Advanced Financial Models Michael Tehranchi

Sample question 5

This is essentially the proof of the 1-period 1FTAP

(a) Let H be as given, and suppose, for the sake of finding a contradiction, that $\rho > 0$ and $\mathbb{E}(\rho X) = 0$. Then

$$0 \leq \mathbb{E}(\rho H \cdot X) = H \cdot \mathbb{E}(\rho X) = 0$$

Hence $\rho\ H \cdot X = 0$ almost surely by the pigeonhole principle, and, since $\rho \neq 0$, we have $H \cdot X = 0$ almost surely, a contradiction.

(b) If $(h_k)_k$ is bounded, then by the Bolzano–Weierstrass theorem there exists a convergent subsequence. We therefore assume the minimising sequence is converges to a point h^*. Since F is smooth

$$F(h_k) \to F(h^*)$$

so h^* is the minimiser of F. By Fermat's first-order condition from calculus, we have

$$0 = \mathrm{grad} F(h^*) = -\mathbb{E}[e^{-h^* \cdot X} \zeta X]$$

and hence

$$\rho = \frac{e^{-h^* \cdot X} \zeta}{F(h^*)}$$

has the correct properties.

(c) Let $(h_k)_k$ be a minimising sequence. Let $\mathcal{U} = \{u \in \mathbb{R}^n : u \cdot X = 0\}$ and $\mathcal{V} = \mathcal{U}^\perp$. Since

$$F(u + v) = F(v)$$

for all $u \in \mathcal{U}$, $v \in \mathcal{V}$, we may assume that $h_k \in \mathcal{V}$ for all k. Since $(h_k)_k$ is unbounded, we may assume $\|h_k\| \to \infty$. Let

$$\hat{h}_k = \frac{h_k}{\|h_k\|}$$

Since $(\hat{h}_k)_k$ is bounded, it has a convergent subsequence. So we suppose $\hat{h}_k$ converges to H. Note $H \neq 0$ and $H \in \mathcal{V}$.

Note that

$$e^{-h_k \cdot X} = (e^{-\hat{h}_k \cdot X})^{\|h_k\|} \to \infty$$

on the event $\{H \cdot X < 0\}$. For the sake of finding a contradiction, suppose $\mathbb{P}(H \cdot X < 0) > 0$. By Fatou's lemma we would have

$$\liminf_k \mathbb{E}[e^{-h_k \cdot X} \zeta] \geq \mathbb{E}[\liminf_k e^{-h_k \cdot X} \zeta] = \infty$$

But this would contradice $F(h_k) \to f < \infty$, and so we conclude that

$$\mathbb{P}(H \cdot X \geq 0) = 1.$$

Since $H \in \mathcal{V}$ and $H \neq 0$, we have

$$\mathbb{P}(H \cdot X = 0) < 1.$$