Advanced Financial Models Michael Tehranchi

Martingales and change of measure

Let Y and N be positive adapted processes. Assume that NY is a martingale. Fix $T > 0$, and let define a measure $\mathbb{Q}$ on $(\Omega, \mathcal{F}_T)$ by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{N_T Y_T}{N_0 Y_0}.$$

Now suppose the adapted process X is such that XY is a $\mathbb{P}$-martingale. Claim: X/N is a $\mathbb{Q}$-martingale.

Proof: By Bayes' formula we have

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\left(\frac{X_T}{N_T}|\mathcal{F}_t\right) &= \frac{\mathbb{E}^{\mathbb{P}}\left(X_T Y_T|\mathcal{F}_t\right)}{\mathbb{E}^{\mathbb{P}}\left(N_T Y_T|\mathcal{F}_t\right)} \\ &= \frac{X_t Y_t}{N_t Y_t} \\ &= \frac{X_t}{N_t}\end{aligned}$$

□

Remark 1. Let Y be a local martingale deflator, N the price of a numéraire asset and X the wealth from a self-financing strategy. In all cases we know that both NY and XY are local martingales. However, suppose that NY happens is a *true* martingale. Then we can define an equivalent martingale measure with respect to the numéraire. If XY also happens to be a true martingale for $\mathbb{P}$, then the above calculation shows that the discounted wealth X/N is a true martingale for $\mathbb{Q}$.

We used this idea around pages 80 of the notes, when when finding minimal cost replicating strategies via the martingale representation theorem. Indeed, the first step was to define a true martingale M via $M_t = \mathbb{E}^{\mathbb{P}}(\xi_T Y_T|\mathcal{F}_t)$ and then show that there exists a self-financing strategy whose weath process sastisfies $X_t Y_t = M_t$. So the claim says $X_t = N_t \mathbb{E}^{\mathbb{Q}}(\xi_T/N_T|\mathcal{F}_t)$.