Problem 1. Let $W$ be a Brownian motion and let $S_t = S_0 e^{\mu t + \sigma W_t}$ for a real constant $\mu$ and positive constants $\sigma, S_0$.

(a) Find $\mu$ such that the process $S$ is a martingale in its natural filtration.

For the rest of the question, let $\mu$ be such that $S$ is a martingale. Further, define a function by

$$F(v, m) = \int (e^{-v/2 + \sqrt{v}z} - m)^+ \phi(z) dz$$

for non-negative $v, m$ where $\phi(z) = e^{-z^2/2} \sqrt{2\pi}$ is the standard normal density.

(b) Fix positive constants $T, K$ and let $C_t = S_t F((T - t)\sigma^2, K/S_t)$ for $0 \leq t \leq T$. Show that $C$ is a martingale.

Now let $\hat{S}_t = \mathbb{1}_{\{t \leq \tau\}} e^{\lambda t} S_t$ where $\tau$ is an exponential random variable with rate $\lambda$, independent of $W$.

(c) Show that $\hat{S}$ is a martingale in its natural filtration.

Problem 2. (a) What does it mean to say that a discrete-time market model is complete? (Assume no asset pays a dividend.)

Consider discrete time model of a market with two assets, a numéraire with price process $N$ and a stock with price process $S$. Suppose the market is complete, and that $N_{t+1} \geq N_t$ almost surely for all $t \geq 0$. Let $C(T, K)$ be the initial replication cost of a European call option on the stock with strike $K$ and maturity $T$.

(b) Show that $T \mapsto C(T, K)$ is increasing for each $K > 0$.

(c) Compute $C(1, 18)$ in the case where $(N_0, S_0) = (10, 10)$ and $P((N_1, S_1) = (15, 20)) = 1/2 = P((N_1, S_1) = (20, 15))$

Problem 3. Let $(Z_t)_{0 \leq t \leq T}$ be a given discrete-time integrable process adapted to the filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$. Let $(U_t)_{0 \leq t \leq T}$ be its Snell envelope defined by

$$U_T = Z_T$$

$$U_t = \max \{Z_t, E[U_{t+1}|\mathcal{F}_t]\} \text{ for } 0 \leq t \leq T - 1.$$ 

(a) Show that $U$ is a supermartingale. Show that $U$ is a martingale if $Z$ is a submartingale.

Let $(S_t)_{0 \leq t \leq T}$ be such that the increments $S_1 - S_0, \ldots, S_T - S_{T-1}$ are independent and identically distributed, and let the filtration be generated by $S$. Fix a measurable function $f : \mathbb{R} \to \mathbb{R}$ and let $Z_t = f(S_t)$. Suppose that $Z_t$ is integrable for each $t \geq 0$, and let $U$ be the Snell envelope of $Z$.

(b) Show that there exists a deterministic function $V$ such that $U_t = V(t, S_t)$.

Problem 4. Suppose $(W_t)_{t \geq 0}$ is a Brownian motion and $(S_t)_{t \geq 0}$ evolves as

$$dS_t = a(S_t) dW_t.$$
Let $V : [0, T] \times \mathbb{R} \to \mathbb{R}_+$ be the unique solution to
\[
\frac{\partial}{\partial t} V(t, S) + \frac{a(S)^2}{2} \frac{\partial^2}{\partial S^2} V(t, S) = 0 \\
V(T, S) = g(S) \text{ for all } S \in \mathbb{R}.
\]
Finally, let $\xi_t = V(t, S_t)$ for $0 \leq t \leq T$. Assume that the functions $a$, $V$, and $g$ are smooth and bounded with bounded derivatives.

(a) Show that
\[
\xi_t = \mathbb{E}[g(S_T) | \mathcal{F}_t]
\]
where $(\mathcal{F}_t)_{t \geq 0}$ is the filtration generated by the Brownian motion.

Let $U : [0, T] \times \mathbb{R} \to \mathbb{R}$ be the unique solution to
\[
\frac{\partial}{\partial t} U(t, S) + a(S) a'(S) \frac{\partial}{\partial S} U(t, S) + \frac{a(S)^2}{2} \frac{\partial^2}{\partial S^2} U(t, S) = 0 \\
U(T, S) = g'(S) \text{ for all } S \in \mathbb{R}.
\]
Let $\pi_t = U(t, S_t)$ for $0 \leq t \leq T$. Assume $U$ is smooth and bounded with bounded derivatives.

(b) Show that
\[
\xi_t = V(0, S_0) + \int_0^t \pi_s dS_s.
\]

**Problem 5.** Let $X$ be a given an $n$-dimensional random vector.

(a) Suppose that $H \in \mathbb{R}^n$ is such that $H \cdot X \geq 0$ almost surely and $\mathbb{P}(H \cdot X > 0) > 0$. Prove that there does not exists a positive random variable $\rho$ such that $\mathbb{E}(\rho) = 1$ and $\mathbb{E}(\rho X) = 0$.

Given a positive random variable $\zeta$, define a function $F$ on $\mathbb{R}^n$ by
\[
F(h) = \mathbb{E}[e^{-h \cdot X \zeta}].
\]
Suppose $F$ is everywhere finite and smooth. Let
\[
f = \inf_{h \in \mathbb{R}^n} F(h).
\]
A sequence $(h_k)_k$ such that $F(h_k) \to f$ is called a minimising sequence.

(b) Suppose there exists a bounded minimising sequence. Show that there exists a positive random variable $\rho$ such that $\mathbb{E}(\rho) = 1$ and $\mathbb{E}(\rho X) = 0$.

(c) Suppose every minimising sequence is unbounded. Show that there exists a vector $H \in \mathbb{R}^n$ such that $H \cdot X \geq 0$ almost surely and $\mathbb{P}(H \cdot X > 0) > 0$.

**Problem 6.** Let $S$ be a positive random variable such that $\mathbb{E}(S) = 1$.

(a) Prove that
\[
M(\theta) = \mathbb{E}(e^{\theta \log S})
\]
is well-defined and bounded for all $\theta \in \{p + i q : 0 \leq p \leq 1, q \in \mathbb{R}\}$, where $i = \sqrt{-1}$.

(b) Prove the identity
\[
\mathbb{E}[(S - K)^+] = 1 - \frac{2\sqrt{K}}{\pi} \int_{-\infty}^{\infty} \frac{M(\frac{1}{2} + iy)}{1 + 4y^2} dy \text{ for all } K > 0.
\]

(c) Explain briefly why the above identity in part (b) is useful in the context of a stochastic volatility model such as the Heston model.
**Problem 7.** Consider a discrete-time market with \( n \) assets with prices \( (P_t)_{t \geq 0} \). No asset pays a dividend.

(a) What is an investment-consumption arbitrage? What is a terminal-consumption arbitrage?

(b) What is a numéraire strategy? Prove that if the market has an investment-consumption arbitrage and a numéraire strategy, then the market has a terminal consumption arbitrage.

**Problem 8.** Consider a discrete-time market model with prices \( (P^T_t)_{t \in [0,T], T \geq 1} \) where \( P^T_t \) is the price at time \( t \) of a risk-free zero-coupon bond of unit face value and maturity \( T \). Assume that the prices are adapted to a filtration \( (\mathcal{F}_t)_{t \geq 0} \), and that the market is free of arbitrage.

(a) Explain why \( P^T_t > 0 \) almost surely for all \( 0 \leq t \leq T \).

(b) Define the spot interest rate \( r_t \) in terms of the bond prices. Define the bank account \( B_t \) in terms of the spot interest rate. What does it mean to say a probability measure \( Q \) is a risk-neutral measure for this model?

(c) Show that \( T \mapsto P^T_t \) is non-increasing almost surely for all \( t \) if and only if \( r_t \geq 0 \) almost surely for all \( t \geq 0 \).

**Problem 9.** Consider a market with two assets, a bank account with time-\( t \) price \( e^{rt} \) and a stock whose price dynamics satisfy

\[
\begin{align*}
    dS_t &= S_t (r \, dt + \sqrt{v_t} \, dW_t) \\
    dv_t &= (a - bv_t) dt + c \sqrt{v_t} (pdW_t + \sqrt{1-\rho^2}dZ_t)
\end{align*}
\]

where \( r, a, b, c \) and \( \rho \) are constants, with \( a, b > 0 \) and \( -1 \leq \rho \leq 1 \), and \( W \) and \( Z \) are independent Brownian motions.

Let \( F : [0,T] \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \) satisfy the partial differential equation

\[
\frac{\partial F}{\partial t} + S_t \frac{\partial F}{\partial S} + (a - bv_t) \frac{\partial F}{\partial v} + \frac{1}{2} S_t^2 v \frac{\partial^2 F}{\partial S^2} + c \rho S_t v \frac{\partial^2 F}{\partial S \partial v} + \frac{1}{2} c^2 v \frac{\partial^2 F}{\partial v^2} = rF
\]

with boundary condition \( F(T, S, v) = \sqrt{S} \).

Introduce a contingent claim with time-\( T \) payout \( \xi_T = \sqrt{S_T} \).

(a) Show that there is no arbitrage relative to the bank account in the augmented market if the time-\( t \) price of the contingent claim is given by \( \xi_t = F(t, S_t, v_t) \). You may use a fundamental theorem of asset pricing as long as it is stated carefully. You may also use standard results from stochastic calculus, such as Itô’s formula, without justification.

Suppose that \( F(t, S, v) = \sqrt{S} e^{A(t)v+B(t)} \) for some functions \( A, B : [0, T] \to \mathbb{R} \).

(b) Show that the function \( A \) satisfies an ordinary differential equation. You should derive the equation, including the boundary conditions, but need not solve it.

**Problem 10.** Now suppose that \( X = (X_t)_{t \geq 0} \) is a discrete-time local martingale adapted to a filtration \( (\mathcal{F}_t)_{t \geq 0} \). Suppose \( \mathcal{F}_0 \) is trivial.

(a) Show that if \( X \) is integrable, then \( X \) is a true martingale.

(b) Show that if \( X \) is non-negative, then \( X \) is a true martingale.

(c) Let \( (K_t)_{t \geq 1} \) be a predictable process. Let \( M_0 = 0 \) and

\[
M_t = \sum_{s=1}^{t} K_s (X_s - X_{s-1})
\]
for $t \geq 1$. Show that $M$ is a local martingale.

**Problem 11.** Let $(S_t)_{t \geq 0}$ be a discrete-time martingale such that $S_0$ is an integer and for all $t \geq 1$ the increment $S_t - S_{t-1}$ is valued in the set $\{-1, 0, 1\}$.

(a) Prove the identity

$$(S_T - K - 1)^+ - 2(S_T - K)^+ + (S_T - K + 1)^+ = 1_{\{S_T = K\}}$$

for integers $K$ and $T \geq 0$.

(b) Prove the identity

$$(S_T - K)^+ = (S_0 - K)^+ + \sum_{t=1}^{T} f(S_{t-1} - K)(S_t - S_{t-1}) + \frac{1}{2} \sum_{t=1}^{T} 1_{\{S_t = K\}}(S_t - S_{t-1})^2$$

for integers $K$ and $T \geq 1$, where $f$ is defined by

$$f(x) = 1_{\{x > 0\}} + \frac{1}{2} 1_{\{x = 0\}}.$$

Let

$$C(T, K) = \mathbb{E}[(S_T - K)^+]$$

for integers $K$ and $T \geq 0$ and

$$\sigma^2(T, K) = \text{Var}(S_{T+1} | S_T = K)$$

for integers $K$ and $T$ such that $|K - S_0| \leq T$.

(c) Using parts (a) and (b), or otherwise, prove the identity

$$C(T + 1, K) - C(T, K) = \frac{1}{2} \sigma^2(T, K)[C(T, K + 1) - 2C(T, K) + C(T, K - 1)]$$

for integers $K$ and $T$ such that $|K - S_0| \leq T$.

**Problem 12.** Let $\xi$ be a random variable with finite exponential moments. Define two functions

$$C(k) = \mathbb{E}[(e^\xi - e^k)^+] \text{ for real } k$$

and

$$M(z) = \mathbb{E}[e^{z\xi}] \text{ for complex } z.$$

(a) Show that the identity

$$M(z) = \int_{-\infty}^{\infty} C(k)f(z, k)dk$$

holds for all complex $z = x + iy$ with $x > 1$, where $f(z, k) = z(z - 1)e^{(x-1)k}$.

(b) Show that the identity

$$C(k) = \frac{1}{2\pi i} \int_{x_0 - i\infty}^{x_0 + i\infty} \frac{M(z)}{f(z, k)}dz$$

holds for all real $k$ and $x_0 > 1$.

[You may assume a complex path integral can be computed as a Lebesgue integral by the formula]

$$\int_{x_0 - i\infty}^{x_0 + i\infty} h(z)dz = i \int_{-\infty}^{+\infty} h(x_0 + iy)dy.$$
Also, you may use the following identity without proof:

\[
\frac{1}{2\pi i} \int_{x_0-i\infty}^{x_0+i\infty} \frac{e^{az}}{z(z-1)} \, dz = (e^a - 1)^+ \]

for real \( a \) and real \( x_0 > 1 \).