Advanced Financial Models Michael Tehranchi
Sample question 2(b)

By the second fundamental theorem of asset pricing, for fixed $T > 0$, there exists a unique equivalent martingale measure $\mathbb{Q}$ on $(\Omega, \mathcal{F}_T)$ for the numéraire asset with price N, so that $(S_t/N_t)_{0\le t\le T}$ is a $\mathbb{Q}$-martingale.

Claim: If ξ_t is the payout of a replicable contingent claim with maturity t, then the intial cost of replication of this claim is $N_0\, \mathbb{E}^{\mathbb{Q}}(\xi_t/N_t)$.

Proof. Suppose that $(H_s)_{1\le s\le t}$ is self-financing, i.e. such that $H_s \cdot P_s = H_{s+1}\cdot P_s$ for all $1 \le s \le t-1$. Let $X_0 = H_1 \cdot P_0$ and $X_s = H_s \cdot P_s$ for $1 \le s \le t$. Then X/N is a $\mathbb{Q}$-martingale. Indeed, for $1 \le s \le t$ we have

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}(X_s/N_s|\mathcal{F}_{s-1}) &= \mathbb{E}^{\mathbb{Q}}(H_s \cdot P_s/N_s|\mathcal{F}_{s-1}) \\
&= H_s \cdot \mathbb{E}^{\mathbb{Q}}(P_s/N_s|\mathcal{F}_{s-1}) \\
&= H_s \cdot P_{s-1}/N_{s-1} \\
&= X_{s-1}
\end{aligned}$$

[Since the market is complete, all random variables are bounded, so there is no concern about integrability in the conditional expectations.] In particular, if $\xi_t = X_t$, then $N_0\mathbb{E}^{\mathbb{Q}}(\xi_t/N_t) = X_0 = H_1 \cdot P_0$, the initial cost of the replicating strategy.

Now, the initial cost of a replicating strategy for a call option with payout $\xi_t = (S_t - K)^+$ for $0 \le t \le T$ is given by

$$C(t,K) = N_0\; \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{N_t}(S_t - K)^+\right]$$

Now

$$\begin{aligned}
C(t+1,K) &= N_0\mathbb{E}^{\mathbb{Q}}\left[\left(\frac{S_{t+1}}{N_{t+1}} - \frac{K}{N_{t+1}}\right)^+\right] \\
&\ge N_0\; \mathbb{E}^{\mathbb{Q}}\left[\left(\frac{S_{t+1}}{N_{t+1}} - \frac{K}{N_t}\right)^+\right] \text{ since } N_{t+1} \ge N_t > 0 \text{ a.s.} \\
&= N_0\; \mathbb{E}^{\mathbb{Q}}\left\{\mathbb{E}^{\mathbb{Q}}\left[\left(\frac{S_{t+1}}{N_{t+1}} - \frac{K}{N_t}\right)^+\right]|\mathcal{F}_t\right\} \text{ tower} \\
&\ge N_0\; \mathbb{E}^{\mathbb{Q}}\left\{\left(\mathbb{E}^{\mathbb{Q}}\left[\frac{S_{t+1}}{N_{t+1}}|\mathcal{F}_t\right] - \frac{K}{N_t}\right)^+\right\} \text{ Jensen} \\
&= N_0\; \mathbb{E}^{\mathbb{Q}}\left[\left(\frac{S_t}{N_t} - \frac{K}{N_t}\right)^+\right] \text{ martingale} \\
&= C(t,K)
\end{aligned}$$

where we have used the $\mathcal{F}_t$-measurability of N_t and the convexity of $x \mapsto x^+$.

Remark 1. One could, in principle, discuss completeness without discussing arbitrage. For instance, consider the one-period case, with no dividends. The market is complete iff for every contingent claim with payout ξ_1 there exists a portfolion H_1 such that $\xi_1 = H_1 \cdot P_1$ almost surely. Note that this definition does not even mention the initial prices P_0. So one could, in principle, even discuss one-period completeness without even assuming that the market has time-0 prices.

However, question 2(b) discussed above asks about the initial replication cost $H_1 \cdot P_0$, so without some assumption on the intial prices, this problem would be impossible.

(By the way, we *were* able to prove that completeness implies that there are only a finite number of disjoint events of positive probablity without without assuming no-arbitrage. But in general, there is little point discussing completeness for a market that has arbitrage.)