matrix by eliminating less meaningful factors (i.e., factors that take into account very little information, the information ratio of a PCA factor is dependent on its eigenvalue : $IR_i = \lambda_i \Big/ \sum_{j=1}^{N} \lambda_j$)
- Disadvantages :
  - The Orthogonal Garch methodology is using unconditional technique (i.e., PCA) to estimate conditional covariances, this is introducing a bias and under crash condition the correlations of the assets will not raise as much as it should with a normal multivariate model. (the model is assuming uncorrelated conditional factors which is not the case).

Multivariate GARCH Models : What are the Alternatives for very large Dimensions ? The orthogonal GARCH model: A FI Example

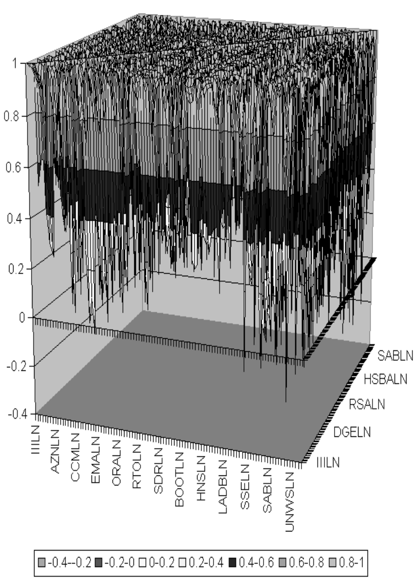


Multivariate GARCH Models : What are the Alternatives for very large Dimensions ? The orthogonal GARCH model: A FI Example

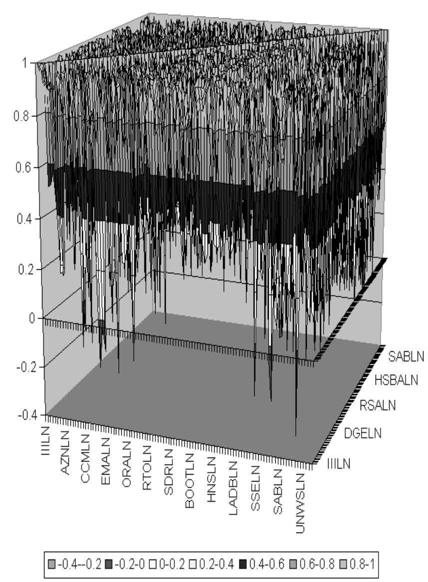


Multivariate GARCH Models:
What are the Alternatives for very large Dimensions?
The orthogonal GARCH Model:An Equity Example:FTSE100

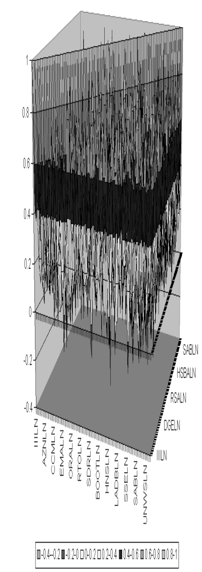
Correlation Matrix for the FTSE100 Stocks with 2 Orthogonal Factors explaining 24.31% of information contained in the naïve correlation matrix as of 24/04/1997



Correlation Matrix for the FTSE100 Stocks with 3 Orthogonal Factors explaining 35.76% of information contained in the naïve correlation matrix as of 24/04/1997

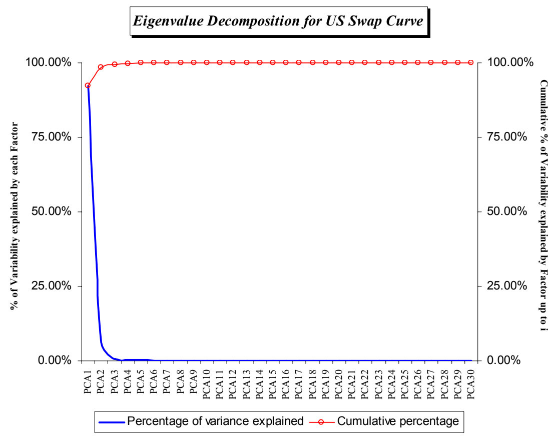


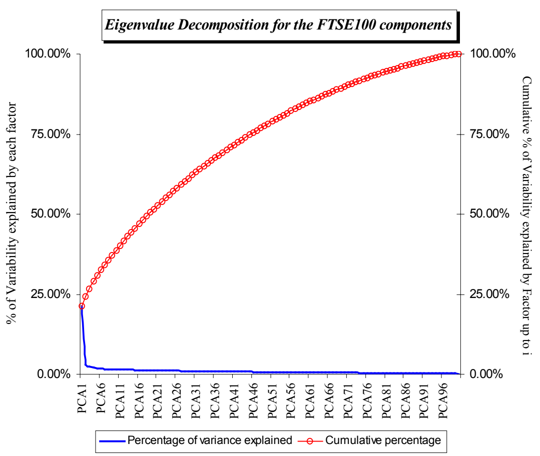
Correlation Matrix for the FTSE100 Stocks with 20 Orthogonal Factors explaining 51.65% of information contained in the naïve correlation matrix as of 24/04/1997



Multivariate GARCH Models: What are the Alternatives for very large Dimensions? The orthogonal GARCH Model

- Multivariate Orthogonal GARCH provide a very good and efficient way to calculate large scale correlation matrices but it depends on the eigenvalue spectra of the unconditional correlation matrix of the assets under analysis.
- For Fixed Income, most of the information is captured by the first 3 factors (99%), for the Ftse100 the first 3 factors capture 26.8% of the variance which insufficient to provide an accurate risk measurement framework.





Multivariate GARCH Models: What are the Alternatives for very large Dimensions? New Solutions No1 (Athayde 2001)

- Can we reduce a n asset problem (n(n-1)/2 covariances and n variances) into a series of 2 asset problems (1 covariance and 2 variances)?
- Usually we end up with a covariance matrix that isn't definite positive.
- Let's assume we want to calculate the correlation matrix of Ftse100, let's start from the index volatility: $\sigma_t^2 = \theta + \alpha\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2 \quad \theta > 0 \; and \; \alpha + \beta < 1$
- The index volatility can also be calculated from its part: $\sigma_t^2 = \omega_t' \Omega_t \omega_t$
- Let's assume that the multivariate process follow a diagonal GARCH which persistence parameters are the one from the overall index volatility, we are left with calculating : θ_{ij}

$$vech\left(H_t\right)' = \begin{vmatrix} h_{11,t} \\ h_{12,t} \\ h_{22,t} \end{vmatrix} = \begin{vmatrix} \theta_{11} \\ \theta_{12} \\ \theta_{22} \end{vmatrix} + \begin{vmatrix} \overline{\alpha} & 0 & 0 \\ 0 & \overline{\alpha} & 0 \\ 0 & 0 & \overline{\alpha} \end{vmatrix} \cdot \begin{vmatrix} \varepsilon_{1,t-1}^2 \\ \varepsilon_{1,t-1}\varepsilon_{2,t-1} \\ \varepsilon_{t-1}^2 \end{vmatrix} + \begin{vmatrix} \overline{\beta} & 0 & 0 \\ 0 & \overline{\beta} & 0 \\ 0 & 0 & \overline{\beta} \end{vmatrix} \cdot \begin{vmatrix} h_{11,t-1} \\ h_{12,t-1} \\ h_{22,t-1} \end{vmatrix}$$

Multivariate GARCH Models: What are the Alternatives for very large Dimensions? New Solutions No1 (Athayde 2001)

- We can back out the covariance term with a small trick:

$$\sigma^2_{\omega_1\varepsilon_1+\omega_2\varepsilon_2} = \omega_1^2\sigma^2_{\varepsilon_1} + \omega_2^2\sigma^2_{\varepsilon_2} + 2\omega_1\omega_2\sigma_{\varepsilon_1\varepsilon_2} \Rightarrow \sigma_{\varepsilon_1\varepsilon_2} = \frac{\sigma^2_{\omega_1\varepsilon_1+\omega_2\varepsilon_2} - \sigma^2_{\omega_1\varepsilon_1-\omega_2\varepsilon_2}}{4\omega_1\omega_2}$$

- For the first asset volatility only θ_{11} is required since α and β are given.
- If a matrix Ω is definite positive then $\Omega = LL'$ with $L = \begin{bmatrix} a_{11} & 0 \\ a_{12} & a_{22} \end{bmatrix}$
- It implies $\Theta = \begin{vmatrix} \theta_{11} & \theta_{21} \\ \theta_{21} & \theta_{22} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 \\ a_{12} & a_{22} \end{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{vmatrix} = \begin{vmatrix} a_{11}^2 & a_{11}a_{12} \\ a_{11}a_{12} & a_{12}^2 + a_{22}^2 \end{vmatrix}$ Engle & Kroner (1993)

$$\sigma^2_{11,t} = \theta_{11} + \bar{\alpha}\varepsilon^2_{1,t-1} + \bar{\beta}\sigma^2_{11,t-1} \Rightarrow a_{11} = \sqrt{\theta_{11}}$$

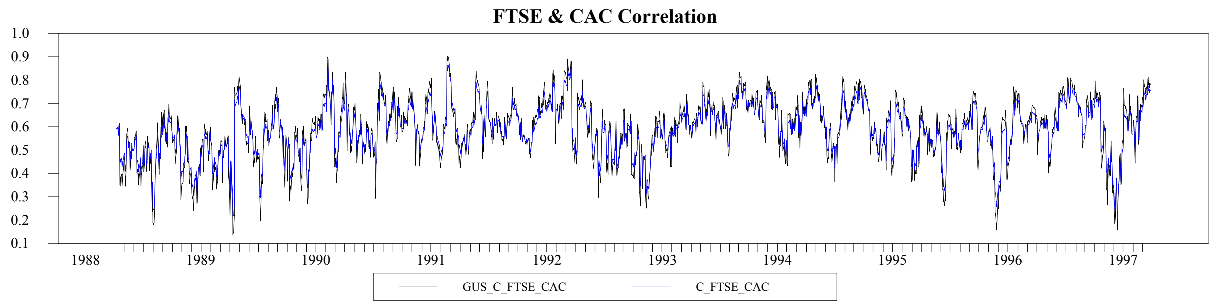
$$\Rightarrow \sigma^2_{12,t} = \theta_{12} + \bar{\alpha}\varepsilon_{1,t-1}\varepsilon_{2,t-1} + \bar{\beta}\sigma^2_{12,t-1} \Rightarrow \theta_{12} = \frac{\theta^+_{12} - \theta^-_{12}}{4\omega_1\omega_2} \Rightarrow a_{12} = \frac{\theta_{12}}{a_{11}}$$

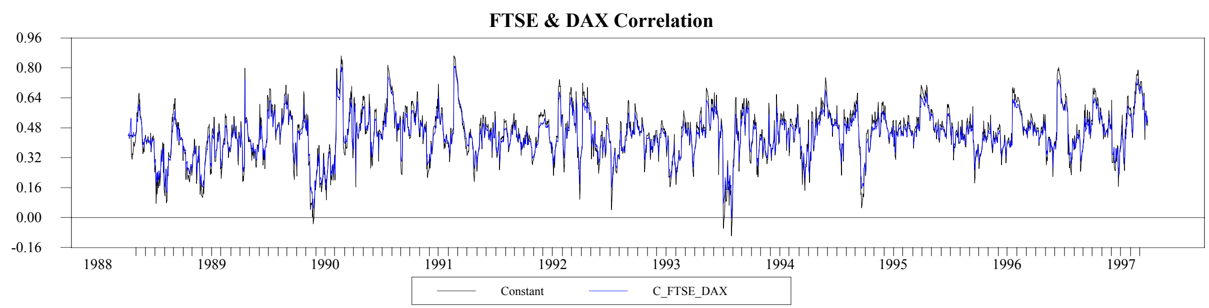
$$\Rightarrow \sigma^2_{22,t} = \theta_{22} + \bar{\alpha}\varepsilon^2_{2,t-1} + \bar{\beta}\sigma^2_{22,t-1} = \left(\bar{a}^2_{12} + \bar{a}^2_{22}\right) + \bar{\alpha}\varepsilon^2_{2,t-1} + \bar{\beta}\sigma^2_{22,t-1}$$

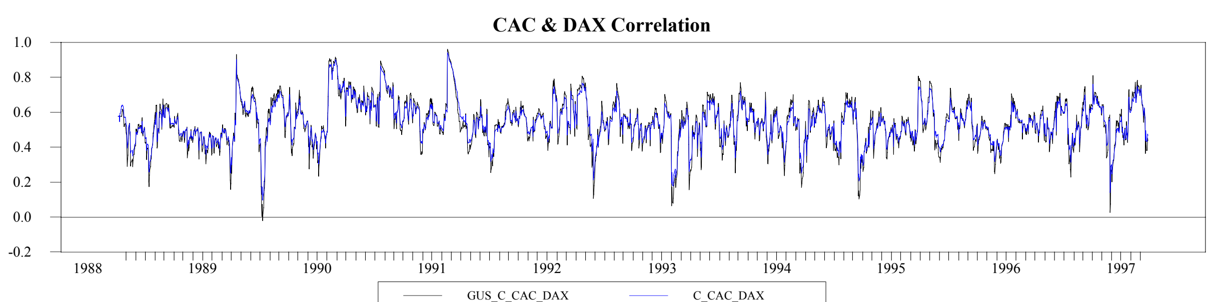
- The algorithm work by building the covariance matrix from NW-SE.

Multivariate GARCH Models:
What are the Alternatives for very large Dimensions?
New Solutions No1 (Athayde 2001)

Difference between Diagonal and Disaggregated GARCH







Frederick Bourgoin
Barclays Global Investors

Multivariate GARCH Models: What are the Alternatives for very large Dimensions? New Solutions No1 (Athayde 2001)

- Avantages:
  - Only univariate GARCH are required to calculate the correlation matrix!
  - The procedure is very quick since there only one parameter to estimate in each equation,
  - Positive definite correlation matrix is constructed regardless of its size.
  - Capture Well the correlation dynamics.
  - Very scalable and fast: large number of variables is no problem (e.g. 3variables: 50% faster than the Diagonal GARCH).
  - Easy to implement in a multithreaded environment.
- Disavantages:
  - Requires a market portfolio, because w_i that must be defined prior to running the algorithm,
  - The persistence of the market portfolio is assumed to valid at the individual asset level (slightly different that the Diagonal GARCH assumption).
  - The estimated correlations are more “noisy”

Multivariate GARCH Models: What are the Alternatives for very large Dimensions? New Solutions No2 Engle's DCC Model

- The constant correlation GARCH (Bollerslev 90) and Orthogonal Garch (Alexander 2000) started a trend try to disentangle volatility and correlation estimation. Usually both are linked in the estimation procedure because the covariance at time t is estimated.
- Let's Assume $r_t | \Im_{t-1} \xrightarrow{L} N(0, H_t)$ where $H_t = D_t R_t D_t$ with $D_t = Diag(h_{i,t})$ and R_t is the time varying correlation matrix. In Bollerslev (90) $D_t = \overline{D}$
- The Loglikelihood can be written:

$$LogL(\phi,\psi) = -\frac{1}{2}\sum_{t=1}^{T}\left(kLog(2\pi) + \log(|H_t|) + r_t' H_t^{-1} r_t\right) = -\frac{1}{2}\sum_{t=1}^{T}\left(kLog(2\pi) + \log(|D_t R_t D_t|) + r_t' D_t^{-1} R_t^{-1} D_t^{-1} r_t\right)$$

$$= -\frac{1}{2}\sum_{t=1}^{T}\left(kLog(2\pi) + 2\log|D_t| + \log(|R_t|) + \varepsilon_t' H_t^{-1} \varepsilon_t\right) \text{ with } \varepsilon_t = D_t^{-1} r_t \text{ (standardised residuals)}$$

Multivariate GARCH Models: What are the Alternatives for very large Dimensions? New Solutions No2 Engle's DCC Model

- The estimation process is done in 2 stages:
  - Estimate the volatilities by replacing R_t by I, the identity matrix.

$$QLog(\phi|r_t) = -\frac{1}{2}\sum_{t=1}^{T}\left(k\log(2\pi)+\log(|I|)+2\log(|D_t|)+r_t'D_t^{-1}ID_t^{-1}r_t\right) = -\frac{1}{2}\sum_{t=1}^{T}\left(k\log(2\pi)+2\log(|D_t|)+r_t'D_t^{-2}r_t\right)$$

$$= -\frac{1}{2}\sum_{n=1}^{k}\left(T\log(2\pi)+\sum_{t=1}^{T}\left(\log(h_{i,t})+\frac{r_{i,t}^2}{h_{i,t}}\right)\right) \text{(Sum of k univariate GARCH loglikelihood functions)}$$

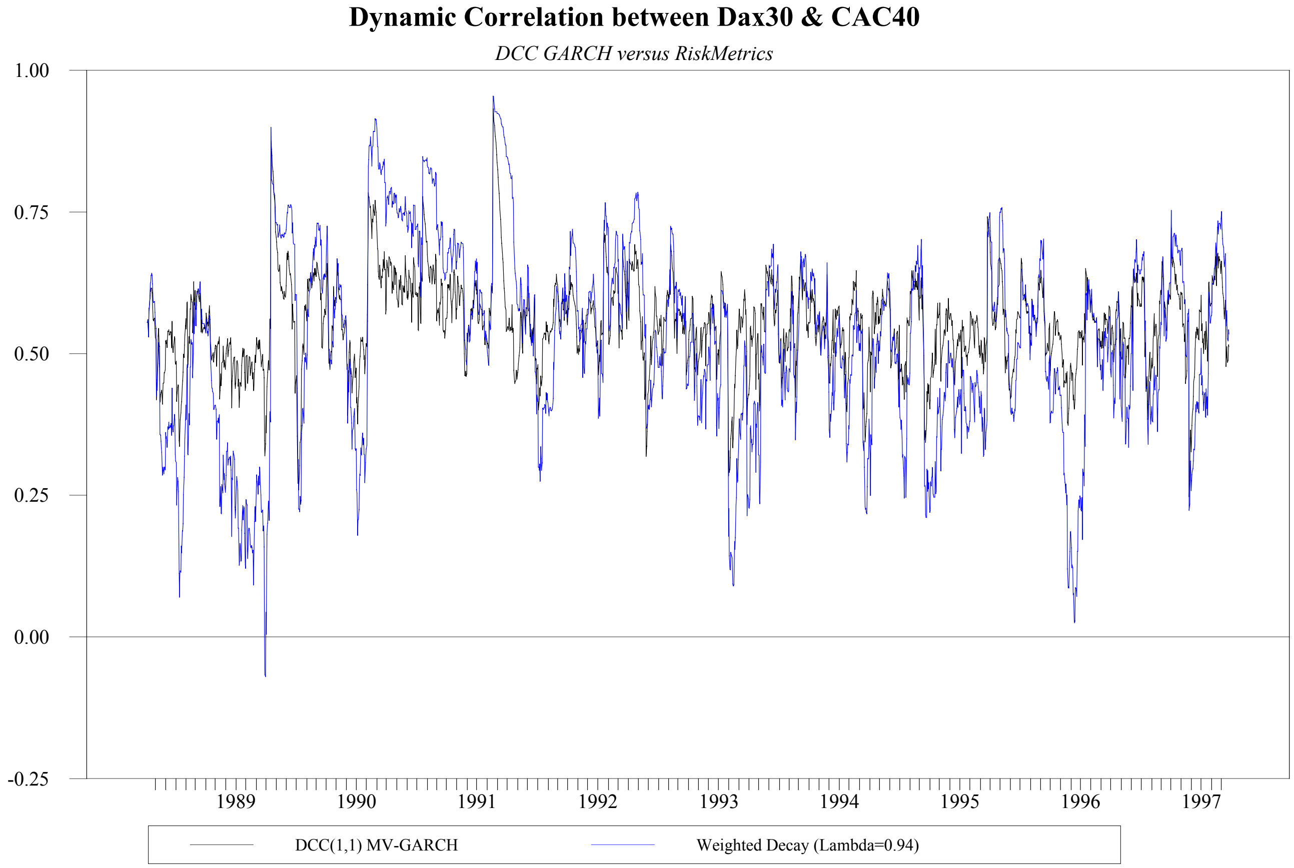
  h_t can have any univariate GARCH specification
  - Estimate the correlation matrix is done next:

$$QLog_2(\psi|\hat{\phi}, r_t) = -\frac{1}{2}\sum_{t=1}^{T}\left(k\log(2\pi)+\log(|R_t|)+2\log(|D_t|)+r_t'D_t^{-1}R_t^{-1}D_t^{-1}r_t\right) \approx -\frac{1}{2}\sum_{t=1}^{T}\left(\log(|R_t|)+\varepsilon_t'R_t^{-1}\varepsilon_t\right)$$

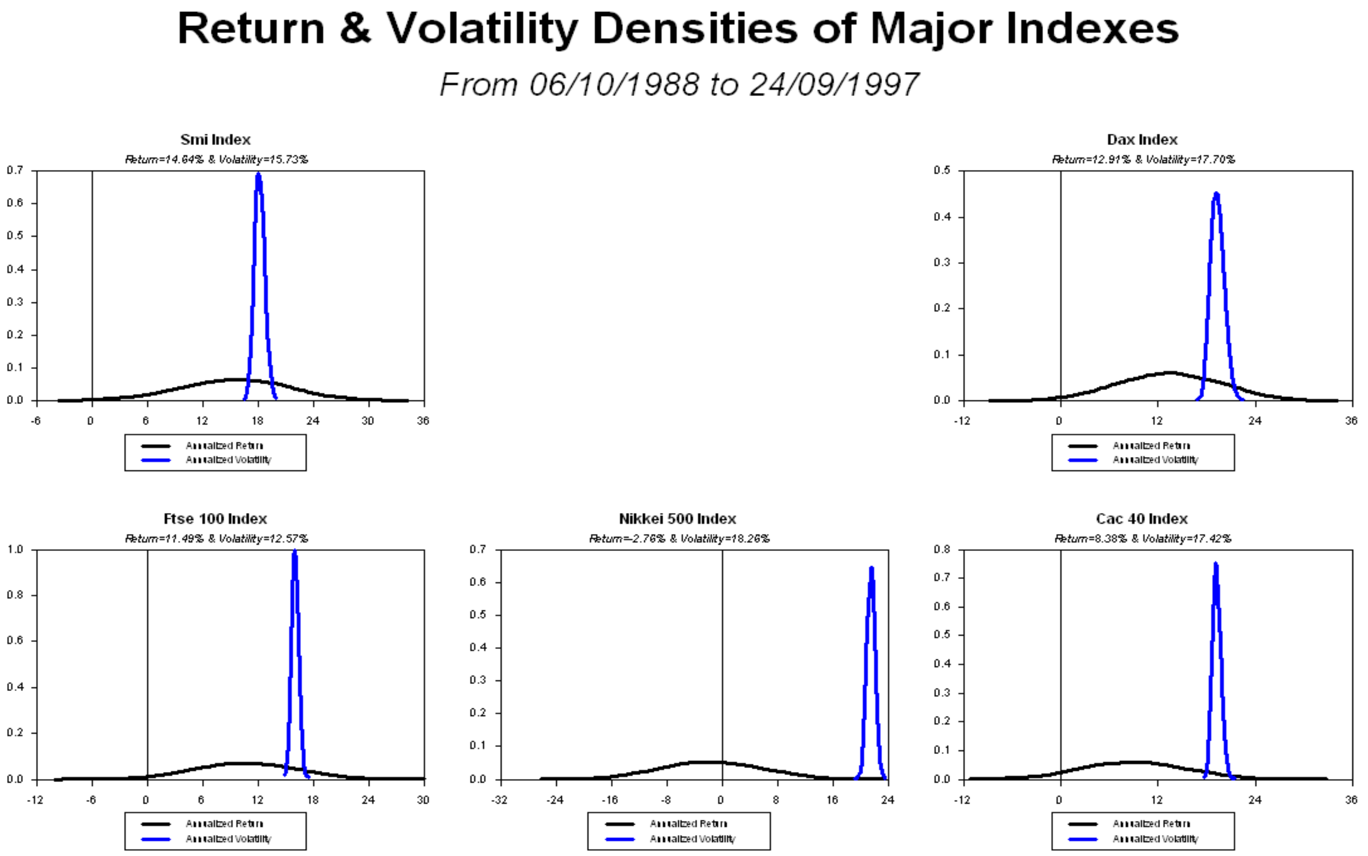
  For a 2x2 problem: $QLog_2(\psi|\hat{\phi}, r_t) = -\frac{1}{2}\sum_{t=1}^{T}\left(\log(1-\rho_t^2)+\frac{\varepsilon_{1,t}^2+\varepsilon_{2,t}^2+2\rho_t\varepsilon_{1,t}\varepsilon_{2,t}}{1-\rho_t^2}\right)$

- The efficient gain in speed is the same as Athayde's technique (50% faster than the diagonal GARCH).

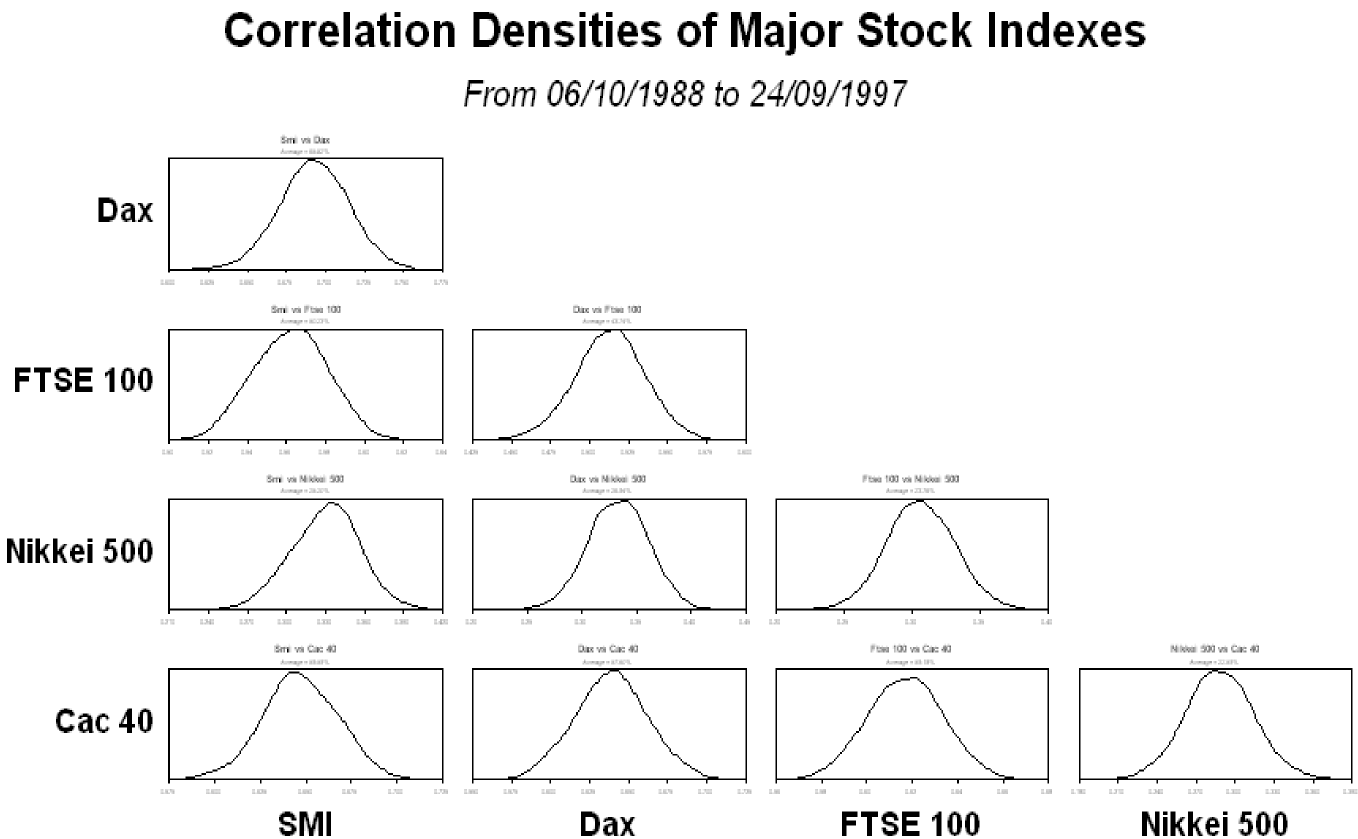
Multivariate GARCH Models: What are the Alternatives for very large Dimensions? New Solutions No2 Engle's DCC Model



Volatility and Correlation: A measurement error problem

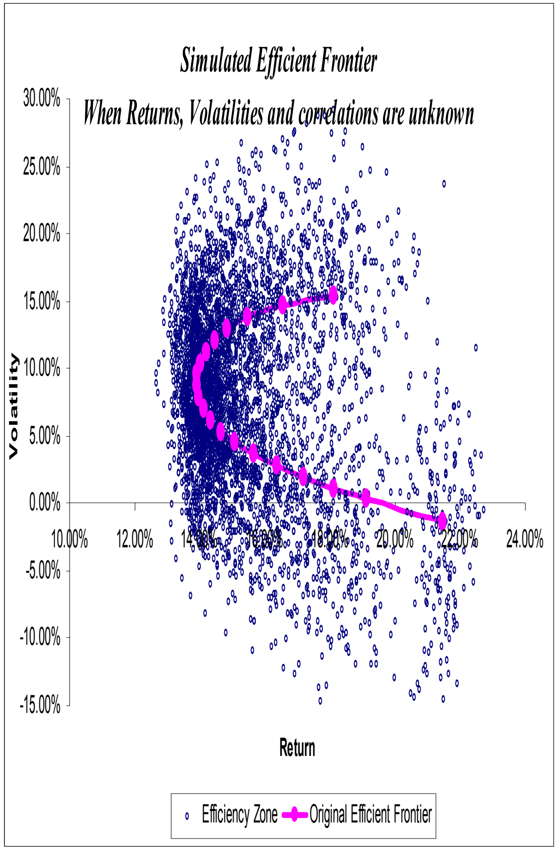


Volatility and Correlation:
A measurement error problem



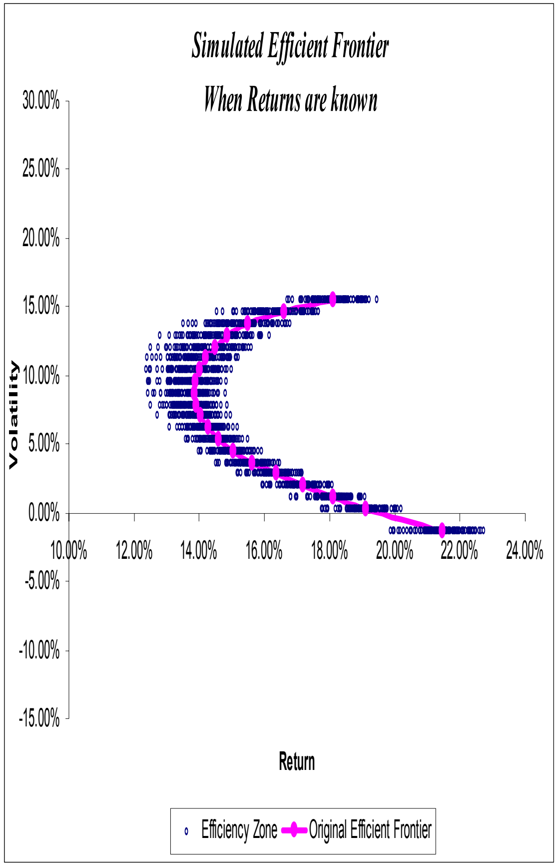
The Impact of the measurement error on the efficient frontier(1)

- If we assume that we are measuring expected returns, volatilities and correlations with error, we don't have a efficient frontier but a zone of efficiency.
- The implications are very important:
  - a lot of portfolios are statistically efficient for the same level of risk or return.
  - Statistical tests can be perform to test the efficiency of a set of portfolio weights Britten-Jones JOF (1999)



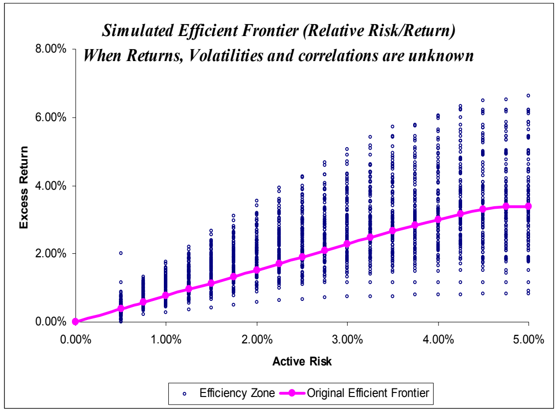
The Impact of the measurement error on the efficient frontier(2)

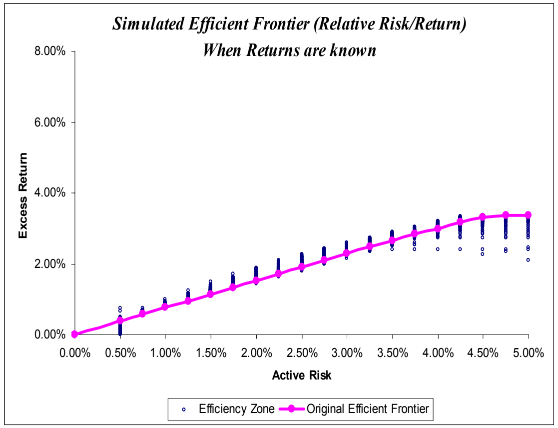
- As we can see the main source of randomness in the efficiency zone is related to the expected return component.
- This is well known by practitioners.
- Academics research has been done in this area: Best and Grauer RFS(1991)



The Impact of the measurement error on the efficient frontier(3)

- The same is true in Excess Return Active Risk space
- The more data we have to calculate the required statistics, the narrower the efficient zone will be.
- Because the standard errors of the statistics (volatility, correlations) depends on the sample size by square root of N.
- To evaluate volatility and correlation models in a markowitz framework will avoid expected return component.





The Minimum VaR portfolio

- We do the following optimisation:

$$\underset{x}{Min}\ x'\Sigma x$$

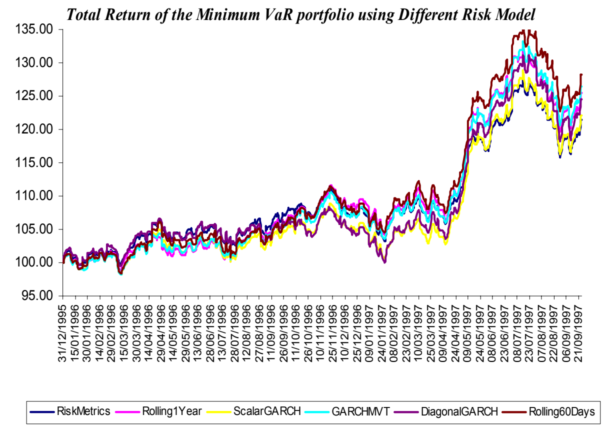
$$s.t.\ x'e = 1 \quad e' = |1 \cdots 1|$$

$$x^* = \frac{\Sigma^{-1} e}{e'\Sigma^{-1} e}$$

- If we are trying to minimise portfolio volatility, conditional models are clearly superior.
- The portfolio are quite similar regardless of the model used.

Risk Summary

	Minimum Variance Portfolio	Nb of breaks of VAR	p-value from BackTest	Accept / Reject
Rolling 60Days	9.90%	33	0.04	reject
Rolling 1Year	9.63%	35	0.01	reject
Riskmetrics	8.77%	25	0.61	accept
Scalar GARCH	9.40%	14	0.05	accept
Scalar GARCH-VT	9.42%	15	0.08	accept
Diagonal GARCH	9.94%	30	0.13	accept



Optimised Portfolio with 10% volatility

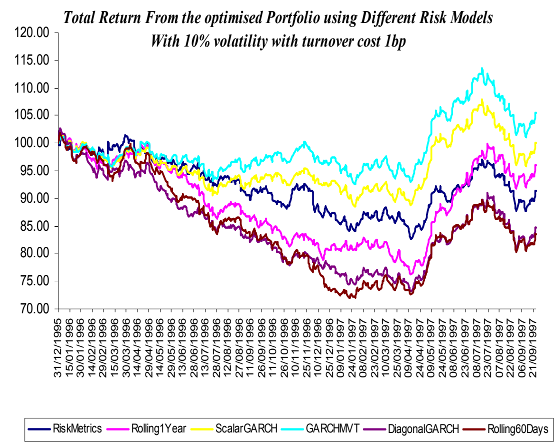
- We do the following optimisation:

$$\max_{x} x'\mu$$
$$s.t.\ x'e = 1 \quad e' = |1 \cdots 1|$$
$$x'\Sigma x = \bar{\sigma}^2$$

- GARCH models are closer to required target volatility.
- When turnover is not constrained in the portfolio optimisation routine and we account for transaction cost (very low), high turnover models under-performed.

Optimised Portfolio with 10% annual volatility

	Realised Portfolio volatility	Realised Turnover
Rolling 60Days	11.41%	58.97%
Rolling 1Year	11.01%	86.99%
Riskmetrics	10.87%	61.18%
Scalar GARCH	9.54%	26.58%
Scalar GARCH-VT	9.45%	17.38%
Diagonal GARCH	10.98%	79.38%



Multivariate GARCH : Conclusions

- Multivariate GARCH models are used to model the covariance dynamics of a set of variables.
- Several techniques are available today to make large size problems feasible.
- Some of them are independent from the structure of the data like RiskMetrics, the Scalar GARCH with VT, Athayde's technique and Engle's DCC.
- The orthogonal GARCH model is very efficient for fixed income risk management, but its application require careful look at the eigenvalue spectrum in order to use it in a different context.
- GARCH models used for VaR or Portfolio Optimisation are more accurate and give better ex-post performance (risk or return) than standard models.

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