Closed Form Solutions for Pricing Asian Options

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Description

Let $[0, T]$ be the time interval into which an Asian option is structured.

- We look at Asian options that can only be exercised at maturity T.

- Their payoff at time T depends on the **average** of the underlying over a time interval $[T_0, T]$.

Different Types of Averages

1. **Discrete Arithmetic Average**

$$Y(T) = \frac{1}{N+1} \sum_{i=0}^{N} X(t_i), \;\; t_0 = T_0$$

2. **Continuous Arithmetic Average**

$$Y(T) = \frac{1}{T - T_0} \int_{T_0}^{T} X(u) du$$

3. **Discrete Geometric Average**

$$G(T) = \left(\prod_{i=0}^{N} X(t_i) \right)^{\frac{1}{N+1}}, \;\; t_0 = T_0$$

4. **Continuous Geometric Average**

$$G(T) = \exp \left[\frac{1}{T - T_0} \int_{T_0}^{T} log(X(u)) du \right]$$

Different Types of Asian Options

- **Fixed-Strike**, if the payoff is

$$P[T] = \begin{cases} \max\,[Y(T) - K, 0] \\ \max\,[G(T) - K, 0] \end{cases}$$

- **Floating-Strike**, if the payoff is

$$P[T] = \begin{cases} \max\,[X(T) - Y(T), 0] \\ \max\,[X(T) - G(T), 0] \end{cases}$$

- **Backward-Started**, if $T_0 \le 0$

- **Forward-Started**, if $T_0 > 0$

Applications of Asian Options

Asian options, are popular contracts in stock, currency, and commodity markets because:

- they are not affected by manipulations of the price of the underlying close to maturity,

- they can be used to hedge a stream of uncertain payoffs over a period of time.

Market Model

1. The market consists of

    - a risk free asset growing at a constant rate, r

2. Short sales are allowed

3. There are no transaction costs or taxes

4. Assets are perfectly divisible

- a risky asset evolving as a geometric Brownian motion,

$$\begin{aligned} dX(t) &= \mu X(t)dt + \sigma X(t)dW(t) \\ X(0) &= X_0 > 0 \end{aligned}$$

- a risky asset evolving as a constant elasticity of variance process,

$$\begin{aligned} dX(t) &= (\mu X(t) + c)\, dt + \sigma X^k(t) dW(t) \\ X(0) &= X_0 > 0 \end{aligned}$$

where $0 \leq k \neq 1$.

- a risky asset evolving as an Ornstein-Uhlenbeck process,

$$\begin{aligned} dX(t) &= \mu X(t)dt + \sigma dW(t) \\ X(0) &= X_0 > 0 \end{aligned}$$

- a risky asset evolving as a generalised square root process,

$$
\begin{aligned}
dX(t) &= (\mu X(t) - c)\,dt + \sigma\sqrt{X(t)}dW(t) \\
X(0) &= X_0 > 0
\end{aligned}
$$

Pricing Asian Options

In such a market, there exist

- explicit solutions for geometric average Asian options,

  (Kemna et al. (1990) and Turnbull et al. (1991))

- no exact solution for arithmetic average Asian options.

Pricing Approximations

- Pseudo-Analytic Formulae

  - Approximations of the unknown distribution of the arithmetic average

    [Levy (1992), Milevsky et al. (1998) (a, b), Turnbull et al. (1991)]

  - Other approximations

  [Bouaziz et al. (1994), Curran (1994), Vorst (1992)]

- Laplace Transforms

  [Geman et al. (1993), Geman et al. (1995), Fu et al. (1999)]

- Numerical Solutions of the Pricing P.D.E [Alziary et al. (1997), Dewynne et al. (1995) (a,b), Rogers er al. (1995)]

Pricing Approximations

- Monte Carlo Simulations

  [Corwin et al. (1996), Haykov (1993), Kemna et al. (1990)]

- Trees & Lattices

  [Hull et al. (1993), Neave et al. (1993)]

- General Numerical Methods

  [Carvehill et al. (1990), Nielsen et al. (1996)]

Is it possible to alter the market model in such a way that

- it becomes no less realistic,
- it allows for the derivation of explicit pricing formulae.

Are there any alternative stochastic processes that could be used for modelling the evolution of the risky asset, such that

- they are realistic,
- they allow for the derivation of explicit pricing formulae.

Support for CEV Processes

As *stock models*, CEV ($k < 1$) processes have been supported by

- theoretical arguments, based on financial and operational leverage*,

- empirical evidence, fitting stock values†,

- empirical evidence comparing CEV and Black and Scholes option values against market values‡.

*Beckers (1980), Black (1976), Cox (1996), Christie (1982) and Geske (1979).

†Beckers (1980), Black (1976), Christie (1982) Emanuel et al. (1982), MacBeth et al. (1980) and Schmalensee et al. (1978).

‡Beckers (1980), Emanuel et al. (1982) and MacBeth et al. (1980).

Support for CEV Processes

CEV processes have also been supported by empirical evidence for modelling the evolution of short term interest rates, currencies and commodities.

[Chan et al. (1992), Choi et al. (1985) and Hauser et al. (1986)]

First Hitting Time of the Origin

Denote by,

$\tilde{T}$, the first time $X(t)$ hits the origin, assuming that it starts at $X_0 > 0$,

$P(\tilde{T} \in (0, t])$, the probability of the process hitting zero in $(0, t]$.

Then, it can be shown that

$$P(\tilde{T} \in (0, t]) = 1 - \mathrm{erf}\left(\frac{X_0}{\sigma}\sqrt{\frac{\mu}{1 - e^{-2\mu t}}}\right)$$

where $\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-u^2} du$.

First Hitting Time of the Origin

TABLE 1

The probability, $P(\tilde{T} \in (0, t])$, that the process hits the origin in $(0, t]$ given that $\mu = 0.07$ and $X_0 = 1$ ($X_0 = 5$).

Vol.	$t = 0.25$ years	$t = 0.5$ years	$t = 1$ year	$t = 2$ years
0.2	0 (0)	0 (0)	0 (0)	0 (0)
0.3	0 (0)	0 (0)	0 (0)	0.00004 (0)
0.4	0 (0)	0.00003 (0)	0.00065 (0)	0.00227 (0)
0.5	0.00001 (0)	0.00085 (0)	0.0064 (0)	0.0146 (0)
0.6	0.00028 (0)	0.00544 (0)	0.02309 (0)	0.04186 (0)

First Hitting Time of the Origin

Denote by,

$\tilde{T}$, the first time $X(t)$ hits the origin, assuming that it starts at $X_0 > 0$,

$E(\tilde{T})$, the expectation of the first hitting time.

Then, it can be shown that

$$E(\tilde{T}) = \begin{cases} \tilde{m}(X_0), & \text{if } \tilde{m}(X_0) \text{ is a positive valued, strictly increasing function of } X_0. \\ +\infty, & \text{otherwise.} \end{cases}$$

where

$$\tilde{m}(X_0) = \sqrt{\frac{\pi}{\mu\sigma^2}} \int_0^{X_0} \left[e^{-\frac{\mu u^2}{\sigma^2}} erfci\left(\sqrt{\frac{\mu u}{\sigma^2}}\right) \right] du$$

and $erfci(x) = 1 - \frac{2}{\sqrt{\pi}} \int_0^x e^{z^2} dz$

First Hitting Time of the Origin

TABLE 2
The expectation, $m(X_0)$, of the first time the process hits the origin given that $\mu = 0.07$

Vol.	$X_0 = 1$	$X_0 = 5$	$X_0 = 10$
0.2	5.76902 years	283.445 years	∞
0.3	8.93492 years	∞	∞
0.4	9.15987 years	∞	∞
0.5	8.59342 years	∞	∞
0.6	7.87536 years	∞	∞

First Hitting Time of the Origin Under the EMM Q

The dynamics of $X(t)$ under Q are given by

$$\begin{aligned} dX(t) &= (r-q)X(t)dt + \sigma d\widehat{W}(t) \\ X(0) &= X_0 \end{aligned}$$

where

r is the (domestic) risk-free rate,

q is the dividend rate or the foreign risk-free rate.

The Value of a European Put Option Struck at Zero

$$
\begin{aligned}
P(t_0, X_0) \quad &= \\
&\frac{\sigma e^{-qt}}{2\sqrt{\pi}\beta(t)} e^{-\beta^2(t)\frac{X_0^2}{\sigma^2}} \\
&+ \frac{e^{-qt} X_0}{2} \text{erfc}\left(-\beta(t)\frac{X_0}{\sigma}\right) - X_0 e^{-qt}
\end{aligned}
$$

where $\beta(t) = \sqrt{\frac{r-q}{1-e^{-2(r-q)t}}}$,

$\text{erfc}(z) = 1 - \text{erf}(z)$.

TABLE 3

Values of a European-type Put struck at zero with maturity t, given that $X_0 = 1$ ($X_0 = 5$).

Panel 1: $r = 0.6,\ q = 0.$

Vol.	$t = 0.25$ years	$t = 0.5$ years	$t = 1$ year	$t = 2$ years
0.2	0 (0)	0 (0)	0 (0)	0 (0)
0.3	0 (0)	0 (0)	0 (0)	0 (0)
0.4	0 (0)	0 (0)	0.00004 (0)	0.0002 (0)
0.5	0 (0)	0.00004 (0)	0.00052 (0)	0.0016 (0)
0.6	0.00001 (0)	0.00036 (0)	0.00234 (0)	0.00557 (0)

Panel 2: $r = 0.6, q = 0.5.$

0.2	0 (0)	0 (0)	0 (0)	0 (0)
0.3	0 (0)	0 (0)	0.00001 (0)	0.0002 (0)
0.4	0 (0)	0 (0)	0.00031 (0)	0.00184 (0)
0.5	0 (0)	0.00014 (0)	0.00186 (0)	0.00606 (0)
0.6	0.00002 (0)	0.00083 (0)	0.00559 (0)	0.01295 (0)

Panel 3: $r = 0.6,\ q = 0.7.$

0.2	0 (0)	0 (0)	0 (0)	0.0001 (0)
0.3	0 (0)	0 (0)	0.00003 (0)	0.0007 (0)
0.4	0 (0)	0.00001 (0)	0.00061 (0)	0.00364 (0)
0.5	0 (0)	0.00022 (0)	0.00289 (0)	0.00927 (0)
0.6	0.00003 (0)	0.00113 (0)	0.00756 (0)	0.01713 (0)

Fixed Strike, Backward Started Asian Options

Let $Y(t)$ denote the continuous type of average of O-U variables.

For every $t \in [0, T]$, the value of a fixed strike, backward started option is given by

$$C(t, X(t), Y(t)) =$$

$$e^{-r(T-t)} E_Q \left[(Y(T) - K)^+ / \mathcal{F}_t \right]$$

$$e^{-r(T-t)} \int_{K-A}^{+\infty} (y + A - K) p(y, T; t) dy$$

where

Q, is the EMM, $\quad A = \frac{1}{T-T_0} \int_0^t X(u) du$,

$$p(y, T; t) = \frac{1}{\sqrt{4d\pi}} e^{-\frac{(y-e)^2}{4d}},$$

$$d = \frac{\sigma^2 \gamma^2}{4r^3} \left((e^{r(T-t)} - 2)^2 - 1 \right) + \frac{\sigma^2 \gamma^2 (T-t)}{2r^2},$$

$$e = \frac{\gamma}{r} (e^{r(T-t)} - 1) X(t), \quad \gamma = \frac{1}{T - T_0}$$

The Price

For any $t \in [0, T]$, the value of a fixed strike, backward started Asian option is given by

$$C(t, X(t), Y(t)) = e^{-r(T-t)} \ \frac{\sqrt{d}}{\sqrt{\pi}} e^{-z^2}$$

$$\frac{e^{-r(T-t)}}{2} (e + A - K)) \left[1 + \text{erf}(z)\right]$$

where $z = \frac{e+A-K}{2\sqrt{d}}$.

The 'Greeks'

For any $t \in [0, T]$, the hedging parameters, Δ and Γ are given by

$$\Delta = \frac{\gamma}{r}(1 - e^{-r(T-t)})\,[1 + \mathrm{erf}(z)]$$

$$\Gamma = \frac{\gamma^2}{2r^2\sqrt{d\pi}}\left(e^{r(T-t)} + e^{-r(T-t)} - 2\right)e^{-z^2}$$

Fixed Strike, Forward-Started Asian Options

$\forall s \in [T_0, T]$, the values of forward and backward options are the same.

$\forall s \in [0, T_0)$, the value of the forward started option is given

$$C(s, X(s)) =$$

$$e^{-r(T_0-s)} E_Q \left[C(T_0, X(T_0), Y(T_0))/\mathcal{F}_s \right] =$$
$$e^{-r(T_0-s)} \int_{-\infty}^{+\infty} C(T_0, x, y; s) p(x, T_0; s) dx$$

where

Q, is the EMM,

$$p(x, T_0; s) \frac{1}{\sqrt{4a\pi}} e^{-\frac{(b-x)^2}{4a}},$$

$$\alpha = \frac{\sigma^2 (e^{2r(T_0-s)} - 1)}{4r}, \quad b = X(s) e^{r(T_0-s)}$$

The Price

For any $s \in [0, T_0)$, the value of the option is

$$C(s, X(s)) =$$

$$\frac{e^{-r(T-s)}}{2} \frac{\sqrt{2d}\sqrt{4\alpha\delta^2+2}}{\sqrt{\pi}} e^{-z^2}$$
$$+\frac{e^{-r(T-s)}}{2}(\delta b\sqrt{2d} - K)\left[1 + \text{erf}(z)\right]$$

where,

$$z = \frac{b\delta\sqrt{2d} - K}{\sqrt{2d}\sqrt{4\alpha\delta^2+2}}, \quad \delta = \frac{\gamma(e^{r(T-T_0)} - 1)}{r\sqrt{2d}},$$

$$d = \frac{\sigma^2\gamma^2}{4r^3}\left((e^{r(T-T_0)} - 2)^2 - 1\right) + \frac{\sigma^2\gamma^2(T-T_0)}{2r^2}$$

The 'Greeks'

For any $s \in [0, T_0)$, the hedging parameter Δ and Γ are given by

$$\Delta = \frac{\delta\sqrt{2d}}{2}\left[1 + \mathrm{erf}(z)\right]$$

$$\Gamma = \frac{\delta^2\sqrt{2d}}{\sqrt{\pi}\sqrt{4\delta^2\alpha + 2}} e^{-r(T-2T_0+s)} e^{-z^2}$$

Floating Strike, Backward Started Asian Options

For every $t \in [0, T]$, the value of a floating strike, backward started option is given by

$$C(t, X(t), Y(t)) =$$

$$e^{-r(T-t)} E_Q \left[(X(T) - Y(T))^+ / \mathcal{F}_t \right]$$
$$e^{-r(T-t)} \int_{-\infty}^{+\infty} \int_{y+A}^{+\infty} (x - y - A) p(x, y, T; t) dx dy$$

where

Q, is the EMM, $\quad A = \frac{1}{T - T_0} \int_0^t X(u) du$,

$$p(x, y, t) = \frac{e^{-\frac{4\alpha d}{4\alpha d - \hat{c}^2}\left[\frac{(b-x)^2}{2\alpha} - \frac{2\hat{c}}{\sqrt{4\alpha d}} \frac{(b-x)}{\sqrt{2\alpha}} \frac{(e-y)}{\sqrt{2d}} + \frac{(e-y)^2}{2d}\right]}}{2\pi \sqrt{4\alpha d - \hat{c}^2}}$$

$$\hat{c} = \frac{\gamma \sigma^2}{2r^2} (e^{r(T-t)} - 1)^2$$

The Price

For any $t \in [0, T]$, the value of a floating strike, backward started Asian option is given by

$$C(t, X(t), Y(t)) =$$

$$\sqrt{\frac{\alpha - \widehat{c} + d}{\pi}} e^{-r(T-t)} e^{-z^2} + \frac{(b - A - e)e^{-r(T-t)}}{2} [1 + \text{erf}(z)]$$

where $z = \frac{b-A-e}{2\sqrt{\alpha+d-\widehat{c}}}$.

The 'Greeks'

For any $t \in [0,T]$, the hedging parameters, Δ and Γ are given by

$$\Delta = \frac{e^{-r(T-t)}}{2}\left[e^{r(T-t)} - \frac{\gamma}{r}(e^{r(T-t)} - 1)\right][1 + \text{erf}(z)]$$

$$\Gamma = \frac{e^{-r(T-t)}}{2\sqrt{\pi}\sqrt{\alpha + d - \widehat{c}}}e^{-z^2}\left[e^{r(T-t)} - \frac{\gamma}{r}(e^{r(T-t)} - 1)\right]^2$$

Floating Strike, Forward-Started Asian Options

$\forall s \in [T_0, T]$, the values of forward and backward options are the same.

$\forall s \in [0, T_0)$, the value of the forward started option is given

$$C(s, X(s)) =$$

$$e^{-r(T_0-s)} E_Q\left[C(T_0, X(T_0), Y(T_0))/\mathcal{F}_s\right] =$$
$$e^{-r(T_0-s)} \int_{-\infty}^{+\infty} C(T_0, x, y; s) p(x, T_0; s) dx$$

where

Q, is the EMM,

$$p(x, T_0; s) \frac{1}{\sqrt{4a\pi}} e^{-\frac{(b-x)^2}{4a}},$$

$$\alpha = \frac{\sigma^2 (e^{2r(T_0-s)} - 1)}{4r}, \;\; b = X(s) e^{r(T_0-s)}$$

The Price

For any $s \in [0, T_0)$, value of the option is

$$C(s, X(s)) = e^{-r(T-s)} \frac{\sqrt{\alpha + d - \hat{c} + \hat{a}\lambda^2}}{\sqrt{\pi}} e^{-z^2} + \frac{\hat{b}\lambda e^{-r(T-s)}}{2} [1 + \text{erf}(z)]$$

where,

$$z = \frac{\hat{b}\lambda}{2\sqrt{\alpha + d - \hat{c} + \hat{a}\lambda^2)}},$$

$$\hat{a} = \frac{\sigma^2 (e^{2r(T_0 - s)} - 1)}{4r}, \quad \hat{b} = X(s) e^{r(T_0 - s)}$$

$$\lambda = e^{r(T - T_0)} - \frac{\gamma}{r} \left(e^{r(T - T_0)} - 1 \right)$$

The 'Greeks'

For any $s \in [0, T_0)$, the hedging parameter Δ and Γ are given by

$$\Delta = \frac{\lambda e^{-r(T-T_0)}}{2}\left[1 + \mathrm{erf}(z)\right]$$

$$\Gamma = \frac{\lambda^2}{2\sqrt{\pi(\hat{a}\lambda^2 + \alpha + d - \hat{c})}} e^{-r(T-2T_0+s)} e^{-z^2}$$

The Generalised Square Root Process

The dynamics of the GSRP are given by

$$\begin{aligned} dX(t) &= (\mu X(t) - c)\, dt + \sigma \sqrt{X(t)} dW(t) \\ X(0) &= X_0 \end{aligned}$$

Its state space is $[0, +\infty)$ and the origin is

- an absorbing boundary, if $c \geq 0$,
- a regular boundary, if $-\frac{\sigma^2}{2} < c < 0$,
- an entrance boundary, if $c \leq -\frac{\sigma^2}{2}$.

The Joint Density of the SR and the Integrated Process

$$p(x, y, T; t) =$$
$$\frac{2X(t)}{\widehat{\rho}\sqrt{\pi}} e^{\frac{r(x-X(t))}{\sigma^2}} e^{-\frac{4r^2 y}{\widehat{\rho}\sigma^4}} \left(\frac{\widehat{\rho}}{4y}\right)^2$$
$$\times \sum_{n=0}^{+\infty} \frac{e^{-a_n^2 \frac{\widehat{\rho}}{4y}} G_n(x, y, T)}{\begin{pmatrix} n+1 \\ n \end{pmatrix}}$$

where $\widehat{\rho} = \frac{8}{\sigma^2(T-T_0)}$,
$a_n = \frac{1}{2}[x + X(t) + \sigma^2(n+1)(T-t)]$, and

$$G_n(x, y, T) =$$
$$\sum_{m=0}^{n} \frac{(-1)^m}{m!} \begin{pmatrix} n+1 \\ n-m \end{pmatrix} \left(\frac{x}{2}\sqrt{\frac{\widehat{\rho}}{y}}\right)^m$$
$$\sum_{k=0}^{n} \frac{(-1)^k}{k!} \begin{pmatrix} n+1 \\ n-k \end{pmatrix} \left(\frac{X(t)}{2}\sqrt{\frac{\widehat{\rho}}{y}}\right)^k$$
$$\times H_{k+m+3}\left(\frac{a_n}{2}\sqrt{\frac{\widehat{\rho}}{y}}\right)$$

See Word Doc: page34.

Floating Strike, Backward Started Asian Options

For every $t \in [0, T]$, the value of a floating strike, backward started option is given by

$$C(t, X(t), Y(t)) =$$

$$e^{-r(T-t)} E_Q\left[(X(T) - Y(T))^+ / \mathcal{F}_t\right]$$
$$e^{-r(T-t)} \int_0^{+\infty} \int_{y+A}^{+\infty} (x - y - A) p(x, y, T; t) dx dy$$

where

Q, is the EMM,

$A = \frac{1}{T - T_0} \int_0^t X(u) du$, and

$p(x, y; t)$, is the joint density function of the pair (X(T), Y(T)) conditioned on information at time t.