Advanced Financial Models Michael Tehranchi

Sample question 1

(b) Note $F(0,m) = (1-m)^+$ and hence $C_T = (S_T - K)^+$. Now

$$\begin{aligned}\mathbb{E}(C_T|\mathcal{F}_t) &= \mathbb{E}[(S_T-K)^+|\mathcal{F}_t] \\ &= S_t\mathbb{E}\left[\left(e^{-\sigma^2(T-t)/2+\sigma(W_T-W_t)} - \frac{K}{S_t}\right)^+ |\mathcal{F}_t\right] \\ &= S_t F\left((T-t)\sigma^2, \frac{K}{S_t}\right) \\ &= C_t\end{aligned}$$

so $(C_t)_{0\le t\le T}$ is a martingale.

(c) For $0 \le s \le t$ write

$$\hat{S}_t = \hat{S}_s \mathbb{1}_{\{t\le\tau\}} e^{\lambda(t-s)} \frac{S_t}{S_s}.$$

Note that $\frac{S_t}{S_s} = e^{\sigma(W_t-W_s)+\mu(t-s)}$ and $\mathbb{1}_{\{t\le\tau\}}$ are independent given $\hat{\mathcal{F}}_s$.

Now

$$\mathbb{E}\left(\frac{S_t}{S_s}|\hat{\mathcal{F}}_s\right) = 1$$

since $\frac{S_t}{S_s}$ is independent of $\mathcal{F}_s$ and τ.

Also, we have

$$\mathbb{P}(\tau \ge t | \tau < s) = 0$$

and

$$\mathbb{P}(\tau \ge t | \tau \ge s) = e^{-\lambda(t-s)}$$

by the lack-of-memory property of exponentials. That is,

$$\mathbb{P}(\tau \ge t|\hat{\mathcal{F}}_s) = e^{-\lambda(t-s)}\mathbb{1}_{\{s\le\tau\}}$$

Hence

$$\mathbb{E}(\hat{S}_t|\hat{\mathcal{F}}_s) = \hat{S}_s$$

from which the conclusion follows.