Advanced Financial Models Michael Tehranchi
Sample questions

Problem 1. Let W be a Brownian motion and let $S_t = S_0 e^{\mu t + \sigma W_t}$ for a real constant μ and positive constants σ, S_0.
(a) Find μ such that the process S is a martingale in its natural filtration.

For the rest of the question, let μ be such that S is a martingale. Further, define a function by

$$F(v,m) = \int (e^{-v/2+\sqrt{v}z} - m)^+ \phi(z) dz$$

for non-negative v, m where $\phi(z) = \frac{e^{-z^2/2}}{\sqrt{2\pi}}$ is the standard normal density.
(b) Fix positive constants T, K and let

$$C_t = S_t F\left((T-t)\sigma^2, \frac{K}{S_t} \right)$$

for $0 \le t \le T$. Show that C is a martingale.

Now let $\hat{S}_t = \mathbb{1}_{\{t \le \tau\}} e^{\lambda t} S_t$ where τ is an exponential random variable with rate λ, independent of W.
(c) Show that $\hat{S}$ is a martingale in its natural filtration.

Problem 2. (a) What does it mean to say that a discrete-time market model is complete? (Assume no asset pays a dividend.)

Consider discrete time model of a market with two assets, a numéraire with price process N and a stock with price process S. Suppose the market is complete, and that $N_{t+1} \ge N_t$ almost surely for all $t \ge 0$. Let $C(T,K)$ be the initial replication cost of a European call option on the stock with strike K and maturity T.
(b) Show that $T \mapsto C(T,K)$ is increasing for each $K > 0$.
(c) Compute $C(1,18)$ in the case where $(N_0, S_0) = (10,10)$ and

$$\mathbb{P}((N_1,S_1) = (15,20)) = 1/2 = \mathbb{P}((N_1,S_1) = (20,15))$$

Problem 3. Let $(Z_t)_{0 \le t \le T}$ be a given discrete-time integrable process adapted to the filtration $(\mathcal{F}_t)_{0 \le t \le T}$. Let $(U_t)_{0 \le t \le T}$ be its Snell envelope defined by

$$\begin{aligned} U_T &= Z_T \\ U_t &= \max\{Z_t, \mathbb{E}[U_{t+1}|\mathcal{F}_t]\} \text{ for } 0 \le t \le T-1. \end{aligned}$$

(a) Show that U is a supermartingale. Show that U is a martingale if Z is a submartingale.

Let $(S_t)_{0 \le t \le T}$ be such that the increments $S_1 - S_0, \ldots, S_T - S_{T-1}$ are independent and identically distributed, and let the filtration be generated by S. Fix a measurable function $f : \mathbb{R} \to \mathbb{R}$ and let $Z_t = f(S_t)$. Suppose that Z_t is integrable for each $t \ge 0$, and let U be the Snell envelope of Z.
(b) Show that there exists a deterministic function V such that $U_t = V(t, S_t)$.

Problem 4. Suppose $(W_t)_{t \ge 0}$ is a Brownian motion and $(S_t)_{t \ge 0}$ evolves as

$$dS_t = a(S_t) dW_t.$$

Let $V : [0,T] \times \mathbb{R} \to \mathbb{R}_+$ be the unique solution to

$$\frac{\partial}{\partial t}V(t,S) + \frac{a(S)^2}{2}\frac{\partial^2}{\partial S^2}V(t,S) = 0$$
$$V(T,S) = g(S) \text{ for all } S \in \mathbb{R}.$$

Finally, let $\xi_t = V(t, S_t)$ for $0 \le t \le T$. Assume that the functions a, V, and g are smooth and bounded with bounded derivatives.
(a) Show that

$$\xi_t = \mathbb{E}[g(S_T)|\mathcal{F}_t]$$

where $(\mathcal{F}_t)_{t\ge 0}$ is the filtration generated by the Brownian motion.
Let $U : [0,T] \times \mathbb{R} \to \mathbb{R}$ be the unique solution to

$$\frac{\partial}{\partial t}U(t,S) + a(S)a'(S)\frac{\partial}{\partial S}U(t,S) + \frac{a(S)^2}{2}\frac{\partial^2}{\partial S^2}U(t,S) = 0$$
$$U(T,S) = g'(S) \text{ for all } S \in \mathbb{R}.$$

Let $\pi_t = U(t, S_t)$ for $0 \le t \le T$. Assume U is smooth and bounded with bounded derivatives.
(b) Show that

$$\xi_t = V(0, S_0) + \int_0^t \pi_s dS_s.$$

Problem 5. Let X be a given an n-dimensional random vector.
(a) Suppose that $H \in \mathbb{R}^n$ is such that $H \cdot X \ge 0$ almost surely and $\mathbb{P}(H \cdot X > 0) > 0$. Prove that there does not exists a positive random variable ρ such that $\mathbb{E}(\rho) = 1$ and $\mathbb{E}(\rho X) = 0$.
Given a positive random variable ζ, define a function F on $\mathbb{R}^n$ by

$$F(h) = \mathbb{E}[e^{-h\cdot X}\zeta].$$

Suppose F is everywhere finite and smooth. Let

$$f = \inf_{h\in\mathbb{R}^n} F(h).$$

A sequence $(h_k)_k$ such that $F(h_k) \to f$ is called a minimising sequence.
(b) Suppose there exists a bounded minimising sequence. Show that there exists a positive random variable ρ such that $\mathbb{E}(\rho) = 1$ and $\mathbb{E}(\rho X) = 0$.
(c) Suppose every minimising sequence is unbounded. Show that there exists a vector $H \in \mathbb{R}^n$ such that $H \cdot X \ge 0$ almost surely and $\mathbb{P}(H \cdot X > 0) > 0$.

Problem 6. Let S be a positive random variable such that $\mathbb{E}(S) = 1$.
(a) Prove that

$$M(\theta) = \mathbb{E}(e^{\theta \log S})$$

is well-defined and bounded for all $\theta \in \{p + \mathrm{i}q : 0 \le p \le 1, q \in \mathbb{R}\}$, where $\mathrm{i} = \sqrt{-1}$.
(b) Prove the identity

$$\mathbb{E}[(S-K)^+] = 1 - \frac{2\sqrt{K}}{\pi}\int_{-\infty}^{\infty} \frac{M\left(\frac{1}{2} + \mathrm{i}y\right)e^{-\mathrm{i}y\log K}}{1+4y^2} dy \text{ for all } K > 0.$$

(c) Explain briefly why the above identity in part (b) is useful in the context of a stochastic volatility model such as the Heston model.

Problem 7. Consider a discrete-time market with n assets with prices $(P_t)_{t\geq 0}$. No asset pays a dividend.
(a) What is an investment-consumption arbitrage? What is a terminal-consumption arbitrage?
(b) What is a numéraire strategy? Prove that if the market has an investment-consumption arbitrage and a numéraire strategy, then the market has a terminal consumption arbitrage.

Problem 8. Consider a discrete-time market model with prices $(P_t^T)_{t\in[0,T],T\geq 1}$ where P_t^T is the price at time t of a risk-free zero-coupon bond of unit face value and maturity T. Assume that the prices are adapted to a filtration $(\mathcal{F}_t)_{t\geq 0}$, and that the market is free of arbitrage.
(a) Explain why $P_t^T > 0$ almost surely for all $0 \leq t \leq T$.
(b) Define the spot interest rate r_t in terms of the bond prices. Define the bank account B_t in terms of the spot interest rate. What does it mean to say a probability measure $\mathbb{Q}$ is a risk-neutral measure for this model?
(c) Show that $T \mapsto P_t^T$ is non-increasing almost surely for all t if and only if $r_t \geq 0$ almost surely for all $t \geq 0$.

Problem 9. Consider a market with two assets, a bank account with time-t price e^{rt} and a stock whose price dynamics satisfy

$$\begin{aligned} dS_t &= S_t(r\ dt + \sqrt{v_t}dW_t) \\ dv_t &= (a - bv_t)dt + c\sqrt{v_t}(\rho dW_t + \sqrt{1-\rho^2}dZ_t) \end{aligned}$$

where r, a, b, c and ρ are contants, with $a, b > 0$ and $-1 \leq \rho \leq 1$, and W and Z are independent Brownian motions.

Let $F : [0,T] \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfy the partial differential equation

$$\frac{\partial F}{\partial t} + Sr\frac{\partial F}{\partial S} + (a - bv)\frac{\partial F}{\partial v} + \frac{1}{2}S^2 v\frac{\partial^2 F}{\partial S^2} + c\rho S v\frac{\partial^2 F}{\partial S \partial v} + \frac{1}{2}c^2 v\frac{\partial^2 F}{\partial v^2} = rF$$

with boundary condition $F(T, S, v) = \sqrt{S}$.

Introduce a contingent claim with time-T payout $\xi_T = \sqrt{S_T}$.

(a) Show that there is no arbitrage relative to the bank account in the augmented market if the time-t price of the contingent claim is given by $\xi_t = F(t, S_t, v_t)$. You may use a fundamental theorem of asset pricing as long as it is stated carefully. You may also use standard results from stochastic calculus, such as Itô's formula, without justification.

Suppose that $F(t, S, v) = \sqrt{S}e^{A(t)v+B(t)}$ for some functions $A, B : [0,T] \to \mathbb{R}$.

(b) Show that the function A satisfies an ordinary differential equation. You should derive the equation, including the boundary conditions, but need not solve it.

Problem 10. Now suppose that $X = (X_t)_{t\geq 0}$ is a discrete-time local martingale adapted to a filtration $(\mathcal{F}_t)_{t\geq 0}$. Suppose $\mathcal{F}_0$ is trivial.
(a) Show that if X is integrable, then X is a true martingale.
(b) Show that if X is non-negative, then X is a true martingale.
(c) Let $(K_t)_{t\geq 1}$ be a predictable process. Let $M_0 = 0$ and

$$M_t = \sum_{s=1}^{t} K_s(X_s - X_{s-1})$$

for $t \geq 1$. Show that M is a local martingale.

Problem 11. Let $(S_t)_{t\geq 0}$ be a discrete-time martingale such that S_0 is an integer and for all $t \geq 1$ the increment $S_t - S_{t-1}$ is valued in the set $\{-1, 0, 1\}$.
(a) Prove the identity

$$(S_T - K - 1)^+ - 2(S_T - K)^+ + (S_T - K + 1)^+ = \mathbf{1}_{\{S_T = K\}}$$

for integers K and $T \geq 0$.
(b) Prove the identity

$$(S_T - K)^+ = (S_0 - K)^+ + \sum_{t=1}^{T} f(S_{t-1} - K)(S_t - S_{t-1}) + \frac{1}{2}\sum_{t=1}^{T} \mathbf{1}_{\{S_t = K\}}(S_t - S_{t-1})^2$$

for integers K and $T \geq 1$, where f is defined by

$$f(x) = \mathbf{1}_{\{x>0\}} + \frac{1}{2}\mathbf{1}_{\{x=0\}}.$$

Let

$$C(T, K) = \mathbb{E}[(S_T - K)^+]$$

for integers K and $T \geq 0$ and

$$\sigma^2(T, K) = \mathrm{Var}(S_{T+1}|S_T = K)$$

for integers K and T such that $|K - S_0| \leq T$.
(c) Using parts (a) and (b), or otherwise, prove the identity

$$C(T+1, K) - C(T, K) = \frac{1}{2}\sigma^2(T, K)[C(T, K+1) - 2C(T, K) + C(T, K-1)]$$

for integers K and T such that $|K - S_0| \leq T$.

Problem 12. Let ξ be a random variable with finite exponential moments. Define two functions

$$C(k) = \mathbb{E}[(e^{\xi} - e^k)^+] \text{ for real } k$$

and

$$M(z) = \mathbb{E}[e^{z\xi}] \text{ for complex } z.$$

(a) Show that the identity

$$M(z) = \int_{-\infty}^{\infty} C(k) f(z, k) dk$$

holds for all complex $z = x + \mathrm{i}y$ with $x > 1$, where $f(z, k) = z(z-1)e^{(z-1)k}$.
(b) Show that the identity

$$C(k) = \frac{1}{2\pi \mathrm{i}} \int_{x_0 - \mathrm{i}\infty}^{x_0 + \mathrm{i}\infty} \frac{M(z)}{f(z, k)} dz$$

holds for all real k and $x_0 > 1$.

[You may assume a complex path integral can be computed as a Lebesgue integral by the formula

$$\int_{x_0 - \mathrm{i}\infty}^{x_0 + \mathrm{i}\infty} h(z) dz = \mathrm{i} \int_{-\infty}^{+\infty} h(x_0 + \mathrm{i}y) dy.$$

Also, you may use the following identity without proof:

$$\frac{1}{2\pi \mathrm{i}} \int_{x_0 - \mathrm{i}\infty}^{x_0 + \mathrm{i}\infty} \frac{e^{az}}{z(z-1)} dz = (e^a - 1)^+$$

for real a and real $x_0 > 1$.]